CS156: The Calculus of Computation

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Chapter 10: Combining Decision Procedures

Combining Decision Procedures: Nelson-Oppen Method

Given

Theories T_i over *signatures* Σ_i
with corresponding decision procedures P_i for T_i-satisfiability.

Goal

Decide satisfiability of a formula F in theory $\cup_i T_i$.

Example: How do we show that

$$F : \; 1 \leq x \;\wedge\; x \leq 2 \;\wedge\; f(x) \neq f(1) \;\wedge\; f(x) \neq f(2)$$

is $(T_E \cup T_{\mathbb{Z}})$-unsatisfiable?

Combining Decision Procedures

Σ_1-theory T_1
$\boxed{P_1}$ for T_1-satisfiability

Σ_2-theory T_2
$\boxed{P_2}$ for T_2-satisfiability

↘ ? ↙

$\boxed{P}$ for $(T_1 \cup T_2)$-satisfiability

Problem:
Decision procedures are domain specific.
How do we combine them?

Nelson-Oppen Combination Method (N-O Method)

$$\Sigma_1 \cap \Sigma_2 = \{=\}$$

Σ_1-theory T_1
stably infinite

Σ_2-theory T_2
stably infinite

$\boxed{P_1}$ for T_1-satisfiability
of quantifier-free Σ_1-formulae

$\boxed{P_2}$ for T_2-satisfiability
of quantifier-free Σ_2-formulae

↘ ↙

$\boxed{P}$ for $(T_1 \cup T_2)$-satisfiability
of quantifier-free $(\Sigma_1 \cup \Sigma_2)$-formulae

Nelson-Oppen: Limitations

Given formula F in theory $T_1 \cup T_2$.

1. F must be quantifier-free.
2. Signatures Σ_i of the combined theory only share $=$, i.e.,

$$\Sigma_1 \cap \Sigma_2 = \{=\}$$

3. Theories must be stably infinite.

Note:

- Algorithm can be extended to combine arbitrary number of theories T_i — combine two, then combine with another, and so on.
- We restrict F to be conjunctive formula — otherwise convert to equivalent DNF and check each disjunct.

Stably Infinite Theories

A Σ-theory T is *stably infinite* iff
for every quantifier-free Σ-formula F:
if F is T-satisfiable
then there exists some T-interpretation that satisfies F
with infinite domain

Example: Σ-theory T

$$\Sigma : \{a, b, =\}$$

Axiom

$$\forall x.\ x = a \ \vee\ x = b$$

For every T-interpretation I, $|D_I| \leq 2$ (by the axiom — at most two elements).
Hence, T is *not* stably infinite.

All the other theories mentioned so far are stably infinite.

Example: T_E is stably infinite

Proof.

Let F be T_E-satisfiable quantifier-free Σ_E-formula
with arbitrary satisfying T_E-interpretation $I : (D_I, \alpha_I)$.
α_I maps $=$ to $=_I$.
Let A be any infinite set disjoint from D_I. Construct new interpretation $J : (D_J, \alpha_J)$ such that

- $D_J = D_I \cup A$
- α_J agrees with α_I: the extension of functions and predicates for A is irrelevant, except $=_J$. For $\mathsf{v}_1, \mathsf{v}_2 \in D_J$,

$$\mathsf{v}_1 =_J \mathsf{v}_2 \equiv \begin{cases} \mathsf{v}_1 =_I \mathsf{v}_2 & \text{if } \mathsf{v}_1, \mathsf{v}_2 \in D_I \\ \text{true} & \text{if } \mathsf{v}_1 \text{ is the same element as } \mathsf{v}_2 \\ \text{false} & \text{otherwise} \end{cases}$$

J is a T_E-interpretation satisfying F with infinite domain.
Hence, T_E is stably infinite.

Example

Consider quantifier-free conjunctive $(\Sigma_E \cup \Sigma_{\mathbb{Z}})$-formula

$$F:\ 1 \leq x \ \wedge\ x \leq 2 \ \wedge\ f(x) \neq f(1) \ \wedge\ f(x) \neq f(2)\ .$$

The signatures of T_E and $T_{\mathbb{Z}}$ only share $=$. Also, both theories are stably infinite. Hence, the N-O combination of the decision procedures for T_E and $T_{\mathbb{Z}}$ decides the $(T_E \cup T_{\mathbb{Z}})$-satisfiability of F.

Intuitively, F is $(T_E \cup T_{\mathbb{Z}})$-unsatisfiable.
For the first two literals imply $x = 1 \ \vee\ x = 2$ so that
$f(x) = f(1) \ \vee\ f(x) = f(2)$.
Contradict last two literals.
Hence, F is $(T_E \cup T_{\mathbb{Z}})$-unsatisfiable.

Nelson-Oppen Method: Overview

Consider quantifier-free conjunctive $(\Sigma_1 \cup \Sigma_2)$-formula F.

Two versions:

- nondeterministic — simple to present, but high complexity
- deterministic — efficient

Nelson-Oppen (N-O) method proceeds in two steps:

- Phase 1 (variable abstraction)
  — same for both versions
- Phase 2
  nondeterministic: guess equalities/disequalities and check
  deterministic: generate equalities/disequalities by equality propagation

Phase 1: Variable abstraction

Given quantifier-free conjunctive $(\Sigma_1 \cup \Sigma_2)$-formula F.
Transform F into two quantifier-free conjunctive formulae

$$\Sigma_1\text{-formula } F_1 \quad \text{and} \quad \Sigma_2\text{-formula } F_2$$

s.t. F is $(T_1 \cup T_2)$-satisfiable iff $F_1 \wedge F_2$ is $(T_1 \cup T_2)$-satisfiable

F_1 and F_2 are linked via a set of shared variables:

$$\text{shared}(F_1, F_2) = \text{free}(F_1) \cap \text{free}(F_2)$$

For term t, let $\text{hd}(t)$ be the root symbol, e.g. $\text{hd}(f(x)) = f$.

Generation of F_1 and F_2

For $i, j \in \{1, 2\}$ and $i \neq j$, repeat the transformations

(1) if function $f \in \Sigma_i$ and $\text{hd}(t) \in \Sigma_j$,

$$F[f(t_1, \ldots, t, \ldots, t_n)] \quad \Rightarrow \quad F[f(t_1, \ldots, w, \ldots, t_n)] \wedge w = t$$

(2) if predicate $p \in \Sigma_i$ and $\text{hd}(t) \in \Sigma_j$,

$$F[p(t_1, \ldots, t, \ldots, t_n)] \quad \Rightarrow \quad F[p(t_1, \ldots, w, \ldots, t_n)] \wedge w = t$$

(3) if $\text{hd}(s) \in \Sigma_i$ and $\text{hd}(t) \in \Sigma_j$,

$$F[s = t] \quad \Rightarrow \quad F[w = t] \wedge w = s$$
$$F[s \neq t] \quad \Rightarrow \quad F[w \neq t] \wedge w = s$$

where w is a fresh variable in each application of a transformation.

Example

Consider $(\Sigma_E \cup \Sigma_{\mathbb{Z}})$-formula

$$F: \ 1 \leq x \ \wedge \ x \leq 2 \ \wedge \ f(x) \neq f(1) \ \wedge \ f(x) \neq f(2) \ .$$

By transformation 1, since $f \in \Sigma_E$ and $1 \in \Sigma_{\mathbb{Z}}$,
replace $f(1)$ by $f(w_1)$ and add $w_1 = 1$. Similarly,
replace $f(2)$ by $f(w_2)$ and add $w_2 = 2$.
Hence, construct the $\Sigma_{\mathbb{Z}}$-formula

$$F_{\mathbb{Z}}: \ 1 \leq x \ \wedge \ x \leq 2 \ \wedge \ w_1 = 1 \ \wedge \ w_2 = 2$$

and the Σ_E-formula

$$F_E: \ f(x) \neq f(w_1) \ \wedge \ f(x) \neq f(w_2) \ .$$

$F_{\mathbb{Z}}$ and F_E share the variables $\{x, w_1, w_2\}$.
$F_{\mathbb{Z}} \wedge F_E$ is $(T_E \cup T_{\mathbb{Z}})$-equisatisfiable to F.

Example

Consider $(\Sigma_E \cup \Sigma_{\mathbb{Z}})$-formula

$$F : f(x) = x + y \land x \leq y + z \land x + z \leq y \land y = 1 \land f(x) \neq f(2) .$$

In the first literal, $\text{hd}(f(x)) = f \in \Sigma_E$ and $\text{hd}(x + y) = + \in \Sigma_{\mathbb{Z}}$; thus, by (3), replace the literal with

$$w_1 = x + y \ \land \ w_1 = f(x) .$$

In the final literal, $f \in \Sigma_E$ but $2 \in \Sigma_{\mathbb{Z}}$, so by (1), replace it with

$$f(x) \neq f(w_2) \ \land \ w_2 = 2 .$$

Now, separating the literals results in two formulae:

$$F_{\mathbb{Z}} : \ w_1 = x + y \ \land \ x \leq y + z \ \land \ x + z \leq y \ \land \ y = 1 \ \land \ w_2 = 2$$

is a $\Sigma_{\mathbb{Z}}$-formula, and

$$F_E : \ w_1 = f(x) \ \land \ f(x) \neq f(w_2)$$

is a Σ_E-formula.

The conjunction $F_{\mathbb{Z}} \ \land \ F_E$ is $(T_E \cup T_{\mathbb{Z}})$-equisatisfiable to F.

Nondeterministic Version

Phase 2: Guess and Check

- Phase 1 separated $(\Sigma_1 \cup \Sigma_2)$-formula F into two formulae:

$$\Sigma_1\text{-formula } F_1 \quad \text{and} \quad \Sigma_2\text{-formula } F_2$$

- F_1 and F_2 are linked by a set of shared variables:

$$V = \text{shared}(F_1, F_2) = \text{free}(F_1) \cap \text{free}(F_2)$$

- Let E be an equivalence relation over V.
- The arrangement $\alpha(V, E)$ of V induced by E is:

$$\alpha(V, E) : \bigwedge_{u,v \,\in\, V.\ uEv} u = v \ \land \bigwedge_{u,v \,\in\, V.\ \neg(uEv)} u \neq v$$

Nondeterministic Version

Lemma

the original formula F is $(T_1 \cup T_2)$-satisfiable iff there exists an equivalence relation E over V s.t.

(1) $F_1 \ \land \ \alpha(V, E)$ is T_1-satisfiable, and

(2) $F_2 \ \land \ \alpha(V, E)$ is T_2-satisfiable.

Otherwise, F is $(T_1 \cup T_2)$-unsatisfiable.

Example 1

Consider $(\Sigma_E \cup \Sigma_{\mathbb{Z}})$-formula

$$F : \ 1 \leq x \ \land \ x \leq 2 \ \land \ f(x) \neq f(1) \ \land \ f(x) \neq f(2)$$

Phase 1 separates this formula into the $\Sigma_{\mathbb{Z}}$-formula

$$F_{\mathbb{Z}} : \ 1 \leq x \ \land \ x \leq 2 \ \land \ w_1 = 1 \ \land \ w_2 = 2$$

and the Σ_E-formula

$$F_E : \ f(x) \neq f(w_1) \ \land \ f(x) \neq f(w_2)$$

with

$$V = \text{shared}(F_1, F_2) = \{x, w_1, w_2\}$$

There are 5 equivalence relations over V to consider, which we list by stating the partitions:

Example 1

1. $\{\{x, w_1, w_2\}\}$, *i.e.*, $x = w_1 = w_2$:
   $x = w_1$ and $f(x) \neq f(w_1) \;\Rightarrow\; F_E \wedge \alpha(V, E)$ is T_E-unsatisfiable.
2. $\{\{x, w_1\}, \{w_2\}\}$, *i.e.*, $x = w_1$, $x \neq w_2$:
   $x = w_1$ and $f(x) \neq f(w_1) \;\Rightarrow\; F_E \wedge \alpha(V, E)$ is T_E-unsatisfiable.
3. $\{\{x, w_2\}, \{w_1\}\}$, *i.e.*, $x = w_2$, $x \neq w_1$:
   $x = w_2$ and $f(x) \neq f(w_2) \;\Rightarrow\; F_E \wedge \alpha(V, E)$ is T_E-unsatisfiable.
4. $\{\{x\}, \{w_1, w_2\}\}$, *i.e.*, $x \neq w_1$, $w_1 = w_2$:
   $w_1 = w_2$ and $w_1 = 1 \wedge w_2 = 2$
   $\Rightarrow F_{\mathbb{Z}} \wedge \alpha(V, E)$ is $T_{\mathbb{Z}}$-unsatisfiable.
5. $\{\{x\}, \{w_1\}, \{w_2\}\}$, *i.e.*, $x \neq w_1$, $x \neq w_2$, $w_1 \neq w_2$:
   $x \neq w_1 \wedge x \neq w_2$ and $x = w_1 = 1 \vee x = w_2 = 2$
   (since $1 \leq x \leq 2$ implies that $x = 1 \vee x = 2$ in $T_{\mathbb{Z}}$)
   $\Rightarrow F_{\mathbb{Z}} \wedge \alpha(V, E)$ is $T_{\mathbb{Z}}$-unsatisfiable.

Hence, F is $(T_E \cup T_{\mathbb{Z}})$-unsatisfiable.

Example 2

Consider the $(\Sigma_{\text{cons}} \cup \Sigma_{\mathbb{Z}})$-formula

$$F : \; \text{car}(x) + \text{car}(y) = z \wedge \text{cons}(x, z) \neq \text{cons}(y, z) \; .$$

After two applications of (1), Phase 1 separates F into the Σ_{cons}-formula

$$F_{cons} : \; w_1 = \text{car}(x) \wedge w_2 = \text{car}(y) \wedge \text{cons}(x, z) \neq \text{cons}(y, z)$$

and the $\Sigma_{\mathbb{Z}}$-formula

$$F_{\mathbb{Z}} : \; w_1 + w_2 = z \; ,$$

with

$$V = \text{shared}(F_{\text{cons}}, F_{\mathbb{Z}}) = \{z, w_1, w_2\} \; .$$

Example 2

Consider the equivalence relation E given by the partition

$$\{\{z\}, \{w_1\}, \{w_2\}\} \; .$$

The arrangement

$$\alpha(V, E) : \; z \neq w_1 \wedge z \neq w_2 \wedge w_1 \neq w_2$$

satisfies both F_{cons} and $F_{\mathbb{Z}}$:

$F_{cons} \wedge \alpha(V, E)$ is T_{cons}-satisfiable, and

$F_{\mathbb{Z}} \wedge \alpha(V, E)$ is $T_{\mathbb{Z}}$-satisfiable.

Hence, F is $(T_{\text{cons}} \cup T_{\mathbb{Z}})$-satisfiable.

Practical Efficiency

Phase 2 was formulated as "guess and check":

1. First, guess an equivalence relation E,
2. then check the induced arrangement.

The number of equivalence relations grows super-exponentially with the # of shared variables. It is given by Bell numbers.
E.g., 12 shared variables ⇒ over four million equivalence relations.

Solution: Deterministic Version

Deterministic Version

Phase 1 as before
Phase 2 asks the decision procedures P_1 and P_2 to propagate new equalities.

Example 3

Theory of equality T_E — P_E

Rational linear arithmethic $T_{\mathbb{Q}}$ — $P_{\mathbb{Q}}$

$$F : \quad f(f(x)-f(y)) \neq f(z) \ \wedge\ x \leq y \ \wedge\ y+z \leq x \ \wedge\ 0 \leq z$$

$(T_E \cup T_{\mathbb{Q}})$-unsatisfiable

Intuitively,
last 3 conjuncts $\Rightarrow$ $x = y \ \wedge\ z = 0$
contradicts 1st conjunct

Phase 1: Variable Abstraction

Example 3

$$F : \ f(f(x) - f(y)) \neq f(z) \ \wedge\ x \leq y \ \wedge\ y+z \leq x \ \wedge\ 0 \leq z$$

Replace $f(x)$ by u, $f(y)$ by v, $u - v$ by w

$F_E :$ $f(w) \neq f(z) \ \wedge\ u = f(x) \ \wedge\ v = f(y)$ ... T_E-formula

$F_{\mathbb{Q}} :$ $x \leq y \ \wedge\ y+z \leq x \ \wedge\ 0 \leq z \ \wedge\ w = u - v$ ... $T_{\mathbb{Q}}$-formula

$$\text{shared}(F_E, F_{\mathbb{Q}}) = \{x, y, z, u, v, w\}$$

Nondeterministic version — over 200 Es!
Let's try the deterministic version.

Phase 2: Equality Propagation

Example 3

$F_E :$ $f(w) \neq f(z) \ \wedge\ u = f(x) \ \wedge\ v = f(y)$
$F_{\mathbb{Q}} :$ $x \leq y \ \wedge\ y+z \leq x \ \wedge\ 0 \leq z \ \wedge\ w = u - v$

$P_{\mathbb{Q}}$ | | P_E

$\{\}$

$F_{\mathbb{Q}} \models x = y$

$\{x = y\}$

$F_E \wedge x = y \models u = v$

$\{x = y, u = v\}$

$F_{\mathbb{Q}} \wedge u = v \models z = w$

$\{x = y, u = v, z = w\}$

$F_E \wedge z = w \models \bot$

$\bot$

Contradiction. Thus, F is $(T_{\mathbb{Q}} \cup T_E)$-unsatisfiable.
(If there were no contradiction, F would be $(T_{\mathbb{Q}} \cup T_E)$-satisfiable.)

Convex Theories

Definition
A Σ-theory T is *convex* iff
for every quantifier-free conjunctive Σ-formula F
and for every disjunction $\bigvee_{i=1}^{n} (u_i = v_i)$
if $F \Rightarrow \bigvee_{i=1}^{n} (u_i = v_i)$
then $F \Rightarrow u_i = v_i$, for some $i \in \{1, \ldots, n\}$

Claim
Equality propagation is a decision procedure for convex theories.

Convex Theories

- T_E, $T_{\mathbb{R}}$, $T_{\mathbb{Q}}$, T_{cons} are convex
- $T_{\mathbb{Z}}$, T_A are not convex

Example: $T_{\mathbb{Z}}$ is not convex
Consider quantifier-free conjunctive $\Sigma_{\mathbb{Z}}$-formula

$$F: \quad 1 \leq z \ \wedge\ z \leq 2 \ \wedge\ u = 1 \ \wedge\ v = 2$$

Then

$$F \quad \Rightarrow \quad z = u \vee z = v$$

but

$$F \quad \not\Rightarrow \quad z = u$$
$$F \quad \not\Rightarrow \quad z = v$$

Convex Theories

Example: Theory of arrays T_A is not convex
Consider the quantifier-free conjunctive Σ_A-formula

$$F: \ a\langle i \triangleleft v\rangle[j] = v \ .$$

Then

$$F \ \Rightarrow \ i = j \ \vee \ a[j] = v \ ,$$

but

$$F \not\Rightarrow i = j$$
$$F \not\Rightarrow a[j] = v \ .$$

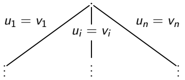
What if T is Not Convex?

Case split when:

$$F \quad \Rightarrow \quad \bigvee_{i=1}^{n} (u_i = v_i)$$

but $F \not\Rightarrow u_i = v_i$ for any $i = 1, \ldots, n$

- For each $i = 1, \ldots, n$, construct a branch on which $u_i = v_i$ is assumed.
- If all branches are contradictory, then **unsatisfiable**. Otherwise, **satisfiable**.



Claim: Equality propagation (with branching) is a decision procedure for non-convex theories too.

Example 1: Non-Convex Theory

$T_{\mathbb{Z}}$ not convex! $\boxed{P_{\mathbb{Z}}}$ | T_E convex $\boxed{P_E}$

$$F \ : \ 1 \leq x \ \wedge\ x \leq 2 \ \wedge\ f(x) \neq f(1) \ \wedge\ f(x) \neq f(2)$$

in $T_{\mathbb{Z}} \cup T_E$.

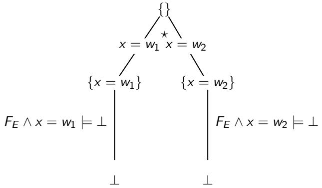
- Replace $f(1)$ by $f(w_1)$, and add $w_1 = 1$.
- Replace $f(2)$ by $f(w_2)$, and add $w_2 = 2$.

Result:

$$F_{\mathbb{Z}} \ : \ 1 \leq x \ \wedge\ x \leq 2 \ \wedge\ w_1 = 1 \ \wedge\ w_2 = 2$$
$$F_E \ : \ f(x) \neq f(w_1) \ \wedge\ f(x) \neq f(w_2)$$

and

$$V = \text{shared}(F_{\mathbb{Z}}, F_E) = \{x, w_1, w_2\}$$



$\star: \; F_{\mathbb{Z}} \models x = w_1 \;\vee\; x = w_2$

All leaves are labeled with $\bot \Rightarrow F$ is $(T_{\mathbb{Z}} \cup T_E)$-unsatisfiable.

Example 4: Non-Convex Theory

Consider

$$F \;:\; 1 \leq x \;\wedge\; x \leq 3 \;\wedge\; f(x) \neq f(1) \;\wedge\; f(x) \neq f(3) \;\wedge\; f(1) \neq f(2)$$

in $T_{\mathbb{Z}} \cup T_E$.

- Replace $f(1)$ by $f(w_1)$, and add $w_1 = 1$.
- Replace $f(2)$ by $f(w_2)$, and add $w_2 = 2$.
- Replace $f(3)$ by $f(w_3)$, and add $w_3 = 3$.

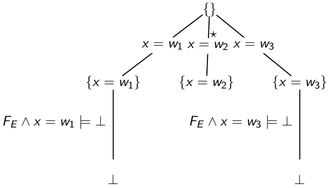
Result:

$$F_{\mathbb{Z}} \;:\; 1 \leq x \;\wedge\; x \leq 3 \;\wedge\; w_1 = 1 \;\wedge\; w_2 = 2 \;\wedge\; w_3 = 3$$
$$F_E \;:\; f(x) \neq f(w_1) \;\wedge\; f(x) \neq f(w_3) \;\wedge\; f(w_1) \neq f(w_2)$$

and

$$V = \text{shared}(F_{\mathbb{Z}}, F_E) = \{x, w_1, w_2, w_3\}$$

Example 4: Non-Convex Theory



$\star: \; F_{\mathbb{Z}} \models x = w_1 \;\vee\; x = w_2 \;\vee\; x = w_3$

No more equations on middle leaf $\Rightarrow F$ is $(T_{\mathbb{Z}} \cup T_E)$-satisfiable.