CS256/Spring 2008 — Lecture #15

Zohar Manna

Particle Tableau

Particle Tableau: Motivation

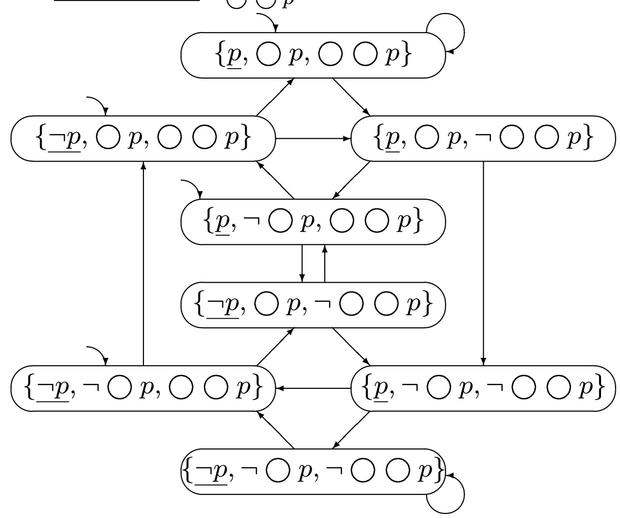
Consider $\boxed{\varphi : \bigcirc \bigcirc p}$

The closure Φ_φ has three basic formulas:

p, $\bigcirc p$, $\bigcirc \bigcirc p$.

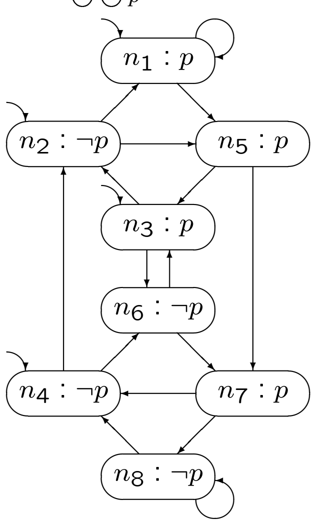
Thus, it has eight atoms.

The atom tableau $T_{\bigcirc \bigcirc p}$ is



Particle Tableau: Motivation

The ω-automaton $A_{\bigcirc \bigcirc p}$:



$$\mathcal{F}_M = \{ \text{ all SCS's } \}$$
$$\mathcal{F}_S = \{\}$$

Note: No promising formulas.

Particle Tableau: Motivation

Because of the atom construction rule:

for every $\psi \in \Phi_\varphi$,

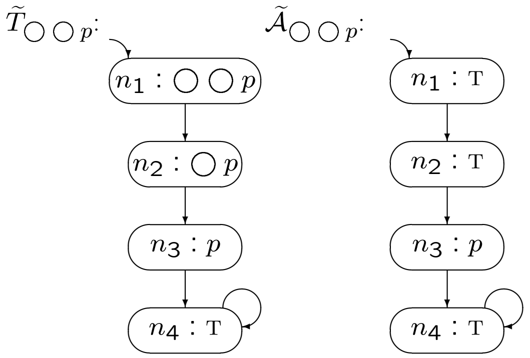
$\psi \in A$ iff $\neg\psi \notin A$,

every atom makes a commitment about every formula in the closure.

Clearly, some of these commitments are irrelevant in determining the satisfiability of the formula.

Particle Tableau: Motivation (Cont'd)

Intuitively, the tableau below should suffice to determine satisfiability. The truth value of p at the first two positions is irrelevant:



If we change the offending rule to

$$\text{if } \psi \in A \text{ then } \neg\psi \notin A$$

we get the particle tableau, which is usually considerably smaller than the atom tableau.

Particles

The idea of a particle is to assert what needs to be true, not what needs to be false, except for state formulas.

Thus, if $\psi \in A$, ψ needs to be true;
if $\psi \notin A$, ψ can be true or false.

Step 0: Push negations inside φ
We push all negations inside the formula such that negations only appear at the state level. This can be done with the help of the following congruences:

$$\begin{array}{rcl} \neg \Diamond p & \approx & \Box \neg p \\ \neg \bigcirc p & \approx & \bigcirc \neg p \\ \neg \Box p & \approx & \Diamond \neg p \\ \neg(p \mathcal{U} q) & \approx & (\neg q) \mathcal{W} (\neg p \wedge \neg q) \\ \neg(p \mathcal{W} q) & \approx & (\neg q) \mathcal{U} (\neg p \wedge \neg q) \end{array}$$

Thus, the closure only needs to contain positive formulas and the negation of state formulas.

Closure $\widetilde{\Phi}_\varphi$

- $\varphi \in \widetilde{\Phi}_\varphi$
- for every $\psi \in \widetilde{\Phi}_\varphi$ and χ a subformula of ψ, $\chi \in \widetilde{\Phi}_\varphi$
- for every ψ of the form

  $$\Box \psi_1,\ \Diamond \psi_1,\ \psi_1 \mathcal{U} \psi_2,\ \psi_1 \mathcal{W} \psi_2,$$

  if $\psi \in \widetilde{\Phi}_\varphi$,
  then $\bigcirc \psi \in \widetilde{\Phi}_\varphi$

Particles: Definition

A particle of φ is any set $P \subseteq \widetilde{\Phi}_\varphi$ that satisfies the following requirements:

- R_{sat}: $state(P)$ is satisfiable
- R_α: for every α-formula $\psi \in \widetilde{\Phi}_\varphi$,
  $$\psi \in P \text{ iff } \kappa(\psi) \in P$$
- R_β: for every β-formula $\psi \in \widetilde{\Phi}_\varphi$,
  $$\psi \in P \text{ iff } \begin{array}{l} \kappa_1(\psi) \in P \\ \text{or } \kappa_2(\psi) \subseteq P \quad \text{(or both)} \end{array}$$

Note: The empty set $\{\}$ is always a particle, denoted by $P_\emptyset$.

Examples:

$$\varphi : \Diamond \Box p$$

$\widetilde{\Phi}_\varphi$: $\{ \Diamond\Box p,\ \bigcirc\Diamond\Box p,\ \Box p,\ \bigcirc\Box p,\ p \}$

Particle: $\{ \Diamond\Box p,\ \bigcirc\Diamond\Box p \}$

Atom: $\{ \Diamond\Box p,\ \bigcirc\Diamond\Box p,\ \neg p,$
$\neg\bigcirc\Box p,\ \neg\Box p \}$

$$\varphi : \bigcirc\bigcirc p$$

$\widetilde{\Phi}_\varphi$: $\{ \bigcirc\bigcirc p,\ \bigcirc p,\ p \}$

Particle: $\{ \bigcirc\bigcirc p \}$

Atom: $\{ \bigcirc\bigcirc p,\ \bigcirc p,\ \neg p \}$

Cover of a Formula Set

Given a set of formulas $B \subseteq \widetilde{\Phi}_\varphi$, we give a procedure for constructing the cover of B, a set of particles of φ that contain B.

Recursive function $cover_\varphi(B$: set of formulas):
set of particles

- if $state(B)$ is not consistent,
  then return $\{\}$

- α-expansion
  if for some α-formula $\psi \in \widetilde{\Phi}_\varphi$,
  $\psi \in B$ but $\kappa(\psi) \not\subseteq B$,
  then return
  $cover_\varphi(B \cup \kappa(\psi))$

- α^{-1}-expansion
  if for some α-formula $\psi \in \widetilde{\Phi}_\varphi$,
  $\kappa(\psi) \subseteq B$ but $\psi \notin B$,
  then return
  $cover_\varphi(B \cup \{\psi\})$

- β-expansion
  if for some β-formula $\psi \in \widetilde{\Phi}_\varphi$,
  $\psi \in B$, but $\kappa_1(\psi) \notin B$ and $\kappa_2(\psi) \not\subseteq B$,
  then return
  $cover_\varphi(B \cup \{\kappa_1(\psi)\})$
  $\cup$
  $cover_\varphi(B \cup \kappa_2(\psi))$

- β^{-1}-expansion
  if for some β-formula $\psi \in \widetilde{\Phi}_\varphi$,
  $\psi \notin B$, but $\kappa_1(\psi) \in B$ or $\kappa_2(\psi) \subseteq B$,
  then return
  $cover_\varphi(B \cup \{\psi\})$

- return $\{B\}$

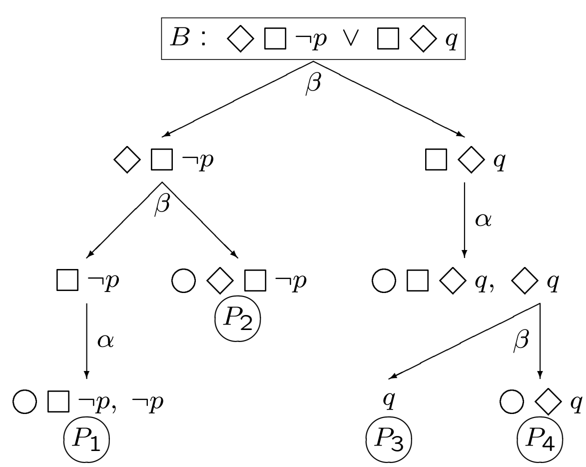
Note: $cover_\varphi(\underbrace{\{\}}_{B}) = \{P_\varnothing\}$

Tree Representation of the Procedure

Example: To find all particles covering

$$B = \varphi : \Diamond\Box\neg p \lor \Box\Diamond q$$

construct the tree:



Example (Cont'd): Particles

Thus,

$$cover_{\varphi}(\underbrace{\{\varphi\}}_{B}) = \{P_1, P_2, P_3, P_4\},$$

where

$P_1: \{\ \varphi,\ \Diamond\Box\neg p,\ \Box\neg p,\ \bigcirc\Box\neg p,\ \neg p\ \}$

$P_2: \{\ \varphi,\ \Diamond\Box\neg p,\ \bigcirc\Diamond\Box\neg p\}$

$P_3: \{\ \varphi,\ \Box\Diamond q,\ \bigcirc\Box\Diamond q,\ \Diamond q,\ q\ \}$

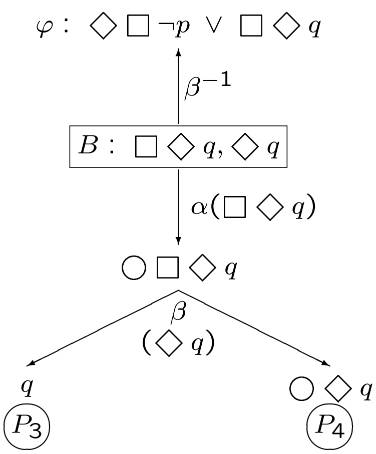
$P_4: \{\ \varphi,\ \Box\Diamond q,\ \bigcirc\Box\Diamond q,\ \Diamond q,\ \bigcirc\Diamond q\ \}$

Example: $\varphi:\ \Diamond\Box\neg p\ \vee\ \Box\Diamond q$

To find all particles of φ covering

$$B:\ \{\Box\Diamond q, \Diamond q\}$$

construct the tree:



Thus, $cover_{\varphi}(\underbrace{\{\Box\Diamond q, \Diamond q\}}_{B}) = \{P_3, P_4\}$

Incremental Particle Tableau Construction

Idea: Start with initial φ-particles and only construct particles that are reachable from previously constructed particles.

Implied successors $imps(P)$ of particle P:

if $\bigcirc\psi \in P$, then $\psi \in imps(P)$

Successors of particle P:

$succ(P) = cover_{\varphi}(imps(P))$

Algorithm for constructing $\widetilde{T}_{\varphi}$:

- initially, $\widetilde{T}_{\varphi} = cover_{\varphi}(\{\varphi\})$
  these are the initial nodes.
- for each particle $P \in \widetilde{T}_{\varphi}$,
  let $S = succ(P)$
  for each $Q \in S$,
  if $Q \notin \widetilde{T}_{\varphi}$, then add it
  draw an edge from P to Q

Example: Construct $\widetilde{T}_{\varphi}$ for $\varphi:\ \Diamond\Box\neg p\ \vee\ \Box\Diamond q$

Particles:

$P_1: \{\ \varphi,\ \Diamond\Box\neg p,\ \Box\neg p,\ \underline{\bigcirc\Box\neg p},\ \neg p\ \}$

$P_2: \{\ \varphi,\ \Diamond\Box\neg p,\ \underline{\bigcirc\Diamond\Box\neg p}\ \}$

$P_3: \{\ \varphi,\ \Box\Diamond q,\ \underline{\bigcirc\Box\Diamond q},\ \Diamond q,\ q\ \}$

$P_4: \{\ \varphi,\ \Box\Diamond q,\ \underline{\bigcirc\Box\Diamond q},\ \Diamond q,\ \underline{\bigcirc\Diamond q}\ \}$

$imps(P_1) = \{\ \Box\neg p\ \}$
$succ(P_1) = cover(\{\Box\neg p\ \}) = \{P_1\}$

$imps(P_2) = \{\ \Diamond\Box\neg p\ \}$
$succ(P_2) = cover(\{\Diamond\Box\neg p\ \}) = \{P_1, P_2\}$

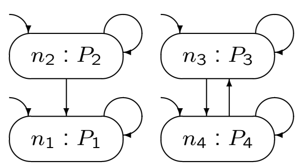
$imps(P_3) = \{\ \Box\Diamond q\ \}$
$succ(P_3) = cover(\{\Box\Diamond q\ \}) = \{P_3, P_4\}$

$imps(P_4) = \{\ \Box\Diamond q,\ \Diamond q\ \}$
$succ(P_4) = cover(\{\Box\Diamond q,\ \Diamond q\}) = \{P_3, P_4\}$

Example (Cont'd):

Particle	*imps*	*succ*
P_1	$\square \neg p$	P_1
P_2	$\diamondsuit \square \neg p$	P_1, P_2
P_3	$\square \diamondsuit q$	$P_3,\ P_4$
P_4	$\square \diamondsuit q,\ \diamondsuit q$	P_3, P_4

$\widetilde{T}_\varphi$:



Fulfillment

A particle P fulfills formula $\psi \in \widetilde{\Phi}_\varphi$, which promises r, if

$$\psi \notin P \quad \text{or} \quad r \in P.$$

An SCS S is fulfilling if every promising formula $\psi \in \widetilde{\Phi}_\varphi$ is fulfilled by some particle $P \in S$.

Proposition:
An LTL formula φ is satisfiable
iff
$\widetilde{T}_\varphi$ has a fulfilling SCS that is reachable from an initial node.

Example: $\varphi : \diamondsuit \square \neg p \ \vee\ \square \diamondsuit q$

Promising formulas:

$$\diamondsuit \square \neg p \quad \text{promises} \quad \square \neg p$$
$$\diamondsuit q \quad \text{promises} \quad q$$

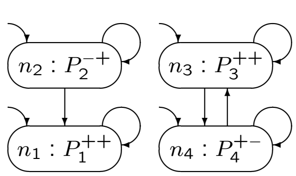
$P_1^{++} : \{ \varphi,\ \diamondsuit \square \neg p,\ \square \neg p,\ \bigcirc \square \neg p,\ \neg p \}$

$P_2^{-+} : \{ \varphi,\ \diamondsuit \square \neg p,\ \bigcirc \diamondsuit \square \neg p \}$

$P_3^{++} : \{ \varphi,\ \square \diamondsuit q,\ \bigcirc \square \diamondsuit q,\ \diamondsuit q,\ q \}$

$P_4^{+-} : \{ \varphi,\ \square \diamondsuit q,\ \bigcirc \square \diamondsuit q,\ \diamondsuit q,\ \bigcirc \diamondsuit q \}$

Particle Tableau $\widetilde{T}_\varphi$:



Fulfilling SCS's :$\{n_1\}, \{n_3\}, \{n_3, n_4\}$
Hence, φ is satisfiable.

From Particle Tableau $\widetilde{T}_\varphi$ to ω-Automaton $\mathcal{A}_\varphi$

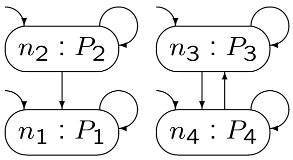
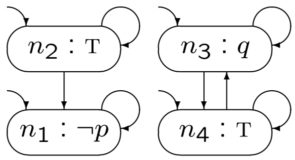
For temporal formula φ, construct the ω-automaton

$$\mathcal{A}_\varphi :\ \langle \underbrace{N,\ N_0,\ E,}_{\substack{\text{Same as} \\ \widetilde{T}_\varphi}}\ \mu,\ \mathcal{F} \rangle$$

where

- Node labeling μ:
  For node $n \in N$ labeled by particle P in $\widetilde{T}_\varphi$,
  $$\mu(n) \ = \ state(P).$$
- Acceptance condition $\mathcal{F}$:
  Muller:
  $\mathcal{F} \ = \ \{\text{SCS } S \mid S \text{ is fulfilling } \}$
  Street:
  $\mathcal{F} \ = \ \{(P_\psi, R_\psi) \mid \psi \in \widetilde{\Phi}_\varphi \text{ promises } r\}$,
  where
  $$P_\psi \ = \ \{ \ P \mid \psi \notin P \ \} \quad \Leftarrow$$
  $$R_\psi \ = \ \{ \ P \mid r \in P \ \}$$

Example (Cont'd): $\varphi: \Diamond \Box \neg p \vee \Box \Diamond q$

Particle Tableau $\widetilde{T}_\varphi$	Corresponding ω-Automaton $\mathcal{A}_\varphi$
	
with fulfilling SCS's $\{n_1\}, \{n_3\}, \{n_3, n_4\}$	$\mathcal{F}_M = \{\{n_1\}, \{n_3\}, \{n_3, n_4\}\}$

$$\mathcal{F}_S = \{(P_{\Diamond \Box \neg p}, R_{\Diamond \Box \neg p}), (P_{\Diamond q}, R_{\Diamond q})\}$$

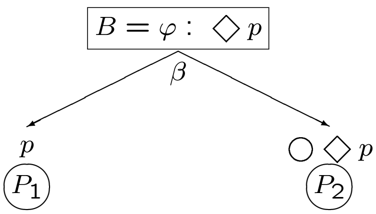
with

$P_{\Diamond \Box \neg p}: \{n_3, n_4\}$
$R_{\Diamond \Box \neg p}: \{n_1\}$
$P_{\Diamond q}: \{n_1, n_2\}$
$R_{\Diamond q}: \{n_3\}$

Example: To find all particles covering

$$\varphi: \Diamond p$$

construct the tree:



Thus, $cover_\varphi(\underbrace{\{\varphi\}}_{B}) = \{P_1, P_2\}$, where

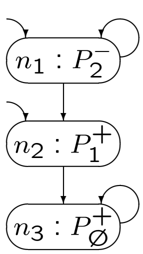
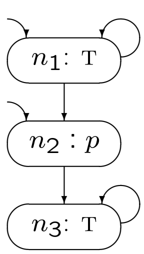
$$P_1: \{\varphi, p\}$$

$$P_2: \{\varphi, \bigcirc \Diamond p\}$$

Example (Cont'd): $\varphi: \Diamond p$

$P_1 : \{\varphi, p\}$	$imps(P_1) = \{\}$	$succ(P_1) = \{P_\varnothing\}$
$P_\varnothing$	$imps(P_\varnothing) = \{\}$	$succ(\{\}) = \{P_\varnothing\}$
$P_2 : \{\varphi, \bigcirc \Diamond p\}$	$imps(P_2) = \{\varphi\}$	$succ(P_2) = \{P_1, P_2\}$

$P_1^+, P_\varnothing^+$ fulfilling $\quad P_2^-$ not fulfilling.

Particle Tableau $\widetilde{T}_\varphi$	Corresponding ω-Automaton $\mathcal{A}_\varphi$
	
with fulfilling SCS's $\{n_3\}$	$\mathcal{F}_M = \{\{n_3\}\}$
	$\mathcal{F}_S = \{(\{n_3\}, \{n_2\})\}$

Hence, φ is satisfiable.

Example: To find all particles covering

$$\varphi: \bigcirc \bigcirc p$$

construct the (trivial) tree:

$$\boxed{B = \varphi : \bigcirc \bigcirc p}$$

(only one node)

Thus,

$$cover_\varphi(\underbrace{\{\bigcirc \bigcirc p\}}_{B}) = \{P_1\},$$

where

$$P_1: \{\varphi\}.$$

Example (Cont'd): $\varphi : \bigcirc \bigcirc p$

$P_1 : \{\varphi\}$ $\quad imps(P_1) = \{\bigcirc p\}$ $\quad succ(P_1) = \{\underbrace{\bigcirc p}_{P_2}\}$

$P_2 : \{\bigcirc p\}$ $\quad imps(P_2) = \{p\}$ $\quad succ(P_2) = \{\underbrace{p}_{P_3}\}$

$P_3 : \{p\}$ $\quad imps(P_3) = \{\}$ $\quad succ(P_3) = \{P_\varnothing\}$

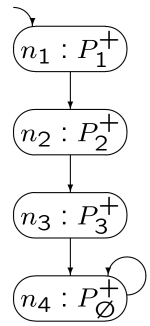
$P_\varnothing$ $\quad imps(P_\varnothing) = \{\}$ $\quad succ(P_\varnothing) = \{P_\varnothing\}$

No promising formulas

$$P_1^+ : \{\varphi\}$$
$$P_2^+ : \{\bigcirc p\}$$
$$P_3^+ : \{p\}$$
$$P_4^+ : P_\varnothing$$

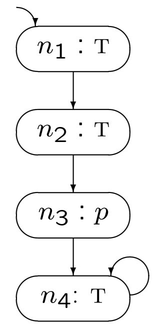
Example (Cont'd): $\varphi : \bigcirc \bigcirc p$

Particle Tableau $\widetilde{T}_\varphi$



with fulfilling SCS $\{n_4\}$

Corresponding ω-Automaton $\mathcal{A}_\varphi$



$\mathcal{F}_M = \{\{n_4\}\}$

$\mathcal{F}_S = \{\}$

Hence, φ is satisfiable.