CS156: The Calculus of Computation

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Autumn 2008

Chapter 9: Quantifier-free Equality and Data Structures

The Theory of Equality T_E

$$\Sigma_E : \{=, a, b, c, \ldots, f, g, h, \ldots, p, q, r, \ldots\}$$

uninterpreted symbols:
- constants $a, b, c, \ldots$
- functions $f, g, h, \ldots$
- predicates $p, q, r, \ldots$

Example:

$x = y \land f(x) \neq f(y)$ T_E-unsatisfiable

$f(x) = f(y) \land x \neq y$ T_E-satisfiable

$f(f(f(a))) = a \land f(f(f(f(f(a))))) = a \land f(a) \neq a$ T_E-unsatisfiable

$x = g(y, z) \rightarrow f(x) = f(g(y, z))$ T_E-valid

Axioms of T_E

1. $\forall x.\ x = x$ (reflexivity)
2. $\forall x, y.\ x = y \rightarrow y = x$ (symmetry)
3. $\forall x, y, z.\ x = y \land y = z \rightarrow x = z$ (transitivity)

define $=$ to be an equivalence relation.

Axiom schema

4. for each positive integer n and n-ary function symbol f,

$$\forall \bar{x}, \bar{y}.\ \left(\bigwedge_{i=1}^{n} x_i = y_i\right) \rightarrow f(\bar{x}) = f(\bar{y})$$

(function)

For example, for unary f, the axiom is

$$\forall x', y'.\ x' = y' \rightarrow f(x') = f(y')$$

Therefore,

$$x = g(y, z) \rightarrow f(x) = f(g(y, z))$$

is T_E-valid. $(x' \rightarrow x, y' \rightarrow g(y, z))$.

Axiom schema

5. for each positive integer n and n-ary predicate symbol p,

$$\forall \bar{x}, \bar{y}.\ \left(\bigwedge_{i=1}^{n} x_i = y_i\right) \rightarrow (p(\bar{x}) \leftrightarrow p(\bar{y}))$$

(predicate)

Thus, for unary p, the axiom is

$$\forall x', y'. x' = y' \rightarrow (p(x') \leftrightarrow p(y'))$$

Therefore,

$$a = b \rightarrow (p(a) \leftrightarrow p(b))$$

is T_E-valid. $(x' \rightarrow a, y' \rightarrow b)$.

We discuss T_E-formulae without predicates

For example, for Σ_E-formula

$$F: \; p(x) \wedge q(x,y) \wedge q(y,z) \rightarrow \neg q(x,z)$$

introduce fresh constant $\bullet$ and fresh functions f_p and f_q, and transform F to

$$G: \; f_p(x) = \bullet \wedge f_q(x,y) = \bullet \wedge f_q(y,z) = \bullet \rightarrow f_q(x,z) \neq \bullet \; .$$

Equivalence and Congruence Relations: Basics

Binary relation R over set S

- is an equivalence relation if
  - reflexive: $\forall s \in S.\; s \; R \; s$;
  - symmetric: $\forall s_1, s_2 \in S.\; s_1 \; R \; s_2 \rightarrow s_2 \; R \; s_1$;
  - transitive: $\forall s_1, s_2, s_3 \in S.\; s_1 \; R \; s_2 \wedge s_2 \; R \; s_3 \rightarrow s_1 \; R \; s_3$.

Example:

Define the binary relation $\equiv_2$ over the set $\mathbb{Z}$ of integers

$$m \equiv_2 n \quad \text{iff} \quad (m \bmod 2) = (n \bmod 2)$$

That is, $m, n \in \mathbb{Z}$ are related iff they are both even or both odd.

$\equiv_2$ is an equivalence relation

- is a congruence relation if in addition

$$\forall \bar{s}, \bar{t}.\; \bigwedge_{i=1}^{n} s_i \; R \; t_i \rightarrow f(\bar{s}) \; R \; f(\bar{t}) \; .$$

Classes

For $\left\{ \begin{array}{c} \text{equivalence} \\ \text{congruence} \end{array} \right\}$ relation R over set S,

the $\left\{ \begin{array}{c} \text{equivalence} \\ \text{congruence} \end{array} \right\}$ class of $s \in S$ under R is

$$[s]_R \stackrel{\text{def}}{=} \{s' \in S \; : \; sRs'\} \; .$$

Example:

The equivalence class of 3 under $\equiv_2$ over $\mathbb{Z}$ is

$$[3]_{\equiv_2} = \{n \in \mathbb{Z} \; : \; n \text{ is odd}\} \; .$$

Partitions

A partition P of S is a set of subsets of S that is

- total $\left(\bigcup_{S' \in P} S' \right) = S$
- disjoint $\forall S_1, S_2 \in P.\; S_1 \neq S_2 \rightarrow S_1 \cap S_2 = \emptyset$

Quotient

The quotient S/R of S by $\left\{ \begin{array}{c} \text{equivalence} \\ \text{congruence} \end{array} \right\}$ relation R is the partition of S into $\left\{ \begin{array}{c} \text{equivalence} \\ \text{congruence} \end{array} \right\}$ classes

$$S/R = \{[s]_R \; : \; s \in S\} \; .$$

It satisfies total and disjoint conditions.

Example: The quotient $\mathbb{Z}/\equiv_2$ is a partition of $\mathbb{Z}$. The set of equivalence classes

$$\{\{n \in \mathbb{Z} \; : \; n \text{ is odd}\}, \; \{n \in \mathbb{Z} \; : \; n \text{ is even}\}\}$$

Note duality between relations and classes

Refinements

Two binary relations R_1 and R_2 over set S.
R_1 is a refinement of R_2, $R_1 \prec R_2$, if

$$\forall s_1, s_2 \in S.\ s_1 R_1 s_2 \ \rightarrow \ s_1 R_2 s_2 \ .$$

R_1 refines R_2.

Examples:

- For $S = \{a, b\}$,
  $R_1 : \{aR_1 b\}$ $\quad$ $R_2 : \{aR_2 b,\ bR_2 b\}$
  Then $R_1 \prec R_2$
- For set $\mathbb{Z}$
  $R_1 : \{xR_1 y \ : \ x \bmod 2 = y \bmod 2\}$
  $R_2 : \{xR_2 y \ : \ x \bmod 4 = y \bmod 4\}$
  Then $R_2 \prec R_1$.

Closures

Given binary relation R over S.
The equivalence closure R^E of R is the equivalence relation s.t.

- R refines R^E, i.e. $R \prec R^E$;
- for all other equivalence relations R' s.t. $R \prec R'$, either $R' = R^E$ or $R^E \prec R'$

That is, R^E is the "smallest" equivalence relation that "covers" R.

Example: If $S = \{a, b, c, d\}$ and $R = \{aRb, bRc, dRd\}$, then

- $aR^E b, bR^E c, dR^E d$ $\quad$ since $R \subseteq R^E$;
- $aR^E a, bR^E b, cR^E c$ $\quad$ by reflexivity;
- $bR^E a, cR^E b$ $\quad$ by symmetry;
- $aR^E c$ $\quad$ by transitivity;
- $cR^E a$ $\quad$ by symmetry.

Similarly, the congruence closure R^C of R is the "smallest" congruence relation that "covers" R.

T_E-satisfiability and Congruence Classes I

Definition: For Σ_E-formula

$$F : \ s_1 = t_1 \ \wedge \ \cdots \ \wedge \ s_m = t_m \ \wedge \ s_{m+1} \neq t_{m+1} \ \wedge \ \cdots \ \wedge \ s_n \neq t_n$$

the subterm set S_F of F is the set that contains precisely the subterms of F.

Example: The subterm set of

$$F : \ f(a, b) = a \ \wedge \ f(f(a, b), b) \neq a$$

is

$$S_F = \{a,\ b,\ f(a, b),\ f(f(a, b), b)\} \ .$$

Note: we consider only quantifier-free conjunctive Σ_E-formulae. Convert non-conjunctive formula F to DNF $\bigvee_i F_i$, where each disjunct F_i is a conjunction of $=, \neq$. Check each disjunct F_i. F is T_E-satisfiable iff at least one disjunct F_i is T_E-satisfiable.

T_E-satisfiability and Congruence Classes II

Given Σ_E-formula F

$$F : \ s_1 = t_1 \ \wedge \ \cdots \ \wedge \ s_m = t_m \ \wedge \ s_{m+1} \neq t_{m+1} \ \wedge \ \cdots \ \wedge \ s_n \neq t_n$$

with subterm set S_F, F is T_E-satisfiable iff there exists a congruence relation $\sim$ over S_F such that

- for each $i \in \{1, \ldots, m\}$, $s_i \sim t_i$;
- for each $i \in \{m+1, \ldots, n\}$, $s_i \not\sim t_i$.

Such congruence relation $\sim$ defines T_E-interpretation $I : (D_I, \alpha_I)$ of F. D_I consists of $|S_F / \sim|$ elements, one for each congruence class of S_F under $\sim$.

Instead of writing $I \models F$ for this T_E-interpretation, we abbreviate $\sim \models F$

The goal of the algorithm is to construct the congruence relation over S_F, or to prove that no congruence relation exists.

Congruence Closure Algorithm

$$F: \underbrace{s_1 = t_1 \wedge \cdots \wedge s_m = t_m}_{\text{generate congruence closure}} \wedge \underbrace{s_{m+1} \neq t_{m+1} \wedge \cdots \wedge s_n \neq t_n}_{\text{search for contradiction}}$$

Decide if F is T_E-satisfiable.

The algorithm performs the following steps:

1. Construct the congruence closure $\sim$ of

$$\{s_1 = t_1, \ldots, s_m = t_m\}$$

over the subterm set S_F. Then

$$\sim \models s_1 = t_1 \wedge \cdots \wedge s_m = t_m .$$

2. If for any $i \in \{m+1, \ldots, n\}$, $s_i \sim t_i$, return unsatisfiable.
3. Otherwise, $\sim \models F$, so return satisfiable.

How do we actually construct the congruence closure in Step 1?

Congruence Closure Algorithm (Details)

Initially, begin with the finest congruence relation $\sim_0$ given by the partition

$$\{\{s\} : s \in S_F\} .$$

That is, let each term over S_F be its own congruence class.

Then, for each $i \in \{1, \ldots, m\}$, impose $s_i = t_i$ by merging the congruence classes

$$[s_i]_{\sim_{i-1}} \quad \text{and} \quad [t_i]_{\sim_{i-1}}$$

to form a new congruence relation $\sim_i$.
To accomplish this merging,

- form the union of $[s_i]_{\sim_{i-1}}$ and $[t_i]_{\sim_{i-1}}$
- propagate any new congruences that arise within this union.

The new relation $\sim_i$ is a congruence relation in which $s_i \sim t_i$.

Congruence Closure Algorithm: Example 1 I

Given Σ_E-formula

$$F: f(a,b) = a \wedge f(f(a,b),b) \neq a$$

Construct initial partition by letting each member of the subterm set S_F be its own class:

1. $\{\{a\}, \{b\}, \{f(a,b)\}, \{f(f(a,b),b)\}\}$

According to the first literal $f(a,b) = a$, merge

$\{f(a,b)\}$ and $\{a\}$

to form partition

2. $\{\{a, f(a,b)\}, \{b\}, \{f(f(a,b),b)\}\}$

According to the (function) congruence axiom,

$f(a,b) \sim a, \ b \sim b$ implies $f(f(a,b),b) \sim f(a,b)$,

resulting in the new partition

3. $\{\{a, f(a,b), f(f(a,b),b)\}, \{b\}\}$

Congruence Closure Algorithm: Example 1 II

This partition represents the congruence closure of S_F.
Is it the case that

$$\{\{a, f(a,b), f(f(a,b),b)\}, \{b\}\} \models F \ ?$$

No, as $f(f(a,b),b) \sim a$ but F asserts that $f(f(a,b),b) \neq a$.
Hence, F is T_E-unsatisfiable.

Congruence Closure Algorithm: Example 2 I

Example: Given Σ_E-formula

$$F: \ f(f(f(a))) = a \ \wedge \ f(f(f(f(f(a))))) = a \ \wedge \ f(a) \neq a$$

From the subterm set S_F, the initial partition is

1. $\{\{a\}, \ \{f(a)\}, \ \{f^2(a)\}, \ \{f^3(a)\}, \ \{f^4(a)\}, \ \{f^5(a)\}\}$

where, for example, $f^3(a)$ abbreviates $f(f(f(a)))$.
According to the literal $f^3(a) = a$, merge

$$\{f^3(a)\} \quad \text{and} \quad \{a\} \ .$$

From the union,

2. $\{\{a, f^3(a)\}, \ \{f(a)\}, \ \{f^2(a)\}, \ \{f^4(a)\}, \ \{f^5(a)\}\}$

deduce the following congruence propagations:

$$f^3(a) \sim a \ \Rightarrow \ f(f^3(a)) \sim f(a) \quad \text{i.e.} \quad f^4(a) \sim f(a)$$

and

$$f^4(a) \sim f(a) \ \Rightarrow \ f(f^4(a)) \sim f(f(a)) \ \text{ i.e. } \ f^5(a) \sim f^2(a)$$

Thus, the final partition for this iteration is the following:

3. $\{\{a, f^3(a)\}, \ \{f(a), f^4(a)\}, \ \{f^2(a), f^5(a)\}\}$.

Congruence Closure Algorithm: Example 2 II

3. $\{\{a, f^3(a)\}, \ \{f(a), f^4(a)\}, \ \{f^2(a), f^5(a)\}\}$.

From the second literal, $f^5(a) = a$, merge

$$\{f^2(a), f^5(a)\} \quad \text{and} \quad \{a, f^3(a)\}$$

to form the partition

4. $\{\{a, f^2(a), f^3(a), f^5(a)\}, \ \{f(a), f^4(a)\}\}$.

Propagating the congruence

$$f^3(a) \sim f^2(a) \ \Rightarrow \ f(f^3(a)) \sim f(f^2(a)) \text{ i.e. } f^4(a) \sim f^3(a)$$

yields the partition

5. $\{\{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a)\}\}$,

which represents the congruence closure in which all of S_F are equal. Now,

$$\{\{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a)\}\} \ \models \ F \ ?$$

No, as $f(a) \sim a$, but F asserts that $f(a) \neq a$. Hence, F is T_E-unsatisfiable.

Congruence Closure Algorithm: Example 3

Given Σ_E-formula

$$F: \ f(x) = f(y) \ \wedge \ x \neq y \ .$$

The subterm set S_F induces the following initial partition:

1. $\{\{x\}, \ \{y\}, \ \{f(x)\}, \ \{f(y)\}\}$.

Then $f(x) = f(y)$ indicates to merge

$$\{f(x)\} \quad \text{and} \quad \{f(y)\} \ .$$

The union $\{f(x), f(y)\}$ does not yield any new congruences, so the final partition is

2. $\{\{x\}, \ \{y\}, \ \{f(x), f(y)\}\}$.

Does

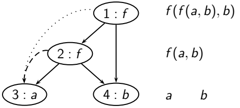
$$\{\{x\}, \ \{y\}, \ \{f(x), f(y)\}\} \ \models \ F \ ?$$

Yes, as $x \not\sim y$, agreeing with $x \neq y$. Hence, F is T_E-satisfiable.

Implementation of Algorithm

Directed Acyclic Graph (DAG)

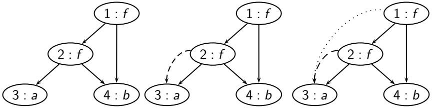
For Σ_E-formula F, graph-based data structure for representing the subterms of S_F (and congruence relation between them).



Efficient way for computing the congruence closure.

Summary of idea

$$f(a,b) = a \ \wedge \ f(f(a,b),b) \neq a$$



Initial DAG

$f(a,b) = a \ \Rightarrow$
MERGE $f(a,b)$ a

$f(a,b) \sim a, \ b \sim b \ \Rightarrow$
$f(f(a,b),b) \sim f(a,b)$
MERGE $f(f(a,b),b)$
$f(a,b)$

- - - explicit equation ...... by congruence

$$\left.\begin{array}{r} \text{FIND } f(f(a,b),b) = a = \text{FIND } a \\ f(f(a,b),b) \neq a \end{array}\right\} \Rightarrow \textbf{Unsatisfiable}$$

DAG representation

type **node** = {
- `id` : **id** — node's unique identification number
- `fn` : **string** — constant or function name
- `args` : **id list** — list of function arguments
- `mutable find` : **id** — the representative of the congruence class
- `mutable ccpar` : **id set** — if the node is the representative for its congruence class, then its `ccpar` (congruence closure parents) are all parents of nodes in its congruence class

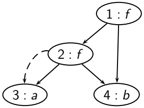
}

DAG Representation of node 2

type **node** = {

`id`	:	**id**	... 2
`fn`	:	**string**	... f
`args`	:	**id list**	... $[3,4]$
`mutable find`	:	**id**	... 3
`mutable ccpar`	:	**id set**	... $\emptyset$

}

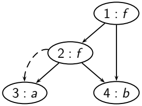


DAG Representation of node 3

type **node** = {

`id`	:	**id**	... 3
`fn`	:	**string**	... a
`args`	:	**id list**	... $[]$
`mutable find`	:	**id**	... 3
`mutable ccpar`	:	**id set**	... $\{1,2\}$

}

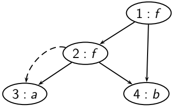


The Implementation I

<u>FIND function</u>

returns the representative of node's congruence class

$$\begin{array}{l} \texttt{let rec}\ \text{FIND}\ i = \\ \quad \texttt{let}\ n\ =\ \text{NODE}\ i\ \texttt{in} \\ \quad \texttt{if}\ n.\texttt{find} = i\ \texttt{then}\ i\ \texttt{else}\ \text{FIND}\ n.\texttt{find} \end{array}$$



<u>Example</u>: $\text{FIND}\ 2 = 3$

$\text{FIND}\ 3 = 3$

3 is the representative of $\{2, 3\}$.

The Implementation II

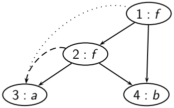
<u>UNION function</u>

$$\begin{array}{l} \texttt{let}\ \text{UNION}\ i_1\ i_2 = \\ \quad \texttt{let}\ n_1\ =\ \text{NODE}\ (\text{FIND}\ i_1)\ \texttt{in} \\ \quad \texttt{let}\ n_2\ =\ \text{NODE}\ (\text{FIND}\ i_2)\ \texttt{in} \\ \quad n_1.\texttt{find}\ \leftarrow\ n_2.\texttt{find}; \\ \quad n_2.\texttt{ccpar}\ \leftarrow\ n_1.\texttt{ccpar}\ \cup\ n_2.\texttt{ccpar}; \\ \quad n_1.\texttt{ccpar}\ \leftarrow\ \emptyset \end{array}$$

n_2 is the representative of the union class

The Implementation III

<u>Example</u>



$\text{UNION}\ 1\ 2 \qquad n_1 = 1 \quad n_2 = 3$

$1.\texttt{find} \leftarrow 3$

$3.\texttt{ccpar} \leftarrow \{1, 2\}$

$1.\texttt{ccpar} \leftarrow \emptyset$

The Implementation IV

<u>CCPAR function</u>

Returns parents of all nodes in i's congruence class

$$\begin{array}{l} \texttt{let}\ \text{CCPAR}\ i = \\ \quad (\text{NODE}\ (\text{FIND}\ i)).\texttt{ccpar} \end{array}$$

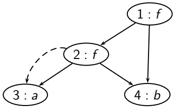
<u>CONGRUENT predicate</u>

Test whether i_1 and i_2 are congruent

$$\begin{array}{l} \texttt{let}\ \text{CONGRUENT}\ i_1\ i_2 = \\ \quad \texttt{let}\ n_1\ =\ \text{NODE}\ i_1\ \texttt{in} \\ \quad \texttt{let}\ n_2\ =\ \text{NODE}\ i_2\ \texttt{in} \\ \quad n_1.\texttt{fn} = n_2.\texttt{fn} \\ \qquad \wedge\ |n_1.\texttt{args}| = |n_2.\texttt{args}| \\ \qquad \wedge\ \forall i \in \{1, \ldots, |n_1.\texttt{args}|\}.\ \text{FIND}\ n_1.\texttt{args}[i] = \text{FIND}\ n_2.\texttt{args}[i] \end{array}$$

The Implementation V

Example:



Are 1 and 2 congruent?

fn fields	— both f
# of arguments	— same
left arguments $f(a,b)$ and a	— both congruent to 3
right arguments b and b	— both 4 (congruent)

Therefore 1 and 2 are congruent.

The Implementation VI

MERGE function

let rec MERGE i_1 i_2 =
  if FIND $i_1 \neq$ FIND i_2 then begin
    let P_{i_1} = CCPAR i_1 in
    let P_{i_2} = CCPAR i_2 in
    UNION i_1 i_2;
    foreach $t_1 \in P_{i_1}, t_2 \in P_{i_2}$ do
      if FIND $t_1 \neq$ FIND t_2 $\wedge$ CONGRUENT t_1 t_2
      then MERGE t_1 t_2
    done
  end

P_{i_1} and P_{i_2} store the values of CCPAR i_1 and CCPAR i_2 (before the union).

Decision Procedure: T_E-satisfiability

Given Σ_E-formula

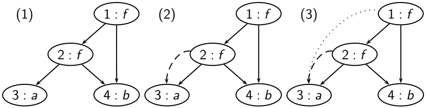
$$F: \; s_1 = t_1 \;\wedge\; \cdots \;\wedge\; s_m = t_m \;\wedge\; s_{m+1} \neq t_{m+1} \;\wedge\; \cdots \;\wedge\; s_n \neq t_n \;,$$

with subterm set S_F, perform the following steps:

1. Construct the initial DAG for the subterm set S_F.
2. For $i \in \{1, \ldots, m\}$, MERGE s_i t_i.
3. If FIND s_i = FIND t_i for some $i \in \{m+1, \ldots, n\}$, return unsatisfiable.
4. Otherwise (if FIND $s_i \neq$ FIND t_i for all $i \in \{m+1, \ldots, n\}$) return satisfiable.

Example 1: T_E-Satisfiability

$$f(a,b) = a \;\wedge\; f(f(a,b),b) \neq a$$



Initial DAG

MERGE 2 3

$P_2 = \{1\}$

$P_3 = \{2\}$

UNION 2 3

CONGRUENT 1 2

MERGE 1 2

$P_1 = \{\}$

$P_2 = \{1, 2\}$

UNION 1 2

FIND $f(f(a,b),b) = a =$ FIND $a \Rightarrow$ **Unsatisfiable**

Given Σ_E-formula

$$F: \; f(a,b) = a \;\wedge\; f(f(a,b),b) \neq a \;.$$

The subterm set is

$$S_F = \{a,\; b,\; f(a,b),\; f(f(a,b),b)\} \;,$$

resulting in the initial partition

$$(1) \quad \{\{a\},\; \{b\},\; \{f(a,b)\},\; \{f(f(a,b),b)\}\}$$

in which each term is its own congruence class. Fig (1).

Final partition (Fig (3))

$$(2) \quad \{\{a, f(a,b), f(f(a,b),b)\},\; \{b\}\}$$

<u>Note</u>: dash edge _ _ _ _ merge dictated by equalities in F
dotted edge .... deduced merge

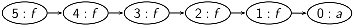
Does

$$\{\{a, f(a,b), f(f(a,b),b)\},\; \{b\}\} \;\models\; F \;?$$

No, as $f(f(a,b),b) \sim a$, but F asserts that $f(f(a,b),b) \neq a$.
Hence, F is T_E-unsatisfiable.

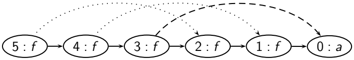
<u>Example 2: T_E-Satisfiability</u>

$$f(f(f(a))) = a \;\wedge\; f(f(f(f(f(a))))) = a \;\wedge\; f(a) \neq a$$



(1)

Initial DAG

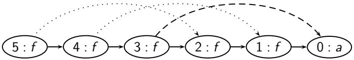


(2)

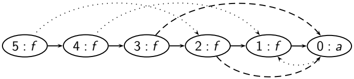
$f(f(f(a))) = a \quad \Rightarrow$ MERGE 3 0 : $P_3 = \{4\}$ $P_0 = \{1\}$ UNION 3 0
$\Rightarrow$ MERGE 4 1 : $P_4 = \{5\}$ $P_1 = \{2\}$ UNION 4 1
$\Rightarrow$ MERGE 5 2 : $P_5 = \{\}$ $P_2 = \{3\}$ UNION 5 2

<u>Example 2: T_E-Satisfiability</u>

$$f(f(f(a))) = a \;\wedge\; f(f(f(f(f(a))))) = a \;\wedge\; f(a) \neq a$$



(2)



(3)

$f(f(f(f(f(a))))) = a \quad \Rightarrow$ MERGE 5 0 : $P_5 = \{3\}$ $P_0 = \{1, 4\}$
UNION 5 0
$\Rightarrow$ MERGE 3 1 : $STOP.Why$?
UNION 3 1

FIND $f(a) = f(a) =$ FIND $a \;\Rightarrow$ **Unsatisfiable**

Given Σ_E-formula

$$F: \; f(f(f(a))) = a \;\wedge\; f(f(f(f(f(a))))) = a \;\wedge\; f(a) \neq a \;,$$

which induces the initial partition

1. $\{\{a\},\; \{f(a)\},\; \{f^2(a)\},\; \{f^3(a)\},\; \{f^4(a)\},\; \{f^5(a)\}\}$.
   The equality $f^3(a) = a$ induces the partition
2. $\{\{a,\; f^3(a)\},\; \{f(a),\; f^4(a)\},\; \{f^2(a),\; f^5(a)\}\}$.
   The equality $f^5(a) = a$ induces the partition
3. $\{\{a,\; f(a),\; f^2(a),\; f^3(a),\; f^4(a),\; f^5(a)\}\}$.
   Now, does

$$\{\{a, f(a), f^2(a), f^3(a), f^4(a), f^5(a)\}\} \;\models\; F \;?$$

No, as $f(a) \sim a$, but F asserts that $f(a) \neq a$. Hence, F is T_E-unsatisfiable.

Theorem (Sound and Complete)

Quantifier-free conjunctive Σ_E-formula F is T_E-satisfiable iff the congruence closure algorithm returns satisfiable.

Recursive Data Structures

Quantifier-free Theory of Lists T_{cons}

Σ_{cons} : $\{\text{cons},\ \text{car},\ \text{cdr},\ \text{atom},\ =\}$

- constructor cons : $\text{cons}(x, y)$ list constructed by appending y to x
- left projector car : $\text{car}(\text{cons}(x, y)) = x$
- right projector cdr : $\text{cdr}(\text{cons}(x, y)) = y$
- atom : unary predicate

Axioms of T_{cons}

- reflexivity, symmetry, transitivity
- function (congruence) axioms:

$$\forall x_1, x_2, y_1, y_2.\ x_1 = x_2 \ \wedge\ y_1 = y_2 \ \rightarrow\ \text{cons}(x_1, y_1) = \text{cons}(x_2, y_2)$$
$$\forall x, y.\ x = y \ \rightarrow\ \text{car}(x) = \text{car}(y)$$
$$\forall x, y.\ x = y \ \rightarrow\ \text{cdr}(x) = \text{cdr}(y)$$

- predicate (congruence) axiom:

$$\forall x, y.\ x = y \ \rightarrow\ (\text{atom}(x) \ \leftrightarrow\ \text{atom}(y))$$

-
  $(A1)$ $\forall x, y.\ \text{car}(\text{cons}(x, y)) = x$ (left projection)
  $(A2)$ $\forall x, y.\ \text{cdr}(\text{cons}(x, y)) = y$ (right projection)
  $(A3)$ $\forall x.\ \neg\text{atom}(x) \rightarrow \text{cons}(\text{car}(x), \text{cdr}(x)) = x$ (construction)
  $(A4)$ $\forall x, y.\ \neg\text{atom}(\text{cons}(x, y))$ (atom)

Simplifications

- Consider only quantifier-free conjunctive Σ_{cons}-formulae. Convert non-conjunctive formula to DNF and check each disjunct.
- $\neg\text{atom}(u_i)$ literals are removed:

  $\boxed{\text{replace} \quad \neg\text{atom}(u_i) \quad \text{with} \quad u_i = \text{cons}(u_i^1, u_i^2)}$

  by the (construction) axiom.
- Result of a conjunctive Σ_{cons}-formula with literals

  $$s = t \qquad s \neq t \qquad \text{atom}(u)$$

- Because of similarity to Σ_E, we sometimes combine $\Sigma_{\text{cons}} \cup \Sigma_E$.

Algorithm: T_{cons}-Satisfiability (the idea)

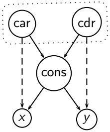
$$F: \quad \underbrace{s_1 = t_1 \ \wedge \ \cdots \ \wedge \ s_m = t_m}_{\text{generate congruence closure}}$$

$$\wedge \quad \underbrace{s_{m+1} \neq t_{m+1} \ \wedge \ \cdots \ \wedge \ s_n \neq t_n}_{\text{search for contradiction}}$$

$$\wedge \quad \underbrace{\text{atom}(u_1) \ \wedge \ \cdots \ \wedge \ \text{atom}(u_\ell)}_{\text{search for contradiction}}$$

where s_i, t_i, and u_i are T_{cons}-terms

Algorithm: T_{cons}-Satisfiability



1. Construct the initial DAG for S_F
2. for each node n with $n.\texttt{fn} = \text{cons}$
   - add car(n) and MERGE car(n) $n.\texttt{args}[1]$
   - add cdr(n) and MERGE cdr(n) $n.\texttt{args}[2]$

   by axioms (A1), (A2)
3. for $1 \leq i \leq m$, MERGE s_i t_i
4. for $m+1 \leq i \leq n$, if FIND s_i = FIND t_i, return **unsatisfiable**
5. for $1 \leq i \leq \ell$, if $\exists v$. FIND v = FIND u_i $\wedge$ $v.\texttt{fn} = \text{cons}$, return **unsatisfiable**
6. Otherwise, return **satisfiable**

Example

Given $(\Sigma_{\text{cons}} \cup \Sigma_E)$-formula

$$F: \quad \begin{array}{l} \text{car}(x) = \text{car}(y) \ \wedge \ \text{cdr}(x) = \text{cdr}(y) \\ \wedge\ \neg\text{atom}(x) \ \wedge \ \neg\text{atom}(y) \ \wedge \ f(x) \neq f(y) \end{array}$$

where the function symbol f is in Σ_E

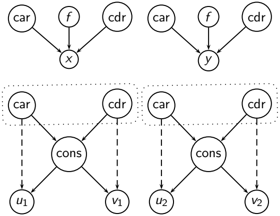
$$F': \quad \begin{array}{lll} \text{car}(x) = \text{car}(y) & \wedge & (1) \\ \text{cdr}(x) = \text{cdr}(y) & \wedge & (2) \\ x = \text{cons}(u_1, v_1) & \wedge & (3) \\ y = \text{cons}(u_2, v_2) & \wedge & (4) \\ f(x) \neq f(y) & & (5) \end{array}$$

Recall the projection axioms:

(A1) $\forall x, y.\ \text{car}(\text{cons}(x, y)) = x$

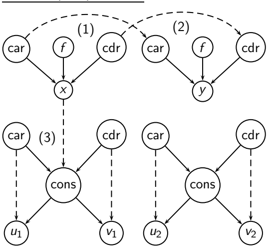
(A2) $\forall x, y.\ \text{cdr}(\text{cons}(x, y)) = y$

Example (cont): Initial DAG



axioms (A1), (A2)

Example (cont): MERGE



-- explicit equation

... by congruence

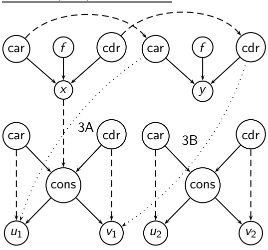
1 : MERGE $\text{car}(x)$ $\text{car}(y)$

2 : MERGE $\text{cdr}(x)$ $\text{cdr}(y)$

3 : MERGE x $\text{cons}(u_1, v_1)$

$\Downarrow$

Example (cont): Propagation



Congruent:

$\text{car}(x)$ $\text{car}(\text{cons}(u_1, v_1))$

FIND $\text{car}(x) = \text{car}(y)$

FIND $\text{car}(\text{cons}(\ldots)) = u_1$

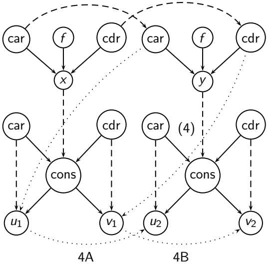
Congruent:

$\text{cdr}(x)$ $\text{cdr}(\text{cons}(u_1, v_1))$

FIND $\text{cdr}(x) = \text{cdr}(y)$

FIND $\text{cdr}(\text{cons}(\ldots)) = v_1$

Example (cont): MERGE



4 : MERGE y $\text{cons}(u_2, v_2)$

$\Downarrow$

Congruent:

$\text{car}(y)$ $\text{car}(\text{cons}(u_2, v_2))$

FIND $\text{car}(y) = u_1$

FIND $\text{car}(\text{cons}(\ldots)) = u_2$

Congruent:

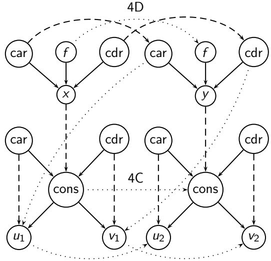
$\text{cdr}(y)$ $\text{cdr}(\text{cons}(u_2, v_2))$

FIND $\text{cdr}(y) = v_1$

FIND $\text{cdr}(\text{cons}(\ldots)) = v_2$

$\Downarrow$

Example (cont): CONGRUENCE



Congruent:

$\text{cons}(u_1, v_1)$ $\text{cons}(u_2, v_2)$

Congruent: $f(x)$ $f(y)$

5 : FIND $f(x) = f(y)$
FIND $f(y) = f(y)$

$\Downarrow$

F is **unsatisfiable**