CS256/Spring 2008 — Lecture #6

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Chapter 1

Invariance: Proof Methods

For assertion q
and SPL program P

show $P \vDash \Box\, q$
(i.e., q is P-invariant)

Proving Invariances

Definitions

Recall:

- the variables of assertion:
  - free (flexible) system variables

    $$V = Y \cup \{\pi\}$$

    where Y are the program variables and π is the control variable
  - quantified (rigid) specification variables

- q' is the primed version of q, obtained by replacing each free occurrence of a system variable $y \in V$ by its primed version y'.

- ρ_τ is the transition relation of τ, expressing the relation holding between a state s and any of its τ-successors $s' \in \tau(s)$.

Verification Conditions

(proof obligations)

standard verification condition

For assertions φ, ψ and transition τ,

$\{\varphi\}\ \tau\ \{\psi\}$ ("Hoare triple") stands for the state formula

$$\boxed{\rho_\tau \ \wedge\ \varphi\ \rightarrow\ \psi'}$$

"Verification condition (VC) of φ and ψ
relative to transition τ"

$$\begin{array}{ccc} \varphi & \tau & \psi \\ j & \text{———} & j+1 \end{array}$$

Verification Conditions (Con't)

`Example:`

ρ_τ: $x \geq 0\ \wedge\ y' = x + y\ \wedge\ x' = x$

φ: $y = 3$ $\qquad$ ψ: $y = x + 3$

Then $\{\varphi\}\ \tau\ \{\psi\}$:

$$\underbrace{x\ \geq\ 0\ \wedge\ y' = x + y\ \wedge\ x' = x}_{\rho_\tau}\ \wedge\ \underbrace{y = 3}_{\varphi}$$
$$\rightarrow\ \underbrace{y' = x' + 3}_{\psi'}$$

Verification Conditions (Con't)

- for $\tau \in \mathcal{T}$ in P

  $\{\varphi\}\tau\{\psi\}$: $\rho_\tau \wedge \varphi \to \psi'$

  "τ leads from φ to ψ in P"

- for $\mathcal{T}$ in P

  $\{\varphi\}\mathcal{T}\{\psi\}$: $\{\varphi\}\tau\{\psi\}$ for every $\tau \in \mathcal{T}$

  "$\mathcal{T}$ leads from φ to ψ in P"

Claim (Verification Condition)

If $\{\varphi\}\tau\{\psi\}$ is P-state valid,
then every τ-successor of a φ-state is a ψ-state.

Verification Conditions (Con't)

Special Cases

- while, conditional ρ_τ: $\rho_\tau^{\mathrm{T}} \vee \rho_\tau^{\mathrm{F}}$

  $\{\varphi\}\tau^{\mathrm{T}}\{\psi\}$: $\rho_\tau^{\mathrm{T}} \wedge \varphi \to \psi'$

  $\{\varphi\}\tau^{\mathrm{F}}\{\psi\}$: $\rho_\tau^{\mathrm{F}} \wedge \varphi \to \psi'$

  $\{\varphi\}\tau\{\psi\}$: $\{\varphi\}\tau^{\mathrm{T}}\{\psi\} \wedge \{\varphi\}\tau^{\mathrm{F}}\{\psi\}$

- idle

  $\{\varphi\}\tau_I\{\varphi\}$: $\rho_{\tau_I} \wedge \varphi \to \varphi'$

  always valid, since

  $\rho_{\tau_I} \to v' = v$ for all $v \in V$,

  so $\varphi' = \varphi$.

Verification Conditions (Con't)

Substituted Form of Verification Condition

Transition relation can be written as
ρ_τ: $C_\tau \wedge (\overline{V}' = \overline{E})$

where
C_τ: enabling condition
$\overline{V}'$: primed variable list
$\overline{E}$: expression list

- The substituted form of verification condition $\{\varphi\}\tau\{\psi\}$:

$$\boxed{C_\tau \wedge \varphi \to \psi[\overline{E}/\overline{V}]}$$

where
$\psi[\overline{E}/\overline{V}]$: replace each variable $v \in \overline{V}$ in ψ by the corresponding $e \in \overline{E}$

Note: No primed variables!

The substituted form of a verification condition is P-state valid iff the standard form is

Verification Conditions (Con't)

Example:

ρ_τ : $x \geq 0 \wedge y' = x + y \wedge x' = x$

φ : $y = 3$ ψ : $y = x + 3$

Standard

$$\underbrace{x \geq 0 \wedge y' = x + y \wedge x' = x}_{\rho_\tau} \wedge \underbrace{y = 3}_{\varphi} \to \underbrace{y' = x' + 3}_{\psi'}$$

Substituted

$$\underbrace{x \geq 0}_{C_\tau} \wedge \underbrace{y = 3}_{\varphi} \to \underbrace{x + y = x + 3}_{\psi[\overline{E}/\overline{V}]}$$

Verification Conditions (Con't)

Example:

φ: $x = y$ $\qquad \psi$: $x = y + 1$

ρ_τ: $\underbrace{x \geq 0}_{C_\tau} \wedge \underbrace{(x', y')}_{\overline{V}'} = \underbrace{(x + 1, y)}_{\overline{E}}$

The substituted form of $\{\varphi\}\tau\{\psi\}$ is

$$\underbrace{x \geq 0}_{C_\tau} \wedge \underbrace{x = y}_{\varphi} \rightarrow \underbrace{(x = y + 1)[(x + 1, y)/(x, y)]}_{\psi[\overline{E}/\overline{V}]}$$

or equivalently

$$x \geq 0 \wedge x = y \rightarrow x + 1 = y + 1$$

Simplifying Control Expressions

$move(L_1, L_2)$: $\quad L_1 \subseteq \pi \wedge \pi' = (\pi - L_1) \cup L_2$

e.g., for $L_1 = \{\ell_1\}, L_2 = \{\ell_2\}$
$move(\ell_1, \ell_2)$: $\quad \ell_1 \in \pi \wedge \pi' = (\pi - \{\ell_1\}) \cup \{\ell_2\}$

Consequences implied by $move(L_1, L_2)$:

- for every $[\ell] \in L_1$
  $at\_\ell = \text{T}$ (i.e., $[\ell] \in \pi$)
- for every $[\ell] \in L_2$
  $at'\_\ell = \text{T}$ (i.e., $[\ell] \in \pi'$)
- for every $[\ell] \in L_1 - L_2$
  $at\_\ell = \text{T}$ (i.e., $[\ell] \in \pi$) and
  $at'\_\ell = \text{F}$ (i.e., $[\ell] \notin \pi'$)
- for every $\ell \notin L_1 \cup L_2$
  $at'\_\ell = at\_\ell$ (i.e., $[\ell] \in \pi, \pi'$ or $[\ell] \notin \pi, \pi'$)

Proving invariance properties: $P \vDash \Box\, q$

We <u>want to show</u> that for every computation of P
$\quad \sigma : s_0, s_1, s_2, \ldots$
assertion q holds in every state s_j, $j \geq 0$,
i.e., $s_j \Vdash q$.

Recall:
A sequence $\sigma : s_0, s_1, s_2, \ldots$ is a <u>computation</u> if the following hold (from Chapter 0):

1. Initiality: $s_0 \Vdash \Theta$
2. Consecution: For each $j \geq 0$,
   s_{j+1} is a τ-successor of s_j for some $\tau \in \mathcal{T}$
   ($s_{j+1} \in \tau(s_j)$)

3, 4. Fairness conditions are respected.

Note: Truth of *safety* properties over programs *does not* depend on fairness conditions.

Proving invariance properties (Con't)

This definition suggests a way to prove invariance properties $\Box\, q$:

1. Base case:
   Prove that q holds initially
   $\quad \Theta \rightarrow q$
   i.e., q holds at s_0.

2. Inductive step:
   prove that q is preserved by all transitions
   $\quad \underbrace{q \wedge \rho_\tau \rightarrow q'}_{\{q\}\tau\{q\}} \quad$ for all $\tau \in \mathcal{T}$

   i.e., if q holds at s_j, then it holds at every τ-successor s_{j+1}.

Rule B-INV (basic invariance)

Show $P \models \Box\, q$ (i.e. q is <u>P-invariant</u>)

For assertion q,

$$\begin{array}{ll} \text{B1.} & P \Vdash \Theta \rightarrow q \\ \text{B2.} & P \Vdash \{q\}\ \mathcal{T}\ \{q\} \\ \hline & P \models \Box\, q \end{array}$$

where B2 stands for

$\boxed{P \Vdash \{q\}\ \tau\ \{q\} \text{ for every } \tau \in \mathcal{T}}$

- The rule states that if we can prove the P-state validity of $\Theta \rightarrow q$ and $\{q\}\mathcal{T}\{q\}$ then we can conclude that $\Box\, q$ is P-valid.
- Thus the proof of a temporal property is reduced to the proof of $1 + |\mathcal{T}|$ first-order verification conditions.

Example 1: REQUEST-RELEASE

local x: **integer** **where** $x = 1$

$$\begin{bmatrix} \ell_0: & \textbf{request } x \\ \ell_1: & \textbf{critical} \\ \ell_2: & \textbf{release } x \\ \ell_3: & \end{bmatrix}$$

Θ: $x = 1 \ \land\ \pi = \{\ell_0\}$

$\mathcal{T}$: $\{\tau_I, \tau_{\ell_0}, \tau_{\ell_1}, \tau_{\ell_2}\}$

Prove

$$P \models \Box\, \underbrace{x \geq 0}_{q}$$

using B-INV.

Example 1: request-release (Con't)

B1: $\underbrace{x = 1 \land \pi = \{\ell_0\}}_{\Theta} \rightarrow \underbrace{x \geq 0}_{q}$

holds since $x = 1 \rightarrow x \geq 0$

B2:

τ_{ℓ_0}: $\underbrace{x \geq 0}_{q} \land \underbrace{move(\ell_0, \ell_1) \land x > 0 \land x' = x - 1}_{\rho_{\tau_{\ell_0}}} \rightarrow \underbrace{x' \geq 0}_{q'}$

holds since $x > 0 \rightarrow x - 1 \geq 0$

τ_{ℓ_1}: $\underbrace{x \geq 0}_{q} \land \underbrace{move(\ell_1, \ell_2) \land x' = x}_{\rho_{\tau_{\ell_1}}} \rightarrow \underbrace{x' \geq 0}_{q'}$

holds since $x \geq 0 \rightarrow x \geq 0$

τ_{ℓ_2}: $\underbrace{x \geq 0}_{q} \land \underbrace{move(\ell_2, \ell_3) \land x' = x + 1}_{\rho_{\tau_{\ell_2}}} \rightarrow \underbrace{x' \geq 0}_{q'}$

holds since $x \geq 0 \rightarrow x + 1 \geq 0$

Example 1: request-release (Con't)

local x: **integer** **where** $x = 1$

$$\begin{bmatrix} \ell_0: & \textbf{request } x \\ \ell_1: & \textbf{critical} \\ \ell_2: & \textbf{release } x \\ \ell_3: & \end{bmatrix}$$

We proved

$$P \models \Box\, x \geq 0$$

using B-INV.

Now we want to prove

$$P \models \Box\, \underbrace{(at\_\ell_1 \rightarrow\ x = 0)}_{q}$$

Example 1: request-release (Con't)

Attempted proof:

B1: $\underbrace{x = 1 \land \pi = \{\ell_0\}}_{\Theta} \;\rightarrow\; (\underbrace{at\_\ell_1 \rightarrow x = 0}_{q})$

holds since $\pi = \{\ell_0\} \;\rightarrow\; at\_\ell_1 = \text{F}$

B2: $\{q\}\tau_{\ell_0}\{q\}$

$$\underbrace{at\_\ell_1 \rightarrow x = 0}_{q} \land \underbrace{move(\ell_0, \ell_1) \land x > 0 \land x' = x - 1}_{\rho_{\tau_{\ell_0}}}$$

$$\rightarrow\; \underbrace{at'\_\ell_1 \rightarrow x' = 0}_{q'}$$

We have $move(\ell_0, \ell_1) \;\rightarrow\; at\_\ell_1 = \text{F},\; at'\_\ell_1 = \text{T}$

BUT

$(\text{F} \rightarrow x = 0) \land x > 0 \land x' = x - 1 \;\rightarrow\; (\text{T} \rightarrow x' = 0)$

Cannot prove: $\boxed{\text{not state-valid}}$

What is the problem?

We need a stronger rule.

Strategies for invariance proofs

Rule B-INV (basic invariance)

For assertion q,

$$\frac{\begin{array}{ll} \text{B1.} & P \Vdash \Theta \rightarrow q \\ \text{B2.} & P \Vdash \{q\}\ \mathcal{T}\ \{q\} \end{array}}{P \vDash \Box\, q}$$

- q is <u>inductive</u> if B1 and B2 are (state) valid
- By rule B-INV,
  <u>every inductive assertion q is P-invariant</u>
- <u>The converse is not true</u>

`Example:` In REQUEST-RELEASE

$$at\_\ell_1 \;\rightarrow\; x = 0$$

is P-invariant, but not inductive

Rule B-INV(Con't)

The problem is:

"The invariant is not inductive"

i.e., it is not strong enough to be preserved by all transitions.

Another way to look at it is to observe that

$$\{q\}\ \tau_{\ell_0}\ \{q\}$$

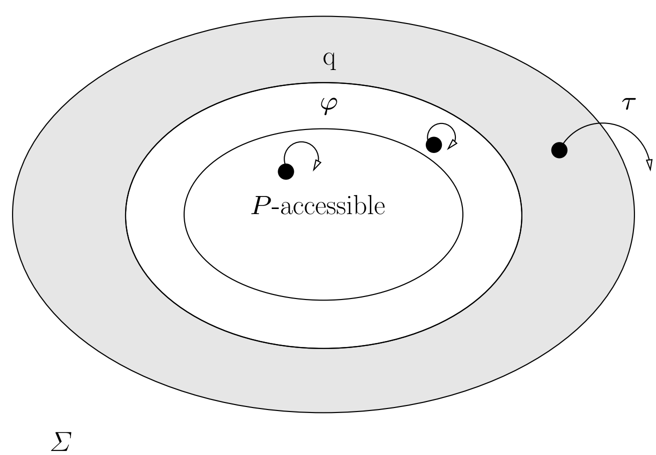
is not state valid, but it is P-state valid,
i.e., it is true in all P-accessible states,
since in all P-accessible states
$x = 1$ when at location ℓ_0.

This suggests two strategies to overcome this problem:

- strengthening
- incremental proof

Strategy 1: Strengthening

Find a stronger assertion φ that is inductive and implies the assertion q we want to prove.



In Chapter 2 it will be shown that there always exists such an assertion φ.

Strategy 1: Strengthening (Con't)

`Example:`

To show

$$\square(\underbrace{at\_\ell_1 \;\rightarrow\; x = 0}_{q})$$

strengthen q to

$$\varphi:\; (at\_\ell_1 \;\rightarrow\; x = 0) \;\wedge\; (at\_\ell_0 \;\rightarrow\; x = 1)$$

and show

$$\square\,\underbrace{(at\_\ell_1 \;\rightarrow\; x = 0) \;\wedge\; (at\_\ell_0 \;\rightarrow\; x = 1)}_{\varphi}$$

by rule B-INV.

Strategy 1: Strengthening (Con't)

The strengthening strategy relies on the following rule, MON-I, which, combined with B-INV leads to the general invariance rule INV.

Rule MON-I (Monotonicity)

For assertions q_1, q_2,

$$\frac{P \vDash \square\, q_1 \qquad P \Vdash\; q_1 \rightarrow q_2}{P \vDash \square\, q_2}$$

Strategy 1: Strengthening (Con't)

Rule INV (general invariance)

For assertions q, φ

$$\begin{array}{ll} \text{I1.} & P \Vdash \varphi \;\rightarrow\; q \\ \text{I2.} & P \Vdash \Theta \;\rightarrow\; \varphi \\ \text{I3.} & P \Vdash \{\varphi\}\; \mathcal{T}\; \{\varphi\} \\ \hline & P \vDash \square\, q \end{array}$$

Soundness: If we manage to prove $\square\, q$ using the INV rule for some program P, is q really an invariant for the program?

We can prove that this is indeed the case. So INV rule is *sound*.

Completeness: What if q is an invariant for a program P but there is **no** way of proving it under the INV rule?

We can prove that this never happens. There always exists an appropriate φ. In other words INV rule is *complete*.

Strategy 1: Strengthening (Con't)

Motivation:

$P \models \square\, \varphi$ (by I2 and I3)

$P \Vdash \varphi \to q$ (by I1)

Therefore,

$P \models \square\, q$ (by MON-I)

i.e., this rule requires that $\square\, \varphi$ holds and φ implies q, then $\square\, q$ can be concluded to hold by monotonicity.

Control Invariants

Some control invariants that can always be used (without mentioning them)

- CONFLICT:
  for labels ℓ_i, ℓ_j that are in conflict (i.e., not $\sim_L$, not parallel):
  $\square\, \neg(at\_\ell_i \wedge at\_\ell_j)$

- SOMEWHERE:
  for the set of labels $\mathcal{L}_i$ in a top-level process:
  $\square \bigvee_{\ell \in \mathcal{L}_i} at\_\ell$

- EQUAL:
  for labels l, m, s.t. $l \sim_L m$:
  $\square(at\_\ell \leftrightarrow at\_m)$

Control Invariants (Con't)

- PARALLEL:
  for substatement $[S_1 || S_2]$:
  $\square(in\_S_1 \leftrightarrow in\_S_2)$
  i.e, if control is in S_1 it must also be in S_2 and vice versa.

`Example:`

Using the invariant CONFLICT,

$move(\ell_2, \ell_3)$ implies $l_0 \notin \pi,\ l_1 \notin \pi,\ l_3 \notin \pi$
$l_0 \notin \pi',\ l_1 \notin \pi',\ l_2 \notin \pi'$

Strategy 1: Strengthening (Con't)

`Example:`

We proposed the strengthened invariant

$$\varphi :\ (at\_\ell_0 \ \to\ x = 1)\ \wedge\ (at\_\ell_1 \ \to\ x = 0)$$

Consider $\{\varphi\}\ \tau_{\ell_0}\ \{\varphi\}$:

$$\underbrace{(at\_\ell_0 \to x = 1)\ \wedge\ (at\_\ell_1 \to x = 0)}_{\varphi}\ \wedge$$
$$\underbrace{move(\ell_0, \ell_1) \wedge x > 0 \wedge x' = x - 1}_{\rho_{\tau_{\ell_0}}}$$
$$\to\ \underbrace{(at'\_\ell_0 \to x' = 1)\ \wedge\ (at'\_\ell_1 \to x' = 0)}_{\varphi'}$$

$move(\ell_0, \ell_1)$ implies $\ell_0 \in \pi, \ell_1 \notin \pi, \ell_1 \in \pi', \ell_0 \notin \pi'$

Therefore

$$(\mathrm{T} \to x = 1) \wedge (\mathrm{F} \to \ldots) \wedge \ldots \wedge x' = x - 1 \wedge \ldots$$
$$\to\ (\mathrm{F} \to \ldots) \wedge (\mathrm{T} \to x' = 0)$$

holds.

Strategy 1: Strengthening (Con't)

`Example (Con't):`

Consider $\{\varphi\}\ \tau_{\ell_2}\ \{\varphi\}$:

$$\underbrace{(at\_\ell_0 \rightarrow x = 1)\ \wedge\ (at\_\ell_1 \rightarrow x = 0)}_{\varphi}\ \wedge$$

$$\underbrace{move(\ell_2, \ell_3) \wedge x' = x + 1}_{\rho_{\tau_{\ell_2}}}$$

$$\rightarrow\ \underbrace{(at'\_\ell_0 \rightarrow x' = 1)\ \wedge\ (at'\_\ell_1 \rightarrow x' = 0)}_{\varphi'}$$

$move(\ell_2, \ell_3)$ implies $\ell_3 \in \pi'$
and by CONFLICT invariants $\ell_0, \ell_1 \notin \pi'$.

Therefore

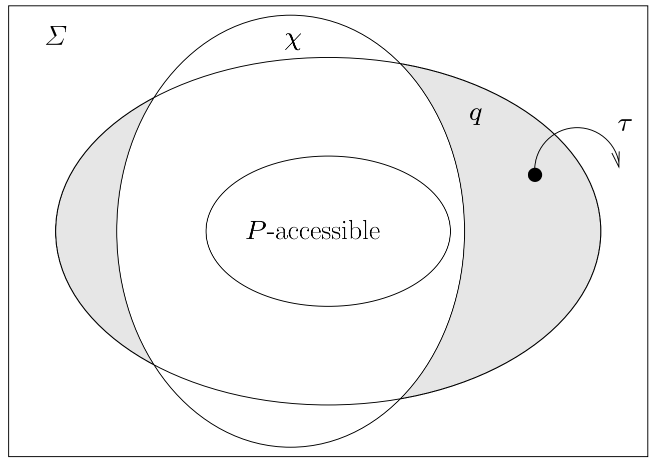
$$\ldots \wedge \ldots\ \rightarrow\ (\mathrm{F} \rightarrow x' = 1) \wedge (\mathrm{F} \rightarrow x' = 0)$$

holds.

$\{\varphi\}\ \tau_{\ell_2}\ \{\varphi\}$ is not state-valid,
but it is P-state valid. Why?

Strategy 2: Incremental proof

Use previously proven invariances χ to exclude parts of the state space from consideration.



Strategy 2: Incremental proof (Con't)

`Example:`

To show

$$\square(\underbrace{at\_\ell_1\ \rightarrow\ x = 0}_{q})$$

prove first (separately) by rule B-INV

$$\square\,\underbrace{(at\_\ell_0\ \rightarrow\ x = 1)}_{\chi},$$

then show

$$\square(\underbrace{at\_\ell_1\ \rightarrow\ x = 0}_{q})$$

by rule B-INV, but add the conjunct

$$at\_\ell_0\ \rightarrow\ x = 1$$

to the antecedent of all verification conditions.

(`Example` continues...)

Strategy 2: Incremental proof (Con't)

`Example:` (cont'd)

e.g., to show $\{\chi \wedge q\}\tau_{\ell_0}\{q\}$, prove

$$\underbrace{at\_\ell_0 \rightarrow x = 1}_{\chi}\ \wedge\ \underbrace{at\_\ell_1 \rightarrow x = 0}_{q}\ \wedge$$

$$\underbrace{move(\ell_0, \ell_1) \wedge x > 0 \wedge x' = x - 1}_{\rho_{\tau_{\ell_0}}}$$

$$\rightarrow\ \underbrace{at'\_\ell_1 \rightarrow x' = 0}_{q'}$$

Strategy 2: Incremental proof (Con't)

In an incremental proof we use previously proven properties to eliminate parts of the state space (non P-accessible states) from consideration, relying on the following rules:

Rule SV-PSV**:** from state validities to P-state validities

For assertions q_1, q_2 and χ,

$$\frac{\begin{array}{l} P \vDash \Box\, \chi \\ P \Vdash \chi \wedge q_1 \;\rightarrow\; q_2 \end{array}}{P \vDash \Box(q_1 \rightarrow q_2)}$$

Rule I-CON**:** Conjunction

For assertions q_1 and q_2,

$$\frac{\begin{array}{l} P \vDash \Box\, q_1 \\ P \vDash \Box\, q_2 \end{array}}{P \vDash \Box(q_1 \wedge q_2)}$$

Strategy 2: Incremental proof (Con't)

`Example:` Program MUX-SEM
(mutual exclusion by semaphores)

$$\textbf{local } y\textbf{: integer where } y = 1$$

$$P_1 :: \left[\begin{array}{l} \ell_0\textbf{: loop forever do} \\ \quad \left[\begin{array}{l} \ell_1 : \textbf{noncritical} \\ \ell_2 : \textbf{request } y \\ \ell_3 : \textbf{critical} \\ \ell_4 : \textbf{release } y \end{array}\right] \end{array}\right] \;||\; P_2 :: \left[\begin{array}{l} m_0\textbf{: loop forever do} \\ \quad \left[\begin{array}{l} m_1\textbf{: noncritical} \\ m_2\textbf{: request } y \\ m_3\textbf{: critical} \\ m_4\textbf{: release } y \end{array}\right] \end{array}\right]$$

Prove mutual exclusion

$\Box\, \underbrace{\neg(at\_\ell_3 \;\wedge\; at\_m_3)}_{q}$

Program MUX-SEM (Con't)

3 steps: $\Box(\underbrace{y \;\geq 0}_{\varphi_1})$

$\Box(\underbrace{at\_\ell_{3,4} + at\_m_{3,4} + y = 1}_{\varphi_2})$

$\Box\, \underbrace{\neg(at\_\ell_3 \;\wedge\; at\_m_3)}_{p}$

where F $= 0$, T $= 1$.

Let π_ℓ: $\quad \pi \cap \{\ell_0, \ldots, \ell_4\}$
π_m: $\quad \pi \cap \{m_0, \ldots, m_4\}$

By control invariants (CONFLICT, SOMEWHERE and PARALLEL)

$$|\pi_\ell| = |\pi_m| = 1$$

Program MUX-SEM (Con't)

Step 1: $\Box(\underbrace{y \geq 0}_{\varphi_1})$

by rule B-INV

B1. $\underbrace{\pi = \{\ell_0, m_0\} \;\wedge\; y = 1}_{\Theta} \;\rightarrow\; \underbrace{y \geq 0}_{\varphi_1}$

B2. $\rho_\tau \;\wedge\; y \geq 0 \;\rightarrow\; y' \geq 0$

check only ℓ_2, ℓ_4, m_2, m_4
("y-modifiable transitions")

ℓ_2: $\underbrace{move(\ell_2,\ell_3) \ \wedge \ y>0 \ \wedge \ y'=y-1}_{\rho_\tau} \wedge \underbrace{y \geq 0}_{\varphi}$

$\rightarrow \underbrace{y' \geq 0}_{\varphi'}$

holds since $y>0 \ \rightarrow \ y-1 \geq 0$

ℓ_4: $\underbrace{move(\ell_4,\ell_0) \ \wedge \ y'=y+1}_{\rho_\tau} \wedge \underbrace{y \geq 0}_{\varphi} \rightarrow \underbrace{y' \geq 0}_{\varphi'}$

holds since $y \geq 0 \ \rightarrow \ y+1 \geq 0$.

Similarly for $m_2, \ m_4$.

Step 2:

$$\Box(\underbrace{at\_\ell_{3,4} + at\_m_{3,4} + y = 1}_{\varphi_2})$$

by rule B-INV

B1. $\underbrace{\pi=\{\ell_0, m_0\} \ \wedge \ y=1}_{\Theta} \ \rightarrow$

$$\underbrace{\underbrace{at\_\ell_{3,4}}_{0} + \underbrace{at\_m_{3,4}}_{0} + \underbrace{y}_{1} \ = \ 1}_{\varphi_2}$$

B2. $\rho_\tau \ \wedge \ \varphi_2 \ \rightarrow \ \varphi_2'$

$\rho_{\ell_0} \ \wedge \ 0+at\_m_{3,4}+y=1 \ \rightarrow$
$0+at\_m_{3,4}+y=1$

$\rho_{\ell_1} \ \wedge \ 0+at\_m_{3,4}+y=1 \ \rightarrow$
$0+at\_m_{3,4}+y=1$

$\rho_{\ell_2} \ \wedge \ 0+at\_m_{3,4}+y=1 \ \rightarrow$
$1+at\_m_{3,4}+(y-1)=1$

$\rho_{\ell_3} \ \wedge \ 1+at\_m_{3,4}+y=1 \ \rightarrow$
$1+at\_m_{3,4}+y=1$

$\rho_{\ell_4} \ \wedge \ 1+at\_m_{3,4}+y=1 \ \rightarrow$
$\underbrace{0}_{at'\_\ell_{3,4}} + \underbrace{at\_m_{3,4}}_{at'\_m_{3,4}} + \underbrace{(y+1)}_{y'} = 1$

Step 3: Show $P \vDash \Box \underbrace{\neg(at\_\ell_3 \ \wedge \ at\_m_3)}_{q}$

- By I-CON

$$\frac{P \vDash \Box\,\varphi_1, \ P \vDash \Box\,\varphi_2}{P \vDash \Box(\varphi_1 \ \wedge \ \varphi_2)}$$

- By MON-I

$$\frac{\begin{array}{l} P \vDash \Box(\varphi_1 \ \wedge \ \varphi_2) \\ P \vDash \underbrace{y \geq 0}_{\varphi_1} \wedge \underbrace{at\_\ell_{3,4}+at\_m_{3,4}+y=1}_{\varphi_2} \\ \qquad \rightarrow \underbrace{\neg(at\_\ell_3 \ \wedge \ at\_m_3)}_{q} \end{array}}{P \vDash \Box \underbrace{\neg(at\_\ell_3 \ \wedge \ at\_m_3)}_{q}}$$