CS256/Spring 2008 — Lecture #3

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TEMPORAL LOGIC(S)

Languages that can specify the behavior of a reactive program.

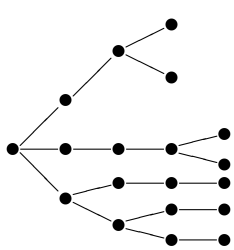
Two views:

(1) the program generates a set of sequences of states

- the models of temporal logic are infinite sequences of states
- LTL (linear time temporal logic) [Manna, Pnueli] approach

(2) the program generates a tree, where the branching points represent nondeterminism in the program

- the models of temporal logic are infinite trees
- CTL (computation tree logic) [Clarke, Emerson] at CMU
  Also CTL*.



Temporal logic: underlying assertion language

Assertion language $\mathcal{L}$:

first-order language over interpreted typed symbols

(functions and relations over concrete domains)

`Example:` $x > 0 \;\rightarrow\; x + 1 > y$

$x, y \in \mathbf{Z}^+$

formulas in $\mathcal{L}$ called:

state formulas or assertions

Temporal logic: underlying assertion language (Con't)

A state formula is evaluated over a single state to yield a truth value.

For state s and state formula p

$$s \Vdash p \quad \text{if} \quad s[p] = \mathrm{T}$$

We say:

p holds at s
s satisfies p
s is a p-state

Example:

For state $s : \{x : 4,\ y : 1\}$

$s \Vdash x = 0 \lor y = 1$
$s \nVdash x = 0 \land y = 1$
$s \Vdash \exists z.\ x = z^2$

Temporal logic: underlying assertion language (Con't)

p is state-satisfiable if

$s \Vdash p$ for some state s

p is state-valid if

$s \Vdash p$ for all states s

p and q are state-equivalent if

$s \Vdash p$ iff $s \Vdash q$ for all states s

Example: $(x, y :$ integer$)$

state-valid: $x \geq y \leftrightarrow x{+}1 > y$

state-equivalent: $x = 0 \rightarrow y = 1$
and
$x \neq 0 \lor y = 1$

TEMPORAL LOGIC (TL)

A formalism for specifying sequences of states

TL = assertions + temporal operators

- assertions (state formulas):
  First-order formulas
  describing the properties of a single state

- temporal operators
  Fig 0.15

Future Temporal Operators

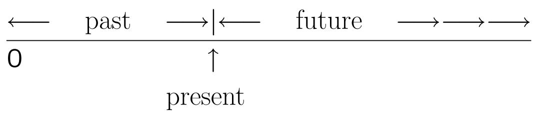
$\square p$	–	Henceforth p
$\diamond p$	–	Eventually p
$p\,\mathcal{U}\,q$	–	p Until q
$p\,\mathcal{W}\,q$	–	p Waiting-for (Unless) q
$\bigcirc p$	–	Next p

Past Temporal Operators

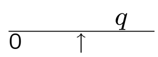
$\boxminus p$	–	So-far p
$\diamondminus p$	–	Once p
$p\,\mathcal{S}\,q$	–	p Since q
$p\,\mathcal{B}\,q$	–	p Back-to q
$\ominus p$	–	Previously p
$\circleddash p$	–	Before p

Fig. 0.15. The temporal operators

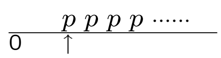
future temporal operators



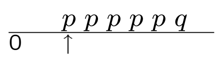
$\Diamond q$ — Eventually q



$\Box p$ — Henceforth p

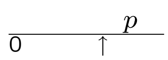


$p\,\mathcal{U} q$ — p Until q



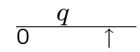
$p\,\mathcal{W}\,q$ — p Wait-for (Unless) q $\quad \Box p \ \vee\ p\,\mathcal{U}\,q$

$\bigcirc p$ — Next p

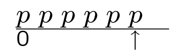


past temporal operators

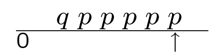
$\diamondminus q$ — Once q



$\boxminus p$ — So-far p

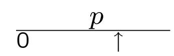


$p\,\mathcal{S}\,q$ — p Since q



$p\,\mathcal{B}\,q$ — p Back-to q $\quad \boxminus p \ \vee\ p\,\mathcal{S}\,q$

$\ominus p$ — Previously p
(false at position 0)



$\widetilde{\ominus}\, p$ — Before p
(true at position 0)

Temporal Logic: Syntax

- Every assertion is a temporal formula
- If p and q are temporal formulas (and u is a variable), so are:

$\neg p \qquad p \vee q \qquad p \wedge q \qquad p \rightarrow q \qquad p \leftrightarrow q$

$\exists u.p \qquad \forall u.p$

$\Box p \qquad \Diamond p \qquad p\,\mathcal{U}\,q \qquad p\,\mathcal{W}\,q \qquad \bigcirc p$

$\boxminus p \qquad \diamondminus p \qquad p\,\mathcal{S}\,q \qquad p\,\mathcal{B}\,q \qquad \ominus p \qquad \widetilde{\ominus}\, p$

`Example:`

$$\Box(x > 0 \rightarrow \diamondminus\, y = x)$$

$$p\,\mathcal{U}\,q \rightarrow \Diamond q$$

Temporal Logic: Semantics

Temporal formulas are evaluated over a model (an infinite sequence of states)

$$\sigma:\ s_0,\ s_1,\ s_2,\ \ldots$$

- The semantics of temporal logic formula p at a position $j \geq 0$ in a model σ,

$$(\sigma, j) \vDash p$$

"formula p holds at position j of model σ", is defined by induction on p:

$$\sigma:\quad s_0,\ \ s_1,\ \ \ldots,\ \ \underset{\substack{\uparrow\\ (\sigma,j)}}{s_j},\ \ \ldots$$

Temporal Logic: Semantics (Con't)

For state formula (assertion) p
(i.e., no temporal operators)

- $(\sigma, j) \models p \iff s_j \Vdash p$

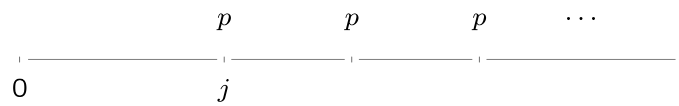
For a temporal formula p:

- $(\sigma, j) \models \neg p \iff (\sigma, j) \not\models p$

- $(\sigma, j) \models p \vee q \iff (\sigma, j) \models p$ or $(\sigma, j) \models q$

Temporal Logic: Semantics (Con't)

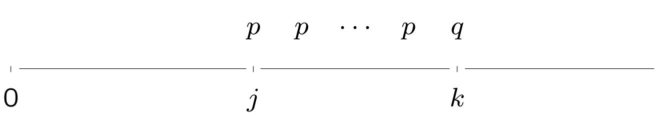
- $(\sigma, j) \models \Box\, p \iff$
  for all $k \geq j,\ (\sigma, k) \models p$



- $(\sigma, j) \models \Diamond\, p \iff$
  for some $k \geq j,\ (\sigma, k) \models p$

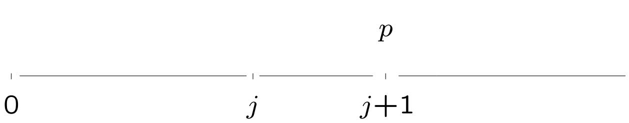
Temporal Logic: Semantics (Con't)

- $(\sigma, j) \models p\,\mathcal{U}\,q \iff$
  for some $k \geq j,\ (\sigma, k) \models q$,
  and for all i, $j \leq i < k,\ (\sigma, i) \models p$



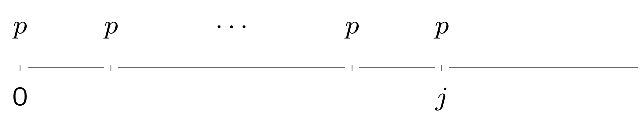
- $(\sigma, j) \models p\,\mathcal{W}\,q \iff$
  $(\sigma, j) \models p\,\mathcal{U}\,q$ or $(\sigma, j) \models \Box\, p$

- $(\sigma, j) \models \bigcirc\, p \iff$
  $(\sigma, j+1) \models p$



Temporal Logic: Semantics (Con't)

- $(\sigma, j) \models \boxminus\, p \iff$
  for all k, $0 \leq k \leq j,\ (\sigma, k) \models p$



- $(\sigma, j) \models \diamondminus\, p \iff$
  for some k, $0 \leq k \leq j,\ (\sigma, k) \models p$

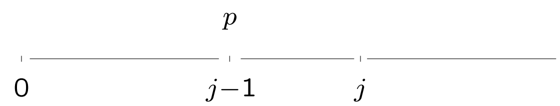
Temporal Logic: Semantics (Con't)

- $(\sigma, j) \vDash p \,\mathcal{S}\, q \quad \Longleftrightarrow$
  for some k, $0 \le k \le j$, $(\sigma, k) \vDash q$
  and for all i, $k < i \le j$, $(\sigma, i) \vDash p$

- $(\sigma, j) \vDash p \,\mathcal{B}\, q \quad \Longleftrightarrow$
  $(\sigma, j) \vDash p \,\mathcal{S}\, q$ or $(\sigma, j) \vDash \boxminus\, p$

Temporal Logic: Semantics (Con't)

- $(\sigma, j) \vDash \ominus\, p \quad \Longleftrightarrow$
  $j \ge 1$ and $(\sigma, j{-}1) \vDash p$



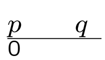
- $(\sigma, j) \vDash \widetilde{\ominus}\, p \quad \Longleftrightarrow$
  either $j = 0$ or else $(\sigma, j{-}1) \vDash p$

Simple Examples

Given temporal formula φ, describe model σ, such that

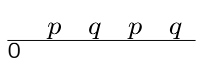
$$(\sigma, 0) \vDash \varphi$$

$p \to \Diamond\, q$



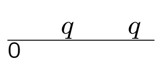
if initially p then eventually q

$\Box(p \to \Diamond\, q)$



every p is eventually followed by a q

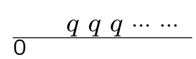
$\Box\, \Diamond\, q$



every position is eventually followed by a q,
i.e.,
infinitely many q's

Simple Examples (Con't)

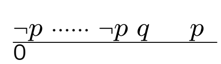
$\Diamond\, \Box\, q$



eventually permanently q,
i.e.,
finitely many $\neg q$'s

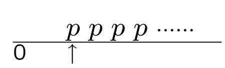
$\Box\, \Diamond\, p \to \Box\, \Diamond\, q$
if there are infinitely many p's
then there are infinitely many q's

$(\neg p)\, \mathcal{W}\, q$



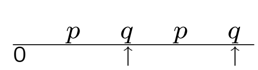
q precedes p (if p occurs)

$\Box(p \to \bigcirc\, p)$



once p, always p

$\Box(q \to \Diamond\!\!\!\!-\, p)$



every q is preceded by a p

Nested Waiting-for Formulas

$$\boxed{q_1 \mathcal{W}\, q_2 \mathcal{W}\, q_3 \mathcal{W} q_4}$$

stands for

$$q_1 \mathcal{W}\, (q_2 \mathcal{W}\, (q_3 \mathcal{W}\, q_4))$$

intervals of continuous q_i

$$\overbrace{q_1 \cdots q_1}\ \overbrace{q_2 \cdots q_2}\ \overbrace{q_3 \cdots q_3}\ \ q_4$$
$$0 \quad \uparrow$$

- possibly empty interval

$$\overbrace{q_1 \cdots q_1}\ \overbrace{q_3 \cdots q_3}\ \ q_4$$
$$0 \qquad \uparrow$$

- possibly infinite interval

$$\overbrace{q_1 \cdots q_1}\ \overbrace{q_2 \cdots q_2}\ \ q_3 q_3 q_3 \cdots\cdots\cdots q_3 \cdots\cdots$$
$$0 \quad \uparrow \qquad\qquad\qquad\qquad\qquad\qquad \rightarrow$$

Abbreviation:

$p \Rightarrow q$ for $\square(p \rightarrow q)$ "p entails q"

Example:

$p \Rightarrow \diamondsuit\, q$

stands for

$\square(p \rightarrow \diamondsuit\, q)$

Past/Future Formulas

Past Formula –
formula with no future operators

Future Formula –
formula with no past operators

A state formula is both a past and a future formula.

Definitions

- For temporal formula p, sequence σ and position $j \geq 0$:

  $(\sigma, j) \vDash p$: p holds at position j of σ
  σ satisfies p at j
  j is a p-position in σ.

- For temporal formula p and sequence σ,

$$\sigma \vDash p \quad \text{iff} \quad (\sigma, 0) \vDash p$$

$\sigma \vDash p$: p holds on σ
σ satisfies p

Satisfiable/Valid

For temporal formula p,

- p is satisfiable if $\sigma \vDash p$ for some sequence (model) σ
- p is valid if $\sigma \vDash p$ for all sequences (models) σ

p is valid iff $\neg p$ is unsatisfiable

Example: (x : integer)

$\diamondsuit(x = 0)$ is satisfiable
$\diamondsuit(x = 0) \vee \square(x \neq 0)$ is valid
$\diamondsuit(x = 0) \wedge \square(x \neq 0)$ is unsatisfiable

Equivalence

For temporal formulas p and q:

p is equivalent to q, written $p \sim q$
if $p \leftrightarrow q$ is valid
(i.e., p and q have the same truth-value at the first position of every model)

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Example:
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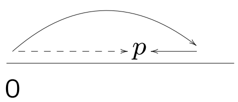
$\Diamond p \sim \Diamond \diamondminus p$



$\varphi \sim \psi$: for any σ,
$(\sigma, 0) \vDash \varphi$ iff $(\sigma, 0) \vDash \psi$.

φ valid: for any σ, $(\sigma, 0) \vDash \varphi$.

Therefore,
φ, ψ valid $\Rightarrow \varphi \sim \psi$.

φ unsatisfiable: for any σ, $(\sigma, 0) \nvDash \varphi$.

For the same reason,
φ, ψ unsatisfiable $\Rightarrow \varphi \sim \psi$.

first

Characterizes the first position.

first: $\neg \ominus \mathrm{T}$

$(\sigma, j) \vDash$ *first*: true for $j = 0$
false for $j > 0$

Then

- $\mathrm{T} \sim \Box \mathrm{T} \sim \mathit{first}$
- T, $\Box \mathrm{T}$, first are valid

Assume V={integer x}
$\mathit{first} : \neg \ominus (x = 0 \lor x \neq 0)$
$\mathrm{T} : (x = 0 \lor x \neq 0)$
$\Box \mathrm{T} : \Box (x = 0 \lor x \neq 0)$

For arbitrary σ:
$(\sigma, 0) \vDash \mathit{first}$ $(\sigma, 0) \vDash \mathrm{T}$ $(\sigma, 0) \vDash \Box \mathrm{T}$
$(\sigma, j) \nvDash \mathit{first}$ $(\sigma, j) \vDash \mathrm{T}$ $(\sigma, j) \vDash \Box \mathrm{T}$ for $j > 0$

Congruence

For temporal formulas p and q:
p is congruent to q, written $p \approx q$
if $\Box (p \leftrightarrow q)$ is valid
$\varphi \approx \psi$: for any σ, j, $(\sigma, j) \vDash \varphi$ iff $(\sigma, j) \vDash \psi$

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Example:
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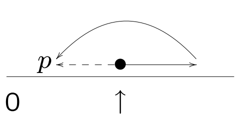
$\mathrm{T} \approx \Box \mathrm{T}$
$\mathrm{T} \not\approx \mathit{first}$ T may be true in the second state, but *first* is not

$\Diamond p \not\approx \Diamond \diamondminus p$ because $\Rightarrow$, but $\not\Leftarrow$



$\Box p \approx \neg \Diamond \neg p$
$\neg \bigcirc p \approx \bigcirc \neg p$

Note
$A \approx B$ *iff* $A \Rightarrow B$ and $B \Rightarrow A$ are valid
$A \sim B$ *iff* $A \rightarrow B$ and $B \rightarrow A$ are valid

Congruences

"conjunction character" — match well with $\land$
"disjunction character" — match well with $\lor$

$\Box$ and $\boxminus$ have conjunction character

$\Diamond$ and $\diamondminus$ have disjunction character

$\mathcal{U}$, $\mathcal{W}$, $\mathcal{S}$, $\mathcal{B}$ first argument has conjunction character
second argument has disjunction character

$$\Box (p \land q) \approx \Box p \land \Box q$$

$$\Diamond (p \lor q) \approx \Diamond p \lor \Diamond q$$

$$p \mathcal{U} (q \lor r) \approx (p \mathcal{U} q) \lor (p \mathcal{U} r)$$

$$(p \land q) \mathcal{U} r \approx (p \mathcal{U} r) \land (q \mathcal{U} r)$$

$$p \mathcal{W} (q \lor r) \approx (p \mathcal{W} q) \lor (p \mathcal{W} r)$$

$$(p \land q) \mathcal{W} r \approx (p \mathcal{W} r) \land (q \mathcal{W} r)$$

Expansions

$\Box p \approx (p \wedge \bigcirc \Box p)$

$\Diamond p \approx (p \vee \bigcirc \Diamond p)$

$p \mathcal{U} q \approx \left[q \vee (p \wedge \bigcirc (p \mathcal{U} q))\right]$

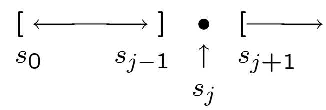
$\boxminus p \approx (p \wedge \tilde{\ominus} \boxminus p)$

$\diamondminus p \approx (p \vee \ominus \diamondminus p)$

$p \mathcal{S} q \approx \left[q \vee (p \wedge \ominus (p \mathcal{S} q))\right]$

Strict Operators

(present not included)



$\widehat{\Box} p$	$\approx$	$\bigcirc \Box p$	$\widehat{\boxminus} p$	$\approx$	$\tilde{\ominus} \boxminus p$
$\widehat{\Diamond} p$	$\approx$	$\bigcirc \Diamond p$	$\widehat{\diamondminus} p$	$\approx$	$\ominus \diamondminus p$
$p \widehat{\mathcal{U}} q$	$\approx$	$\bigcirc (p \mathcal{U} q)$	$p \widehat{\mathcal{S}} q$	$\approx$	$\ominus (p \mathcal{S} q)$
$p \widehat{\mathcal{W}} q$	$\approx$	$\bigcirc (p \mathcal{W} q)$	$p \widehat{\mathcal{B}} q$	$\approx$	$\tilde{\ominus} (p \mathcal{B} q)$

Next and Previous Values of Exps

When evaluating x at position $j \geq 0$

x refers to $s_j[x]$

x^+ refers to $s_{j+1}[x]$

x^- refers to $\begin{cases} s_{j-1}[x] & \text{if } j > 0 \\ s_0[x] & \text{if } j = 0 \end{cases}$

Example:

$\sigma\colon \langle x\colon 0\rangle, \langle x\colon 1\rangle, \langle x\colon 2\rangle, \ldots$

satisfies

$x = 0 \wedge \Box(x^+ = x+1) \wedge \bigcirc \Box(x = x^- + 1)$

Temporal Logic: Substitutivity

The ability to substitute equals for equals in a formula and obtain a formula with identical meaning.

- For state formula $\phi(u)$

  if $p \sim q$ then $\phi(p) \sim \phi(q)$

Example:

Consider state formula $\phi(u)\colon\ r \wedge u$

Since $\Diamond p \sim \Diamond \diamondminus p$

then $r \wedge \Diamond p \sim r \wedge \Diamond \diamondminus p$.

Temporal Logic: Substitutivity (Con't)

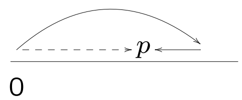
This does not hold if $\phi(u)$ is a temporal formula.

`Example:`

Consider temporal formula $\phi(u)$: $\Box\, u$

$\Diamond p \;\sim\; \Diamond \diamondminus p$

but $\Box \Diamond p \not\sim \Box \Diamond \diamondminus p$



- For <u>temporal formula</u> $\phi(u)$

  if $p \approx q$ then $\phi(p) \approx \phi(q)$

`Example:`

Consider the temporal formula $\phi(u)$: $q\mathcal{U}u$

Since $\Box\, p \;\approx\; \neg \Diamond \neg p$

therefore $q\mathcal{U}(\Box\, p) \;\approx\; q\mathcal{U}(\neg \Diamond \neg p)$