CS256/Spring 2008 — Lecture #8

Zohar Manna

Finding Inductive Assertions

Top-Down Approach

Assertion propagation

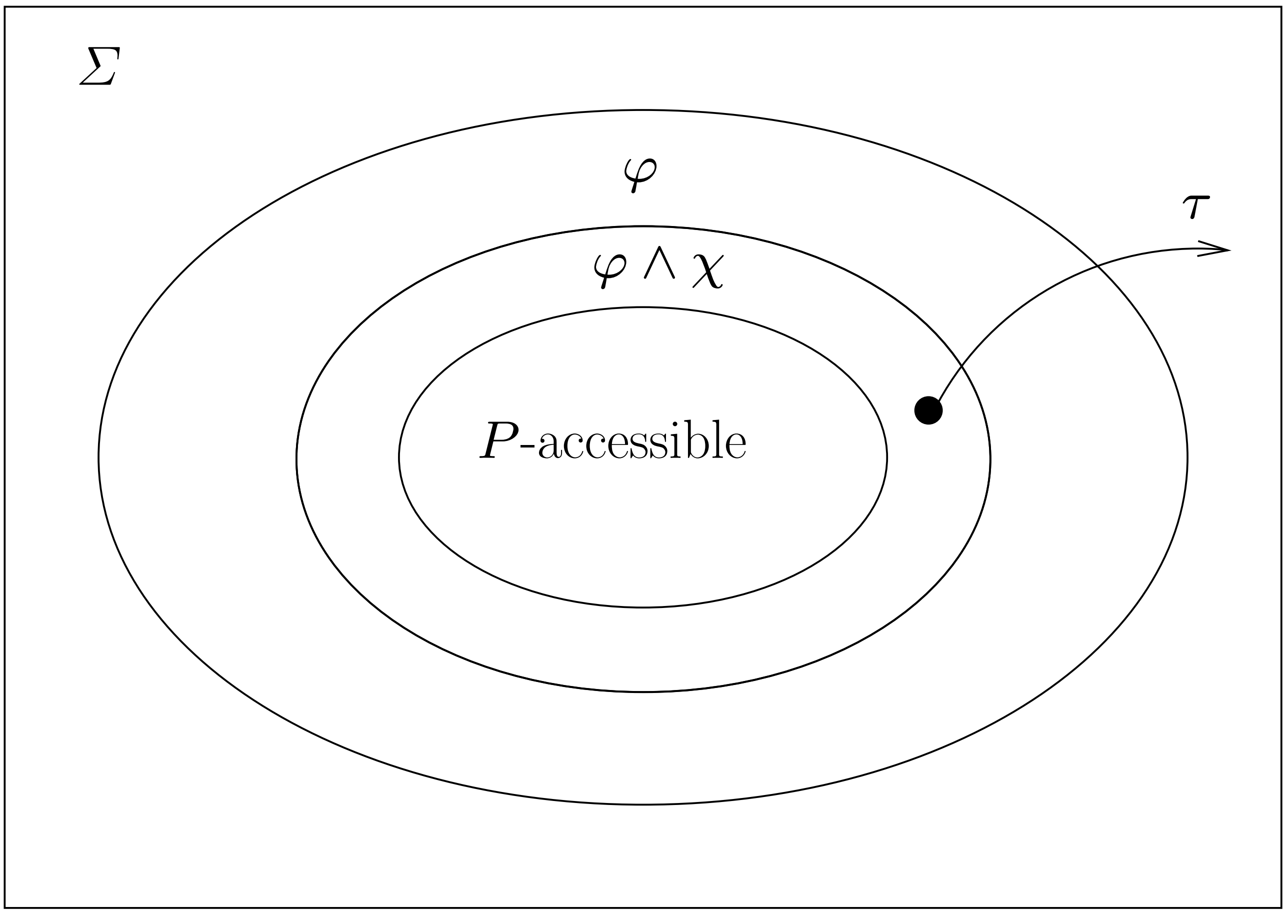
we have previously proven $\Box\,\chi$
and we want to prove $\Box\,\varphi$
but

$$\{\chi \wedge \varphi\}\tau\{\varphi\}$$

is not state-valid for some $\tau \in \mathcal{T}$.

What is the problem?
(assuming that φ is indeed an invariant)

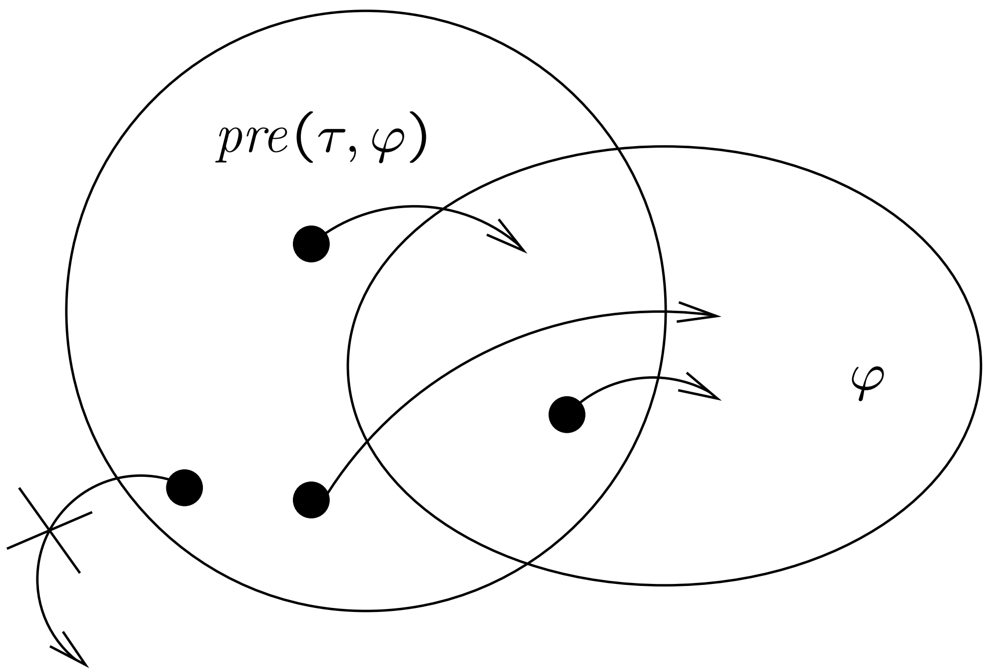
Top-Down Approach (Con'd)



Solution: Take the largest set of states that will result in a φ-state when τ is taken. How?

Precondition of φ w.r.t. τ

$pre(\tau, \varphi) : \forall V' . \rho_\tau \rightarrow \varphi'$



a state s satisfies $pre(\tau, \varphi)$
iff
all τ-successors of s satisfy φ.

Note:
s trivially satisfies $pre(\tau, \varphi)$ if it does not have any τ-successors (i.e., τ is not enabled in s).

Precondition of φ w.r.t. τ (Con'd)

`Example:`

$V : \{x\}$ integer

$\rho_\tau : x > 0 \;\wedge\; x' = x - 1$

$\varphi : x \geq 2$

$pre(\tau, \varphi) :$

$$\forall x' .\ \underbrace{x > 0 \;\wedge\; x' = x - 1}_{\rho_\tau} \;\rightarrow\; \underbrace{x' \geq 2}_{\varphi'}$$

$$x > 0 \;\rightarrow\; x - 1 \geq 2$$

$$x \leq 0 \;\vee\; x \geq 3$$

$j \xrightarrow{\ \tau\ } j{+}1$

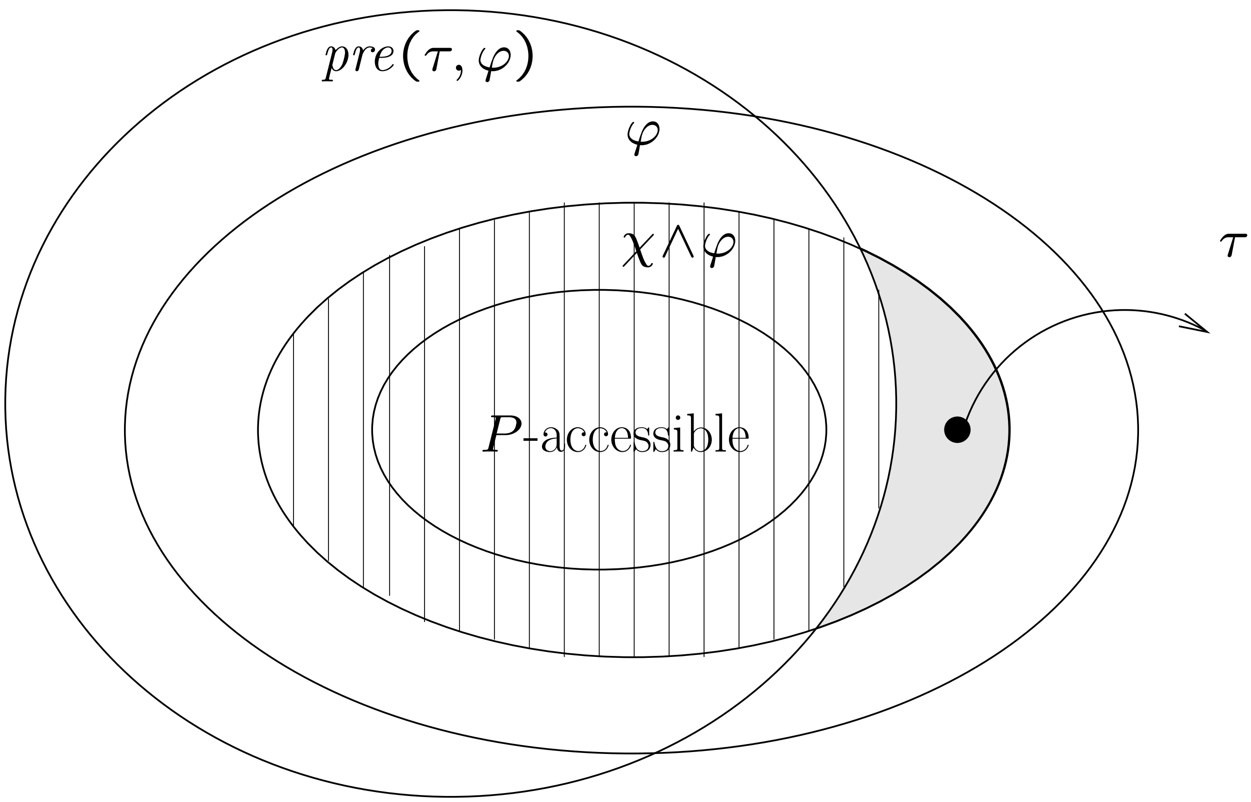
$x \leq 0 \;\vee\; x \geq 3$ (at j) $\qquad x \geq 2$ (at $j{+}1$)

Properties of $\boldsymbol{pre}(\tau, \varphi)$

By the definition of $pre(\tau, \varphi)$,

$$\{\chi \wedge \varphi \wedge pre(\tau, \varphi)\}\ \tau\ \{\varphi\}$$

is guaranteed to be state-valid.



But we have to justify adding the conjunct $pre(\tau, \varphi)$ to the antecedent.

This can be done in two ways:

1. Incremental: prove $\square\, pre(\tau, \varphi)$
2. Strengthening: prove $\square(\varphi \wedge pre(\tau, \varphi))$

Properties of $\boldsymbol{pre}(\tau, \varphi)$ (Con'd)

Claim: If φ is P-invariant then so is $\mathit{pre}(\tau, \varphi)$ for every $\tau \in \mathcal{T}$.

Proof:
Suppose φ is P-invariant, but $\mathit{pre}(\tau, \varphi)$ is not P-invariant.

Then there exists a P-accessible state s such that

$$s \not\models \mathit{pre}(\tau, \varphi).$$

But then, by the definition of $\mathit{pre}(\tau, \varphi)$, there exists a τ-successor s' of s such that $s' \not\models \varphi$.

Since s is P-accessible, s' is also P-accessible, contradicting that φ is a P-invariant.

Properties of $\boldsymbol{pre}(\tau, \varphi)$ (Con'd)

Definition: A transition τ is said to be self-disabling if for every state s, τ is disabled in all τ-successors of s.

Claim: For every assertion φ and self-disabling transition τ

$$\{\varphi \wedge pre(\tau, \varphi)\} \ \tau \ \{\varphi \wedge pre(\tau, \varphi)\}$$

is state-valid.

Proof:
Assume $s \vDash \varphi \wedge pre(\tau, \varphi)$.

Then by definition of $pre(\tau, \varphi)$, for every s', τ-successor of s,

$$s' \vDash \varphi.$$

Since τ is self-disabling, τ is disabled in all τ-successors s' of s, and so trivially

$$s' \vDash pre(\tau, \varphi)$$

Thus for all τ-successors s' of s,

$$s' \vDash \varphi \wedge pre(\tau, \varphi).$$

Heuristic

If the verification condition

$$\{\chi \wedge \varphi\}\tau\{\varphi\}$$

is not state-valid:

Find $pre(\tau, \varphi)$ and then

- Strengthening approach:

strengthen φ by adding the conjunct $pre(\tau, \varphi)$
prove $\square(\varphi \ \wedge \ pre(\tau, \ \varphi))$
or,

- Incremental approach:

prove $\square \ pre(\tau, \varphi)$
and add $pre(\tau, \varphi)$ to χ.

Note:
$pre(\tau, \varphi)$ is not guaranteed to be an inductive invariant, so the premises of INV have to be checked again.

Example:

$$\textbf{local } x\textbf{: integer where } x = 1$$

$$\left[\begin{array}{l} \ell_0 : \textbf{request } x \\ \ell_1 : \textbf{critical} \\ \ell_2 : \textbf{release } x \end{array}\right]$$

We want to prove

$$\boxed{\Box \underbrace{(at\_\ell_1 \rightarrow x = 0)}_{\varphi}}$$

Problem:

$$\{at\_\ell_1 \rightarrow x = 0\}\ \tau_{\ell_0}\ \{at\_\ell_1 \rightarrow x = 0\}$$

is not state-valid.

If we use the above heuristic we get

$$pre(\tau_{\ell_0}, \varphi) = \\ \forall x', \pi' .\ \underbrace{(move(\ell_0, \ell_1) \land x > 0 \land x' = x - 1)}_{\rho_{\ell_0}} \\ \rightarrow\ \underbrace{(at'\_\ell_1 \rightarrow x' = 0)}_{\varphi'}$$

`Example (Con'd):`

$$pre(\tau_{\ell_0}, \varphi) = \\ \forall x', \pi' . \underbrace{(move(\ell_0, \ell_1) \wedge x > 0 \wedge x' = x - 1)}_{\rho_{\ell_0}} \rightarrow \underbrace{(at'\_\ell_1 \rightarrow x' = 0)}_{\varphi'}$$

Since

$$move(\ell_0, \ell_1) \;\rightarrow at\_\ell_0 = \mathrm{T}, at'\_\ell_1 = \mathrm{T}$$

$$x' = x - 1 \wedge x' = 0 \;\rightarrow\; x = 1$$

it simplifies to

$$pre(\tau_{\ell_0}, \varphi):\; at\_\ell_0 \wedge x > 0 \;\rightarrow\; x = 1$$

Strengthened assertion
$\varphi \wedge pre(\tau_{\ell_0}, \varphi)$: $(at\_\ell_1 \rightarrow x = 0) \wedge (at\_\ell_0 \rightarrow x = 1)$
what we "guessed" before

Show that $\varphi \wedge pre(\tau_{\ell_0}, \varphi)$ is inductive
("strengthening approach")

Substituted form of $\boldsymbol{pre}(\tau, \varphi)$

Many transition relations have the form

$$\rho_\tau : \quad C_\tau \ \wedge \ \overline{V}' = \overline{E}$$

where C_τ is the enabled condition of τ.

And so

$$pre(\tau, \varphi) : \ \forall \overline{V}' \,.\, C_\tau \ \wedge \ \overline{V}' = \overline{E} \ \rightarrow \ \varphi'$$

can be simplified to

$$\forall \overline{V}' \,.\, C_\tau \ \rightarrow \ \varphi[\overline{E}/\overline{V}]$$

replacing all primed variables by its corresponding expression,
thus the quantifier can be eliminated to obtain

$$\boxed{pre(\tau, \varphi) : \ C_\tau \rightarrow \ \varphi[\overline{E}/\overline{V}]}$$

Example: Program mux-pet1(Fig. 2.25)

(Peterson's Algorithm for mutual exclusion)

local y_1, y_2: **boolean** **where** $y_1 = \text{F}, y_2 = \text{F}$
s : **integer** **where** $s = 1$

$$P_1 :: \quad \ell_0: \ \textbf{loop forever do} \begin{bmatrix} \ell_1: & \textbf{noncritical} \\ \ell_2: & (y_1,\ s) := (\text{T},\ 1) \\ \ell_3: & \textbf{await}\ (\neg y_2) \vee (s \neq 1) \\ \ell_4: & \textbf{critical} \\ \ell_5: & y_1 := \text{F} \end{bmatrix}$$

$||$

$$P_2 :: \quad m_0: \ \textbf{loop forever do} \begin{bmatrix} m_1: & \textbf{noncritical} \\ m_2: & (y_2,\ s) := (\text{T},\ 2) \\ m_3: & \textbf{await}\ (\neg y_1) \vee (s \neq 2) \\ m_4: & \textbf{critical} \\ m_5: & y_2 := \text{F} \end{bmatrix}$$

Example: Program mux-pet1 (Fig. 2.25) (Con'd)

We want to prove mutual exclusion:

$$\boxed{\Box \underbrace{\neg(at\_\ell_4 \ \wedge \ at\_m_4)}_{\psi}}$$

Bottom-up invariants:

$$\varphi_0: \quad s = 1 \ \vee \ s = 2$$

$$\varphi_1: \quad y_1 \ \leftrightarrow \ at\_\ell_{3..5}$$

$$\varphi_2: \quad y_2 \ \leftrightarrow \ at\_m_{3..5}$$

Problem: the verification conditions

$$\{\varphi_0 \ \wedge \ \varphi_1 \ \wedge \ \varphi_2 \ \wedge \ \psi\} \ \ell_3 \ \{\psi\}$$
$$\{\varphi_0 \ \wedge \ \varphi_1 \ \wedge \ \varphi_2 \ \wedge \ \psi\} \ m_3 \ \{\psi\}$$

are not state-valid

Example: Program mux-pet1 (Fig. 2.25) (Con'd)

$$pre(\tau_{\ell_3}, \psi)\colon \forall \pi'\colon \underbrace{move(\ell_3, \ell_4) \ \land \ (\neg y_2 \ \lor \ s \neq 1)}_{\rho_{\ell_3}} \ \rightarrow$$

$$\underbrace{\neg(at'\_\ell_4 \ \land \ at'\_m_4)}_{\psi'}$$

since

$move(\ell_3, \ell_4)$ implies $at'\_\ell_4 = \mathrm{T}$, $at'\_m_4 = at\_m_4$

$pre(\tau_{\ell_3}, \psi)$ simplifies to:

$$at\_\ell_3 \ \land \ (\neg y_2 \ \lor \ s \neq 1) \ \rightarrow \ \neg at\_m_4$$

$$\boxed{\varphi_3\colon \ at\_\ell_3 \ \land \ at\_m_4 \ \rightarrow \ y_2 \ \land \ s = 1}$$

$pre(\tau_{m_3}, \psi)$: $\forall \pi' \ldots\ldots$

simplifies to:

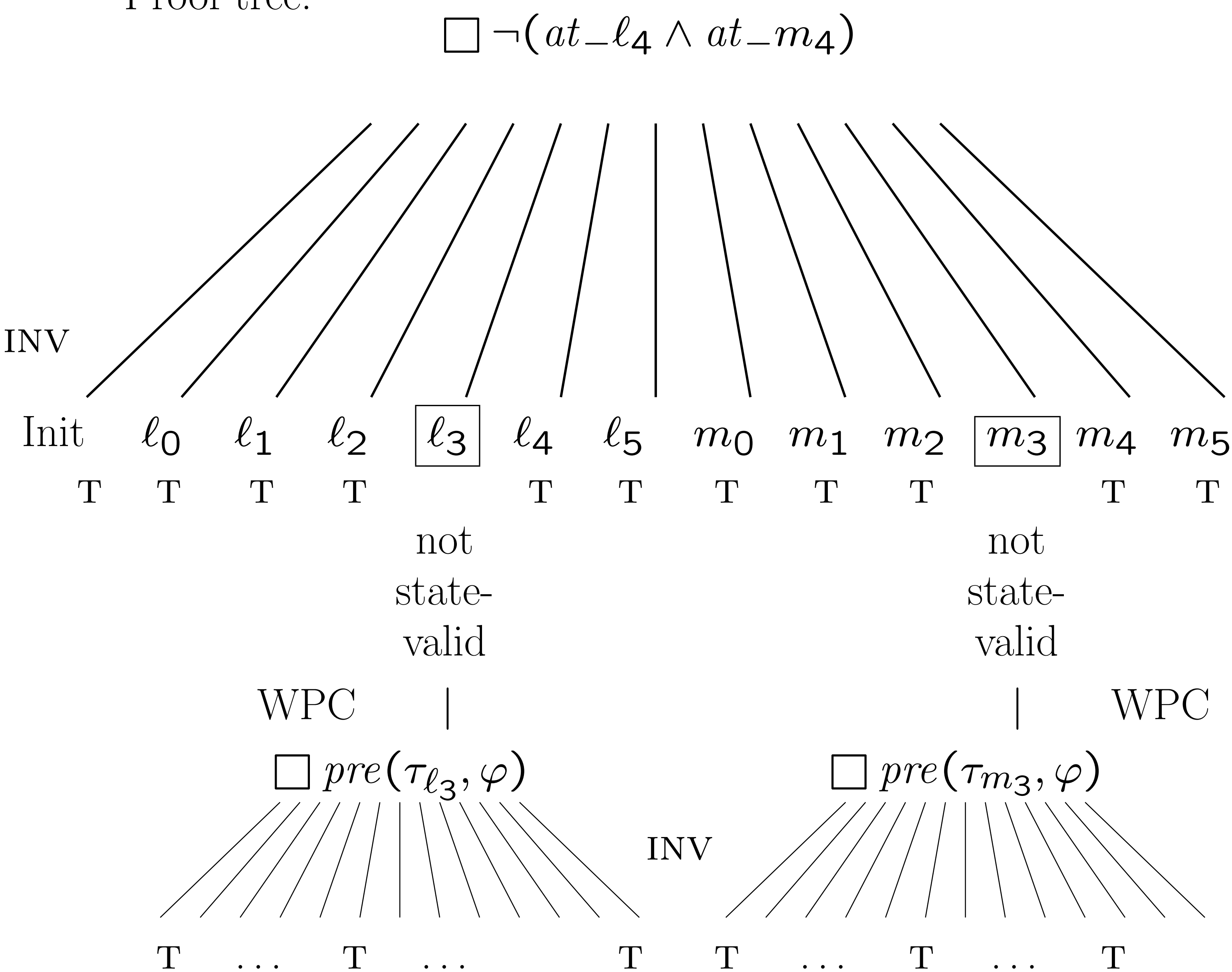
$$\boxed{\varphi_4\colon \ at\_\ell_4 \ \land \ at\_m_3 \ \rightarrow \ y_1 \ \land \ s = 2}$$

Show that φ_3: $pre(\tau_{\ell_3}, \psi)$ and φ_4: $pre(\tau_{m_3}, \psi)$ are inductive relative to $\varphi_0 \land \varphi_1 \land \varphi_2$ ("incremental approach")

Then show that ψ is inductive relative to $\varphi_0 \land \ldots \land \varphi_4$.

Example: Program mux-pet1 (Fig. 2.25) (Con'd)

Proof tree:



T = state-valid (relative to the bottom-up invariants)

Example: *pre* may never terminate

The transition is

$$\rho_\tau : \; x' = x + y \;\wedge\; y' = y$$

The property is

$$\varphi : \; x \geq 0$$

The VC is

$$\underbrace{x' = x + y \;\wedge\; y' = y}_{\rho_\tau} \;\wedge\; \underbrace{x \geq 0}_{\varphi} \;\rightarrow\; \underbrace{x' \geq 0}_{\varphi'}$$

which is not state valid.

Step 1: The precondition is

$$pre(\tau, x \geq 0) : \; \forall x', y' \colon x' = x + y \wedge y' = y \;\rightarrow\; x' \geq 0$$

that is $y \geq -x$.

Attempting to prove $\square\, pre(\tau, \varphi)$ state valid, the VC

$$\underbrace{x' = x + y \;\wedge\; y' = y}_{\rho_\tau} \;\wedge\; \underbrace{y \geq -x}_{pre} \;\rightarrow\; \underbrace{y' \geq -x'}_{pre'}$$

is not state-valid.

Step 2: Compute $pre(\tau,\ y \geq -x)$

$$\forall x', y':\ \underbrace{x' = x + y\ \wedge\ y' = y}_{\rho_\tau}\ \rightarrow\ \underbrace{y' \geq -x'}_{pre'}$$

that is $y \geq -\frac{x}{2}$.

In general the precondition

$$pre\left(\tau,\ y \geq -\frac{x}{n}\right):\ y \geq -\frac{x}{n+1}$$

Taking the limit as n approaches infinity, we obtain

$$y \geq 0$$

which is what we want.

Finite-State Algorithmic Verification

finite-state program P

each $x \in V$ assumes only finitely many values in all P-computations

Therefore,

there are only finitely many distinct P-accessible states.

Example:

MUX-PET1 (Fig 2.25) is finite-state program:

$s = 1, 2$

$y_1 = \mathrm{T}, \mathrm{F} \qquad y_2 = \mathrm{T}, \mathrm{F}$

π can assume at most 36 different values

Example: Program mux-pet1 (Fig. 2.25)

(Peterson's Algorithm for mutual exclusion)

local y_1, y_2: **boolean** **where** $y_1 = \text{F}, y_2 = \text{F}$
s : **integer** **where** $s = 1$

$$P_1 :: \quad \ell_0 : \ \textbf{loop forever do} \begin{bmatrix} \ell_1 : & \textbf{noncritical} \\ \ell_2 : & (y_1,\ s) := (\text{T},\ 1) \\ \ell_3 : & \textbf{await}\ (\neg y_2) \vee (s \neq 1) \\ \ell_4 : & \textbf{critical} \\ \ell_5 : & y_1 := \text{F} \end{bmatrix}$$

$$||$$

$$P_2 :: \quad m_0 : \ \textbf{loop forever do} \begin{bmatrix} m_1 : & \textbf{noncritical} \\ m_2 : & (y_2,\ s) := (\text{T},\ 2) \\ m_3 : & \textbf{await}\ (\neg y_1) \vee (s \neq 2) \\ m_4 : & \textbf{critical} \\ m_5 : & y_2 := \text{F} \end{bmatrix}$$

Algorithm (transition-graph)

For a given finite-state program P
Incrementally construct the
state-transition graph G_P, where each node represents a state.

- Initially
  Place as nodes in G_P all initial states (satisfy Θ)

- Repeat until no new nodes or
  new edges can be added to G_P

  $$\left[\begin{array}{l} \text{For some } s \in G_P\text{, let } s_1, \ldots, s_k \text{ be its successors} \\ \text{Add to } G_P \text{ all new nodes in } \{s_1, \ldots, s_k\} \\ \text{and draw edges connecting } s \text{ to } s_i, \\ \hfill i = 1, \ldots, k \end{array}\right]$$

Algorithmic Verification of Invariance

For assertion q,
To check validity of $\Box\, q$ over finite-state program P:

1. Construct the state-transition graph G_{P}.

2. Check if q holds in each state of the graph.

Example: Program MUX-SEM (Fig 2.26)

Generates finite state-transition graph (Fig 2.27)

Check assertion

$$\varphi: \quad \neg(at\_\ell_3 \ \wedge\ at\_m_3)$$

in the graph.

φ holds over <u>all</u> accessible states.
Thus, $\Box\, \varphi$ for MUX-SEM.

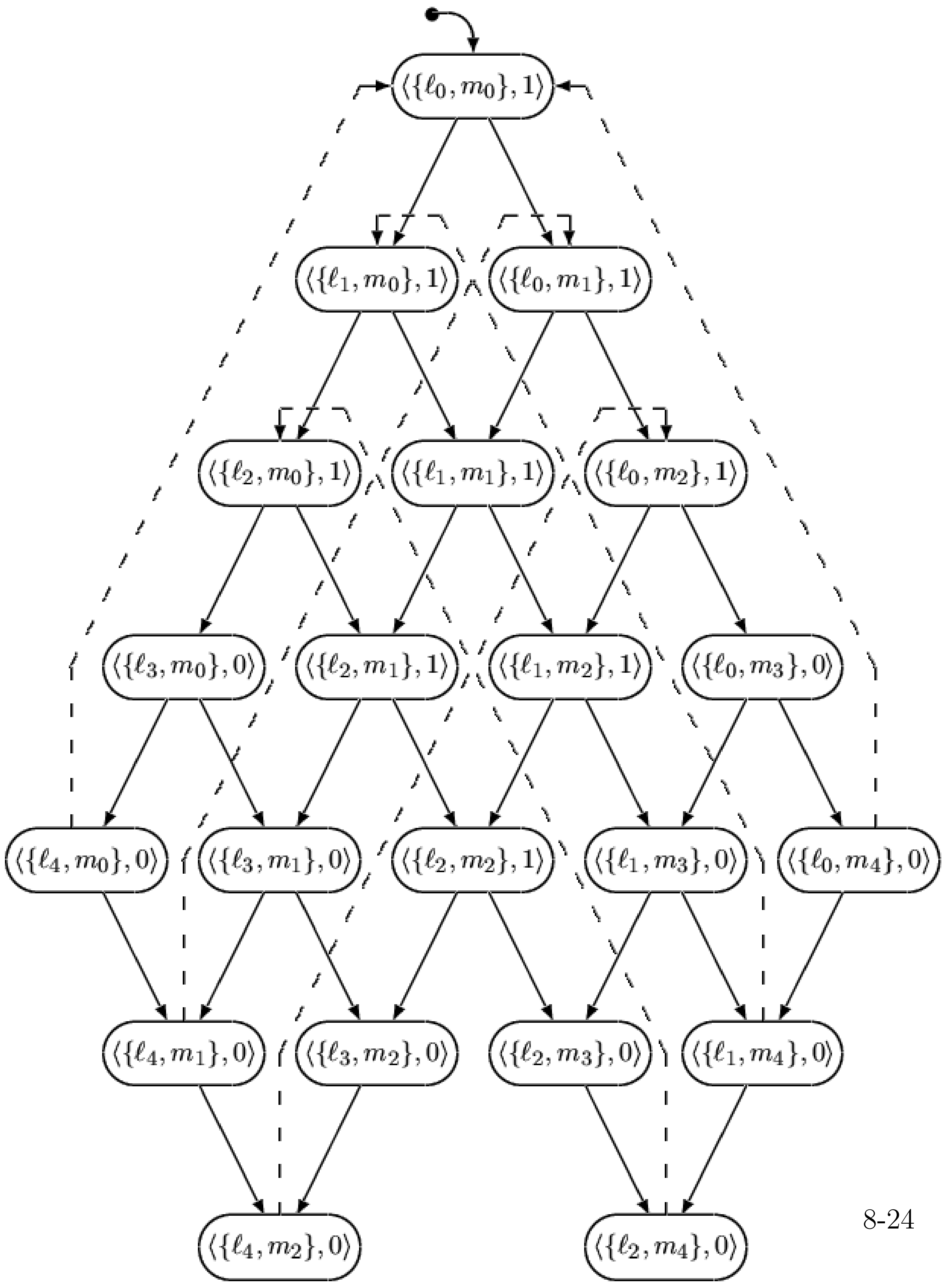
Program MUX-SEM (Fig. 2.26)

(mutual exclusion by semaphores)

$$
\textbf{local } y\text{: }\textbf{integer where } y = 1
$$

$$
P_1 :: \left[\begin{array}{l} \ell_0\text{: } \textbf{loop forever do} \\ \quad \left[\begin{array}{l} \ell_1 : \textbf{noncritical} \\ \ell_2 : \textbf{request } y \\ \ell_3 : \textbf{critical} \\ \ell_4 : \textbf{release } y \end{array} \right] \end{array} \right] \; || \; P_2 :: \left[\begin{array}{l} m_0\text{: } \textbf{loop forever do} \\ \quad \left[\begin{array}{l} m_1\text{: } \textbf{noncritical} \\ m_2\text{: } \textbf{request } y \\ m_3\text{: } \textbf{critical} \\ m_4\text{: } \textbf{release } y \end{array} \right] \end{array} \right]
$$

Program MUX-SEM state-transition graph (Fig. 2.27)



`Example:` Program MUX-PET1 (Fig 2.25)

State-transition graph G_P (Fig 2.28)

(i, j, v) means $\pi\colon \{\ell_i, m_j\},\ s\colon v$

No y_1, y_2 since

$$\begin{array}{lll} y_1 = \text{T} & \text{iff} & 3 \leq i \leq 5 \\ y_2 = \text{T} & \text{iff} & 3 \leq j \leq 5 \end{array}$$

Property checked

$$\Box \underbrace{\neg(at\_\ell_4 \ \wedge \ at\_m_4)}_{\psi}$$

Example: Program mux-pet1(Fig. 2.25)

(Peterson's Algorithm for mutual exclusion)

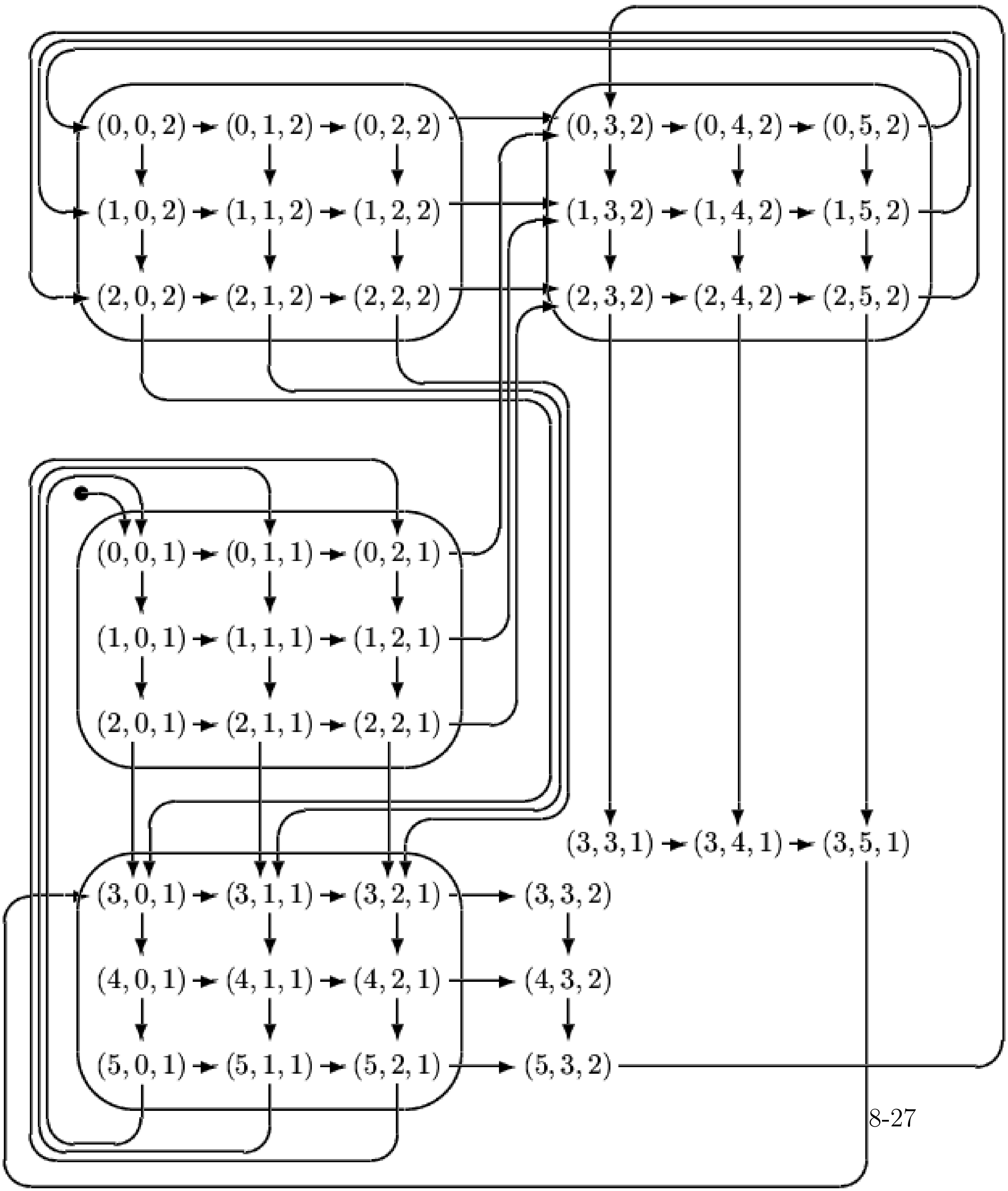
local y_1, y_2: **boolean** **where** $y_1 = \text{F}, y_2 = \text{F}$
s : **integer** **where** $s = 1$

$$
\begin{array}{l}
\ell_0: \ \textbf{loop forever do} \\
P_1 :: \left[\begin{array}{ll}
\ell_1: & \textbf{noncritical} \\
\ell_2: & (y_1,\ s) := (\text{T},\ 1) \\
\ell_3: & \textbf{await}\ (\neg y_2) \vee (s \neq 1) \\
\ell_4: & \textbf{critical} \\
\ell_5: & y_1 := \text{F}
\end{array}\right]
\end{array}
$$

$$||$$

$$
\begin{array}{l}
m_0: \ \textbf{loop forever do} \\
P_2 :: \left[\begin{array}{ll}
m_1: & \textbf{noncritical} \\
m_2: & (y_2,\ s) := (\text{T},\ 2) \\
m_3: & \textbf{await}\ (\neg y_1) \vee (s \neq 2) \\
m_4: & \textbf{critical} \\
m_5: & y_2 := \text{F}
\end{array}\right]
\end{array}
$$

MUX-PET1 State-transition graph (Fig 2.28)



Completeness of rule INV

Rule INV (general invariance)

For assertions φ, q,

$$\begin{array}{ll} \text{I1.} & \Vdash \varphi \rightarrow q \\ \text{I2.} & \Vdash \Theta \rightarrow \varphi \\ \text{I3.} & \Vdash \{\varphi\}\ \mathcal{T}\ \{\varphi\} \\ \hline & \vDash \Box q \end{array}$$

Theorem (Relative completeness of rule INV)

For every assertion q such that

$\Box q$ is P-valid

there exists an assertion φ such that I1 – I3 are provable from state validities

We actually show

"completeness relative to
first-order reasoning"

taking all state-valid assertions as axioms

Outline of proof

Given FTS P with system variables (program + control variables)

$$\overline{y} = (y_1, \ldots, y_m)$$

- Assume $\Box\, q$ is P-valid, i.e.,
  (†) q holds over every P-accessible state

- Construct (to be shown) accessibility assertion
  $$acc_P(\overline{y})$$
  such that for any state s,
  (*) s is P-accessible state iff $s \Vdash acc_P$

- Take $\varphi = acc_P$

We have to show :

1. acc_P satisfies I1 – I3
2. acc_P can be "constructed"

1. acc_P satisfies I1 – I3

- Premise I1: $\underbrace{acc_P}_{\varphi} \rightarrow q$

$$s \Vdash acc_P \quad \overset{(*)}{\Rightarrow} \quad s \text{ is } P\text{-accessible state}$$

$$\overset{(\dagger)}{\Rightarrow} \quad s \Vdash q$$

Thus

$$\underbrace{acc_P}_{\varphi} \rightarrow q$$

is state-valid

- Premise I2: $\Theta \rightarrow \underbrace{acc_P}_{\varphi}$

$$s \Vdash \Theta \quad \Rightarrow \quad s \text{ is } P\text{-accessible}$$

$$\overset{(*)}{\Rightarrow} \quad s \Vdash \underbrace{acc_P}_{\varphi}$$

Thus

$$\Theta \rightarrow \underbrace{acc_P}_{\varphi}$$

is state-valid

- Premise I3: for every $\tau \in \mathcal{T}$,

$$\rho_\tau \ \wedge \ acc_P \ \rightarrow \ acc'_P$$

where $acc'_P = acc_P(\overline{y}')$.

Take s' to be a $\overline{y}$-variant of s (s agrees with s' on all variables other than $\overline{y}$) and for each y_i take

$$s'[y_i] = s[y'_i]$$

Then

$$\left.\begin{array}{lcl} s \Vdash \rho_\tau & \Rightarrow & s' \text{ is a } \tau\text{-successor of } s \\ s \Vdash acc_P & \overset{(*)}{\Rightarrow} & s \text{ is } P\text{-accessible} \end{array}\right\} \Rightarrow$$

$$\begin{array}{ll} \Rightarrow & s' \text{ is } P\text{-accessible} \\ \overset{(*)}{\Rightarrow} & s' \Vdash acc_P \\ \Rightarrow & s \Vdash acc'_P \end{array}$$

`Example:`

V: $\{y\}$ $\quad\Theta$: $y = 0$

$\mathcal{T}$: $\{\tau_I, \tau\}$, where ρ_τ: $y' = y + 2$

For this program: $acc_P(y)$: $y \geq 0 \wedge even(y)$

2. Construction of acc_P

Assume assertion language includes
dynamic array $\underline{a}$ over D

Array $\underline{a}$ is viewed as function,

$$a\colon [1..n] \mapsto D$$

where n is the size of the array

The assumption is <u>not essential</u>

We can use Gödel numbering

$$(k_1, \ldots, k_n) \quad \mapsto \quad n = p_1^{k_1} \cdots p_n^{k_n}$$

where p_i is the ith prime number

Case: single-variable y

Define

$$acc_P(y): \quad (\exists n > 0) \ (\exists a \in [1..n] \mapsto D) \,. \\ init \ \wedge \ last \ \wedge \ evolve$$

where

$init$: $\Theta(a[1])$

$last$: $a[n] = y$

$evolve$: $\forall i \,.\, 1 \leq i < n \,.\, \bigvee_{\tau \in \mathcal{T}} \rho_\tau(a[i], a[i+1])$

i.e., there exists an array a, such that

- $a[1]$ is an initial state
- $a[n]$ has value y (last element)
- every two consecutive elements are related by some transition relation

array a represents a prefix

$$s_1, \ldots, s_n$$

of a computation where $a[i]$ stands for

the value of y at state s_i

<u>Claim</u>:
For any value $d \in D$,

$acc_P(d) = \mathrm{T}$

iff

d is a possible value of y in a P-accessible state

$acc_P(d)$ asserts the existence of a computation prefix that leads to a state where $y = d$.

`Example:` Transition system EVEN

V: $\{y\}$ ranges over $\mathbb{Z}$ (the integers)	
Θ: $y = 0$	
ρ_τ: $y' = y + 2$	

$acc_P(y)$:

$$(\exists n > 0)(\exists a \in [1..n] \mapsto \mathbb{Z}) \,. \begin{pmatrix} a[1] = 0 \;\wedge\; a[n] = y \;\wedge \\ \forall i \,.\, 1 \le i < n \,.\, a[i+1] = a[i] + 2 \end{pmatrix}$$

simplifies to

$$(\exists n > 0)(\exists a \in [1..n] \mapsto \mathbb{Z}) \,. \begin{pmatrix} a[n] = y \;\wedge \\ \forall i \,.\, 1 \le i \le n \,.\, a[i] = 2 \cdot (i - 1) \end{pmatrix}$$

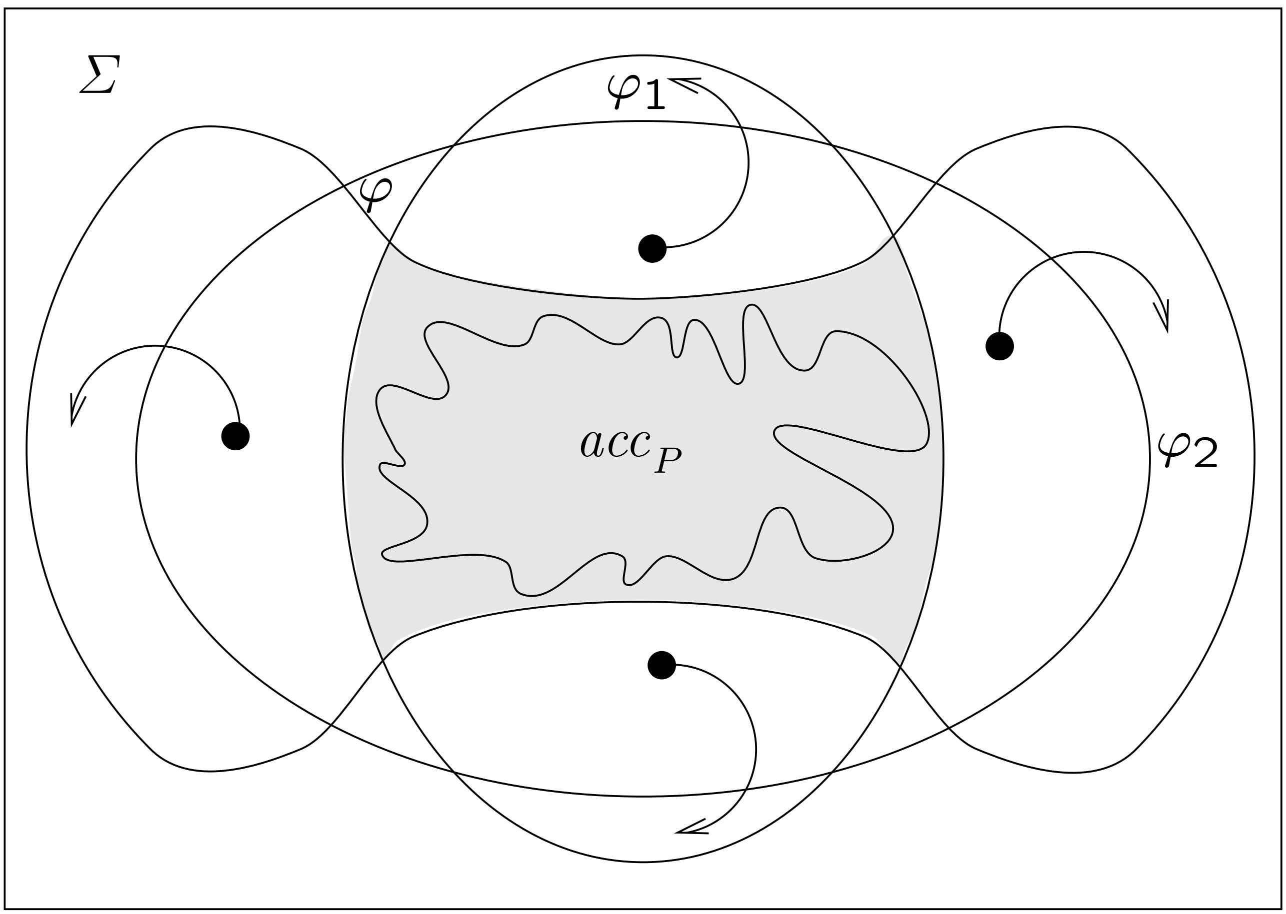
simplifies to

$$\boxed{y \ge 0 \;\wedge\; even(y)}$$

Precisely characterizes the values that y may assume in P-accessible states of EVEN

Discussion

Although the assertion acc_P is inductive and strengthens any P-invariant, it is not very useful in practice.



The shaded area is preserved by all transitions. Its description is much simpler than that of acc_P.

Multivariable $\overline{y} = (y_1, \ldots, y_m)$ case

Use 2-dimensional array a

$$
\begin{array}{ccc}
\underline{y_1} & & \underline{y_m} \\
a[1,1] & \ldots\ldots & a[1,m] \\
a[2,1] & \ldots\ldots & a[2,m] \\
\vdots & & \vdots
\end{array}
$$

`Example:` Transition system FACT

y,z ranging over $\mathbb{N}$ (the nonnegative integers)

Θ: $y = 1 \ \wedge \ z = 1$

ρ_τ: $y' = y + 1 \ \wedge \ z' = (y+1) \cdot z$

Construction of acc_P:

$(\exists n > 0)(\exists a \in [1..n] \times [1,2] \mapsto \mathbb{N})\,.$

$$\left(\begin{array}{c} a[1,1] = 1 \ \wedge \ a[1,2] = 1 \ \wedge \\ a[n,1] = y \ \wedge \ a[n,2] = z \\ \bigwedge \\ \forall i\colon\ 1 \le i < n\colon\ a[i+1,1] = a[i,1] + 1 \ \wedge \\ a[i+1,2] = (a[i,1]+1) \cdot a[i,2] \end{array}\right)$$

$$(\exists n > 0)(\exists a \in [1..n] \times [1,2] \mapsto \mathbb{N})\,.$$

$$\left(\begin{array}{c} a[1,1] = 1 \;\wedge\; a[1,2] = 1 \;\wedge \\ a[n,1] = y \;\wedge\; a[n,2] = z \\ \bigwedge \\ \forall i\colon\; 1 \leq i < n\colon\; a[i+1,1] = a[i,1] + 1 \;\wedge \\ a[i+1,2] = (a[i,1]+1) \cdot a[i,2] \end{array}\right)$$

simplifies to

$$(\exists n > 0)(\exists a \in [1..n] \times [1,2] \mapsto \mathbb{N})\,.$$

$$\left(\begin{array}{c} a[n,1] = y \;\wedge\; a[n,2] = z \\ \wedge \\ \forall i\colon\; 1 \leq i \leq n\colon\; a[i,1] = i \;\wedge\; a[i,2] = i! \end{array}\right)$$

simplifies to

$$\boxed{y \geq 1 \;\wedge\; z = y!}$$

Precisely characterizes the P-accessible states for the transition system FACT