

Termination Analysis of Loops

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Example: GCD Algorithm

$$\gcd(y_1, y_2) = \begin{cases} \gcd(y_1 - y_2, y_2) & \text{if } y_1 > y_2 \\ \gcd(y_1, y_2 - y_1) & \text{if } y_1 < y_2 \\ y_1 & \text{if } y_1 = y_2 \end{cases}$$

Example:

$$\begin{aligned} \gcd(77, 112) &= \gcd(77, 35) = \gcd(42, 35) = \gcd(7, 35) \\ &= \gcd(7, 28) = \gcd(7, 21) = \gcd(7, 14) \\ &= \gcd(7, 7) = 7 \end{aligned}$$

Example: GCD Program

```
int gcd(int  $y_1 > 0$ , int  $y_2 > 0$ )  
  while  $y_1 \neq y_2$  do  
    if  $y_1 > y_2$  then  $y_1 := y_1 - y_2$  else  $y_2 := y_2 - y_1$   
  done  
  return  $y_1$ 
```

Abstract program:

$$\Theta : \{y_1 \geq 1, y_2 \geq 1\}$$

$$\tau_1 : \{y_1 \geq y_2 + 1\} \Rightarrow \{y'_1 = y_1 - y_2, y'_2 = y_2\}$$

$$\tau_2 : \{y_2 \geq y_1 + 1\} \Rightarrow \{y'_2 = y_2 - y_1, y'_1 = y_1\}$$

for $y_1, y_2 \in \mathbb{R}$

Example: Termination of GCD

$$\Theta : \{y_1 \geq 1, y_2 \geq 1\}$$

$$\tau_1 : \{y_1 \geq y_2 + 1\} \Rightarrow \{y'_1 = y_1 - y_2, y'_2 = y_2\}$$

$$\tau_2 : \{y_2 \geq y_1 + 1\} \Rightarrow \{y'_2 = y_2 - y_1, y'_1 = y_1\}$$

$\delta(y_1, y_2) = y_1 + y_2$ is a **ranking function**

$y_1 \geq 1 \wedge y_2 \geq 1$ is a **loop invariant**

- δ is **bounded from below**:

if τ_1 or τ_2 can be taken, $\delta(y_1, y_2) \geq 0$

- δ **decreases** on each iteration:

if τ_1 or τ_2 is taken, $\delta(y'_1, y'_2) \leq \delta(y_1, y_2) - 1$

Therefore, GCD terminates.

Goal:

Find ranking functions and supporting invariants automatically.

Ranking Functions

Loops

Loop Abstraction:

GCD

$L : \langle \mathcal{V}, \Theta, \mathcal{T} \rangle :$

- **variables** \mathcal{V} range over \mathbb{R} $\{y_1, y_2\}$
- **initial condition** Θ is assertion over \mathcal{V} $y_1 \geq 1 \wedge y_2 \geq 1$
- **transitions** $\tau \in \mathcal{T}$ are assertions $\{\tau_1, \tau_2\}$
 $\tau(\mathcal{V}, \mathcal{V}')$ over $\mathcal{V} \cup \mathcal{V}'$

Loop Validity:

Assertion φ is **valid over loop** L

$$L \models \varphi$$

if φ holds on all **reachable states** \mathcal{S}_L of L . *values of* (y_1, y_2)

In practice, replace “ $L \models$ ” with **loop invariants**. $y_1 \geq 1 \wedge y_2 \geq 1$

Well-founded Relation

(D, \prec) : \prec is **well-founded** if there is no infinite sequence

$$d_1, d_2, d_3, \dots \quad \text{where } d_i \in D$$

such that

$$(\forall i) d_i \succ d_{i+1}$$

$$(d_2 \prec d_1 \iff d_1 \succ d_2)$$

Examples:

- $(\mathbb{Z}^+, <)$
- $(\mathbb{R}^+, \prec_\epsilon)$ for $\epsilon > 0$ $x \prec_\epsilon y \iff x \leq y - \epsilon$
- (\mathbb{L}, \prec) for lists \mathbb{L} $\ell_1 \prec \ell_2 \iff |\ell_1| < |\ell_2|$

Ranking Function

Consider loop $L : \langle \mathcal{V}, \Theta, \mathcal{T} \rangle$.

$\delta : \mathcal{S}_L \rightarrow \mathbb{R}$ is a **ranking function** of L if

(Bounded) $(\forall \tau \in \mathcal{T})$

$$L \models \tau(\mathcal{V}, \mathcal{V}') \rightarrow \delta(\mathcal{V}) \geq 0$$

(Ranking) $(\exists \epsilon > 0)(\forall \tau \in \mathcal{T})$

$$L \models \tau(\mathcal{V}, \mathcal{V}') \rightarrow \delta(\mathcal{V}') \leq \delta(\mathcal{V}) - \epsilon$$

δ, ϵ induce a well-founded relation over \mathcal{S}_L :

for $s, t \in \mathcal{S}_L$,

$$s \prec t \iff \delta(s) \leq \delta(t) - \epsilon$$

Thus, L always terminates.

Example: GCD

Prove $\delta(y_1, y_2) = y_1 + y_2$ is a ranking function for GCD.

- Take loop invariant $y_1 \geq 1 \wedge y_2 \geq 1$.
- Choose $\epsilon = 1$.

Bounded τ_1

$$\underbrace{y_2 \geq 1}_{\text{invariant}} \wedge \underbrace{y_1 \geq y_2 + 1}_{\text{guard of } \tau_1} \rightarrow y_1 + y_2 \geq 0$$

Ranking τ_1

$$\underbrace{y_2 \geq 1}_{\text{invariant}} \rightarrow \underbrace{(y_1 - y_2) + (y_2)}_{\text{substitution by } \tau_1} \leq y_1 + y_2 - \underbrace{1}_{\epsilon}$$

Example: GCD

Bounded τ_2

$$\underbrace{y_1 \geq 1}_{\text{invariant}} \wedge \underbrace{y_2 \geq y_1 + 1}_{\text{guard of } \tau_2} \rightarrow y_1 + y_2 \geq 0$$

Ranking τ_2

$$\underbrace{y_1 \geq 1}_{\text{invariant}} \rightarrow \underbrace{(y_1) + (y_2 - y_1)}_{\text{substitution by } \tau_2} \leq y_1 + y_2 - \underbrace{1}_{\epsilon}$$

Assertions are valid, so GCD always terminates.

Lexicographic Well-founded Relation

Given well-founded relations over domains

$$(D_1, \prec_1), (D_2, \prec_2), \dots, (D_k, \prec_k)$$

define **lexicographic well-founded relation** \prec over

$$D = D_1 \times D_2 \times \dots \times D_k$$

For $d = \langle d_1, d_2, \dots, d_k \rangle$, $e = \langle e_1, e_2, \dots, e_k \rangle \in D$

$$d \prec e \iff (\exists i) [d_i \prec_i e_i \wedge (\forall j < i) d_j = e_j]$$

$$\langle d_1, \dots, d_i, \dots, d_k \rangle$$

$$= \prec_i$$

$$\langle e_1, \dots, e_i, \dots, e_k \rangle$$

Lexicographic Ranking Function

Consider loop $L : \langle \mathcal{V}, \Theta, \mathcal{T} \rangle$.

Tuple of functions $\delta : \langle \delta_1, \delta_2, \dots, \delta_k \rangle$ where $\delta_i : \mathcal{S}_L \rightarrow \mathbb{R}$ is a **lexicographic ranking function** for L if there is a map $\pi : \mathcal{T} \rightarrow \{1, \dots, k\}$ s.t.

(Bounded) $(\forall \tau \in \mathcal{T})$

$$L \models \tau(\mathcal{V}, \mathcal{V}') \rightarrow \delta_{\pi(\tau)}(\mathcal{V}) \geq 0$$

(Ranking) $(\exists \epsilon > 0)(\forall \tau \in \mathcal{T})$

$$L \models \tau(\mathcal{V}, \mathcal{V}') \rightarrow \delta_{\pi(\tau)}(\mathcal{V}') \leq \delta_{\pi(\tau)}(\mathcal{V}) - \epsilon$$

(Nonincreasing) $(\forall \tau \in \mathcal{T})$

$$L \models (\forall j < \pi(\tau))[\tau(\mathcal{V}, \mathcal{V}') \rightarrow \delta_j(\mathcal{V}') \leq \delta_j(\mathcal{V})]$$

Induced Lexicographic Well-founded Relation

δ, ϵ induce a lexicographic well-founded relation over \mathcal{S}_L :

for $s, t \in \mathcal{S}_L$,

$$s \prec t \quad \Rightarrow \quad (\exists i) [\delta_i(s) \leq \delta_i(t) - \epsilon \wedge (\forall j < i) \delta_j(s) \leq \delta_j(t)]$$

Also, for any $\tau \in \mathcal{T}$,

$$\text{if } \tau(t, s), \text{ then } s \prec t$$

Thus, L always terminates.

Example: McCarthy 91

For $n \in \mathbb{Z}^+$,

$$f(n) = \begin{cases} f(f(n + 11)) & \text{if } n \leq 100 \\ n - 10 & \text{if } n > 100 \end{cases}$$

For every $1 \leq n \leq 92$, $f(n) = 91$, if it terminates.

We prove termination for all $n \in \mathbb{Z}^+$.

Example:

$$\begin{aligned} f(89) &= f(f(100)) = f(f(f(111))) = f(f(101)) \\ &= f(91) = f(f(102)) = \dots = 91 \end{aligned}$$

Example: Imperative McCarthy 91

```
int f(int x)
  int s = 1
  while true do
    if x > 100
    then  if s = 1
          then return x - 10
          else  x := x - 10
              s := s - 1
    else  x := x + 11
         s := s + 1
  done
```

Abstract program:

$$\begin{aligned}\Theta &: \{s = 1\} \\ \tau_1 &: \{x \geq 101, s \neq 1\} \Rightarrow \{x' = x - 10, s' = s - 1\} \\ \tau_2 &: \{x \leq 100\} \Rightarrow \{x' = x + 11, s' = s + 1\}\end{aligned}$$

for $x, s \in \mathbb{R}$

Example: McCarthy 91

Prove

$$\langle \underbrace{10s - x + 90}_{\delta_1}, \underbrace{x}_{\delta_2} \rangle$$

is a lexicographic ranking function for McCarthy 91, with

$$\pi(\tau_1) = 2, \pi(\tau_2) = 1$$

- Take loop invariant $s \geq 1$.
- Choose $\epsilon = 1$.

Show

$$\begin{array}{ll} \tau_1 \rightarrow \delta_2 \geq 0 & \tau_2 \rightarrow \delta_1 \geq 0 \\ \tau_1 \rightarrow \delta'_2 \leq \delta_2 - \epsilon & \tau_2 \rightarrow \delta'_1 \leq \delta_1 - \epsilon \\ \tau_1 \rightarrow \delta'_1 \leq \delta_1 & \end{array}$$

Example: McCarthy 91

Bounded τ_1 : $\pi(\tau_1) = 2$

$$\underbrace{x \geq 101}_{\text{guard of } \tau_1} \rightarrow \underbrace{x}_{\delta_2} \geq 0$$

Ranking τ_1 : $\pi(\tau_1) = 2$

$$\underbrace{x \geq 101}_{\text{guard of } \tau_1} \rightarrow \underbrace{(x - 10)}_{\text{substitution into } \delta_2 \text{ by } \tau_1} \leq \underbrace{x}_{\delta_2} - \underbrace{1}_{\epsilon}$$

Nonincreasing τ_1 : $1 < \pi(\tau_1) = 2$

$$\underbrace{x \geq 101}_{\text{guard of } \tau_1} \rightarrow \underbrace{10(s - 1) - (x - 10) + 90}_{\text{substitution into } \delta_1 \text{ by } \tau_1} \leq \underbrace{10s - x + 90}_{\delta_1}$$

Example: McCarthy 91

Bounded τ_2 : $\pi(\tau_2) = 1$

$$\underbrace{s \geq 1}_{\text{invariant}} \wedge \underbrace{x \leq 100}_{\text{guard of } \tau_2} \rightarrow \underbrace{10s - x + 90}_{\delta_1} \geq 0$$

Ranking τ_2 : $\pi(\tau_2) = 1$

$$\underbrace{10(s + 1) - (x + 11) + 90}_{\text{substitution into } \delta_1 \text{ by } \tau_2} \leq \underbrace{10s - x + 90}_{\delta_1} - \underbrace{1}_{\epsilon}$$

Assertions are valid, so McCarthy 91 always terminates.

The Theoretical Landscape

Ranking Functions

Theorem Every terminating loop has a ranking function.

But in general, expressing a ranking function requires **FOL with fixpoints**, which is **incomplete**.

Therefore, termination is not necessarily semi-decidable.

In fact, termination is not semi-decidable for a simple class of loops.

Interlude: Linear Loops

Consider variables $\mathcal{V} = \{x_1, x_2, \dots, x_m\}$.

homogenous vector:

$$\mathbf{x} = (x_1, \dots, x_m, 1)^T$$

linear assertion:

$$\bigwedge_{i \in \{1, \dots, k\}} (a_{i,1}x_1 + \dots + a_{i,m}x_m + a_{i,m+1} \geq 0)$$
$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,m} & a_{1,m+1} \\ & & \vdots & \\ a_{k,1} & \cdots & a_{k,m} & a_{k,m+1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$
$$\mathbf{Ax} \geq 0$$

Interlude: Linear Loops

Consider variables $\mathcal{V} = \{x_1, x_2, \dots, x_m\}$.

linear loop: $L : \langle \mathcal{V}, \Theta, \mathcal{T} \rangle$ in which all assertions are linear

- initial condition: $\Theta \mathbf{x} \geq 0$

- transitions: $\tau_i(\mathbf{xx}') \geq 0$

where $(\mathbf{xx}') = (x_1, \dots, x_m, x'_1, \dots, x'_m, 1)^T$

Theoretical Limitation

Consider loops of form:

$$\Theta : \bigwedge_{x_i \in \bar{\mathcal{V}} \subseteq \mathcal{V}} x_i = c_i$$

while $\mathbf{g}^T \mathbf{x} \geq 0$ do

$$\mathbf{x} := \underbrace{(\mathbf{A}_1 | \mathbf{A}_2 | \cdots | \mathbf{A}_k)}_{\text{nondeterministic choice}} \mathbf{x}$$

done

for $\mathbf{x} \in \mathbb{R}^n$, $k \geq 1$.

Restricted subset of linear loops.

Theorem Termination of such loops is not semi-decidable
(not recursively enumerable).

No complete method.

Synthesis Problem

Identify class of loops L ,

class of functions F ,

and class of assertions A

such that

synthesis of ranking functions of form F

with supporting invariants A is complete for L .

Example:

(Lexicographic) linear ranking functions

and linear invariants

for **linear** loops over the reals.

Recent Work

Colón & Sipma 2001, 2002

Linear ranking functions for linear loops.

Colón, Sankaranarayanan & Sipma 2003

Constraint-based linear invariant generation.

Podelski & Rybalchenko 2004

Complete method for one-transition linear loops.

Bradley, Manna & Sipma 2005

Lexicographic linear ranking functions, supporting invariants.

Bradley, Manna & Sipma 2005

Linear ranking functions, supporting invariants,
for *integer* linear loops.

**Synthesis of
Linear Ranking Functions
with Supporting Invariants**

CONSTRAINT-BASED APPROACH:
TEMPLATES WITH UNKNOWN COEFFICIENTS

Linear Ranking Function

Consider linear loop $L : \langle \mathcal{V}, \Theta, \mathcal{T} \rangle$.

For $\mathbf{r}^T \mathbf{x}$ a template (r_i are unknown coefficients)

$$r_1 x_1 + \cdots + r_m x_m + r_{m+1}$$

$\mathbf{r}^T \mathbf{x}$ is a **linear ranking function** if

(Bounded) $(\forall \tau \in \mathcal{T})$

$$L \models \tau(\mathbf{x}\mathbf{x}') \geq 0 \rightarrow \underbrace{\mathbf{r}^T \mathbf{x}}_{\delta} \geq 0$$

(Ranking) $(\exists \epsilon > 0)(\forall \tau \in \mathcal{T})$

$$L \models \tau(\mathbf{x}\mathbf{x}') \geq 0 \rightarrow \mathbf{r}^T \mathbf{x} - \mathbf{r}^T \mathbf{x}' \geq \epsilon$$

Lexicographic Linear Ranking Function

Consider linear loop $L : \langle \mathcal{V}, \Theta, \mathcal{T} \rangle$.

$$\langle \underbrace{\mathbf{r}_1^T \mathbf{x}}_{\delta_1}, \dots, \underbrace{\mathbf{r}_k^T \mathbf{x}}_{\delta_k} \rangle \quad \text{with} \quad \pi : \mathcal{T} \rightarrow \{1, \dots, k\}$$

for unknown coefficients \mathbf{r}_{ij} and unknown π ,
is a k -component lexicographic linear ranking function if

(Bounded) $(\forall \tau \in \mathcal{T})$

$$L \models \tau(\mathbf{x}\mathbf{x}') \geq 0 \rightarrow \mathbf{r}_{\pi(\tau)}^T \mathbf{x} \geq 0$$

(Ranking) $(\exists \epsilon > 0)(\forall \tau \in \mathcal{T})$

$$L \models \tau(\mathbf{x}\mathbf{x}') \geq 0 \rightarrow \mathbf{r}_{\pi(\tau)}^T \mathbf{x} - \mathbf{r}_{\pi(\tau)}^T \mathbf{x}' \geq \epsilon$$

(Nonincreasing) $(\forall \tau \in \mathcal{T})$

$$L \models (\forall j < \pi(\tau)) [\tau(\mathbf{x}\mathbf{x}') \geq 0 \rightarrow \mathbf{r}_j^T \mathbf{x} - \mathbf{r}_j^T \mathbf{x}' \geq 0]$$

Linear Supporting Invariant

Ranking functions often require **supporting invariants**.

Consider linear loop $L : \langle \mathcal{V}, \Theta, \mathcal{T} \rangle$.

For $\mathbf{Ix} \geq \mathbf{0}$ a template (\mathbf{I}_{ij} are unknown coefficients),

$\mathbf{Ix} \geq \mathbf{0}$ is an ℓ -**conjunct linear invariant** if

(Initiation)

$$\Theta \mathbf{x} \geq \mathbf{0} \rightarrow \mathbf{Ix} \geq \mathbf{0}$$

(Consecution) ($\forall \tau \in \mathcal{T}$)

$$\mathbf{Ix} \geq \mathbf{0} \wedge \tau(\mathbf{xx}') \geq \mathbf{0} \rightarrow \mathbf{Ix}' \geq \mathbf{0}$$

Inductive assertion \Rightarrow invariant

How do we find the unknown coefficients in the templates?

Farkas Lemma (1894)

System of linear inequalities over real variables $\mathbf{x} = \{x_1, \dots, x_n\}$:

$$S : \begin{bmatrix} a_{1,1}x_1 + \cdots + a_{1,n}x_n + b_1 \geq 0 \\ \vdots \\ a_{m,1}x_1 + \cdots + a_{m,n}x_n + b_m \geq 0 \end{bmatrix}$$

S entails linear inequality

$$\psi : c_1x_1 + \cdots + c_nx_n + d \geq 0$$

$S \models \psi$ if and only if S is unsatisfiable

or there exist **real** numbers $\lambda_1, \dots, \lambda_m \geq 0$ such that

$$c_1 = \sum_{i=1}^m \lambda_i a_{i,1} \quad \cdots \quad c_n = \sum_{i=1}^m \lambda_i a_{i,n} \quad d \geq \left(\sum_{i=1}^m \lambda_i b_i \right)$$

Synthesis Overview

Consider loop $L : \langle \mathcal{V} = \{x_1, x_2, \dots, x_m\}, \Theta, \mathcal{T} \rangle$.

Given templates

- ℓ -conjunct invariant template $\boxed{\mathbf{I}\mathbf{x} \geq \mathbf{0}}$ (\mathbf{I} has ℓ rows)
- k -component lexicographic ranking function templates

$$\boxed{\{\mathbf{c}_1^T \mathbf{x}, \dots, \mathbf{c}_k^T \mathbf{x}\}}$$

Apply Farkas Lemma **rules** to encode

- conditions for the unknown invariant coefficients \mathbf{I}_{ij}
- conditions for the unknown ranking function coefficients \mathbf{c}_{ij}

Solve the generated constraint system.

Farkas Lemma Rules: Invariant

(Initiation)

$$\text{II} : \frac{\Theta \mathbf{x} \geq \mathbf{0}}{\mathbf{I} \mathbf{x} \geq \mathbf{0}}$$

(Consecution)

$$\text{C}_i : \frac{\mathbf{I} \mathbf{x} \geq \mathbf{0} \quad \tau_i(\mathbf{x} \mathbf{x}') \geq \mathbf{0}}{\mathbf{I} \mathbf{x}' \geq \mathbf{0}}$$

(Disabled)

$$\text{D}_i : \frac{\mathbf{I} \mathbf{x} \geq \mathbf{0} \quad \tau_i(\mathbf{x} \mathbf{x}') \geq \mathbf{0}}{-1 \geq \mathbf{0} \quad \leftarrow \text{false}}$$

Farkas Lemma Rules: Ranking Function

(Bounded)

$$\begin{array}{r} \mathbf{I}\mathbf{x} \geq \mathbf{0} \\ \mathbb{B}_i : \frac{\tau_i(\mathbf{x}\mathbf{x}') \geq \mathbf{0}}{\mathbf{c}^T\mathbf{x} \geq \mathbf{0}} \end{array}$$

(Ranking)

$$\begin{array}{r} \mathbf{I}\mathbf{x} \geq \mathbf{0} \\ \mathbb{R}_i : \frac{\tau_i(\mathbf{x}\mathbf{x}') \geq \mathbf{0}}{\mathbf{c}^T\mathbf{x} - \mathbf{c}^T\mathbf{x}' - \epsilon \geq \mathbf{0}} \end{array}$$

Generated Constraints

- Constraints are **linear** if no invariant template is given.
- Constraints are **parametric linear** otherwise.
 - Linear, except for a few bilinear quadratic terms
(a λ and a supporting invariant template coefficient,
e.g., $\lambda_1 i_{1,1}$)
 - Decidable [**Tarski 1951**]
 - Generic solvers based on
 - * CAD (Cylindrical Algebraic Decomposition) [**Collins 1975**]
(implementation in, *e.g.*, Mathematica)
 - * Redlog [**Dolzmann & Sturm 1997**]
 - Specialized solvers
[**Sankaranarayanan et al. 2004**], [**Bradley et al. 2005**]

Synthesis: Soundness and Completeness

Special Case

Linear loop $L : \langle \mathcal{V}, \Theta, \mathcal{T} \rangle$ has a
linear ranking function
supported by an ℓ -conjunct linear invariant

\Leftrightarrow

the constraint system generated by

$$\mathbb{I} \wedge \bigwedge_{\tau_i \in \mathcal{T}} (\mathbb{D}_i \vee (\mathbb{C}_i \wedge \mathbb{B}_i \wedge \mathbb{R}_i))$$

is **satisfiable**.

Example: GCD

$$\Theta : \{y_1 \geq 1, y_2 \geq 1\}$$

$$\tau_1 : \{y_1 \geq y_2 + 1\} \Rightarrow \{y'_1 = y_1 - y_2, y'_2 = y_2\}$$

$$\tau_2 : \{y_2 \geq y_1 + 1\} \Rightarrow \{y'_2 = y_2 - y_1, y'_1 = y_1\}$$

Find

- a linear ranking function $\mathbf{c}^T \mathbf{x}$

$$c_1 y_1 + c_2 y_2 + c_3$$

- with a 2-conjunct supporting invariant $\mathbf{I} \mathbf{x} \geq \mathbf{0}$

$$\begin{bmatrix} i_{1,1} & i_{1,2} & i_{1,3} \\ i_{2,1} & i_{2,2} & i_{2,3} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix} \geq \mathbf{0}$$

Example: GCD Invariant

Initiation

$$\begin{array}{l} y_1 \geq 1 \\ y_2 \geq 1 \\ \hline \text{II : } i_{1,1}y_1 + i_{1,2}y_2 + i_{1,3} \geq 0 \\ i_{2,1}y_1 + i_{2,2}y_2 + i_{2,3} \geq 0 \end{array}$$

Consecution

$$\begin{array}{l} i_{1,1}y_1 + i_{1,2}y_2 + i_{1,3} \geq 0 \\ i_{2,1}y_1 + i_{2,2}y_2 + i_{2,3} \geq 0 \\ y_1 \geq y_2 + 1 \\ \text{C}_1 : \quad y'_1 = y_1 - y_2 \\ \quad \quad y'_2 = y_2 \\ \hline i_{1,1}y'_1 + i_{1,2}y'_2 + i_{1,3} \geq 0 \\ i_{2,1}y'_1 + i_{2,2}y'_2 + i_{2,3} \geq 0 \end{array}$$

$$\begin{array}{l} i_{1,1}y_1 + i_{1,2}y_2 + i_{1,3} \geq 0 \\ i_{2,1}y_1 + i_{2,2}y_2 + i_{2,3} \geq 0 \\ y_2 \geq y_1 + 1 \\ \text{C}_2 : \quad y'_2 = y_2 - y_1 \\ \quad \quad y'_1 = y_1 \\ \hline i_{1,1}y'_1 + i_{1,2}y'_2 + i_{1,3} \geq 0 \\ i_{2,1}y'_1 + i_{2,2}y'_2 + i_{2,3} \geq 0 \end{array}$$

Example: GCD Ranking Function

Bounded

$$\begin{array}{l}
 \mathbb{B}_1 : \quad i_{1,1}y_1 + i_{1,2}y_2 + i_{1,3} \geq 0 \\
 \quad \quad i_{2,1}y_1 + i_{2,2}y_2 + i_{2,3} \geq 0 \\
 \quad \quad y_1 \geq y_2 + 1 \\
 \hline
 \quad \quad c_1y_1 + c_2y_2 + c_3 \geq 0
 \end{array}
 \qquad
 \begin{array}{l}
 \mathbb{B}_2 : \quad i_{1,1}y_1 + i_{1,2}y_2 + i_{1,3} \geq 0 \\
 \quad \quad i_{2,1}y_1 + i_{2,2}y_2 + i_{2,3} \geq 0 \\
 \quad \quad y_2 \geq y_1 + 1 \\
 \hline
 \quad \quad c_1y_1 + c_2y_2 + c_3 \geq 0
 \end{array}$$

Ranking

$$\begin{array}{l}
 \mathbb{R}_1 : \quad i_{1,1}y_1 + i_{1,2}y_2 + i_{1,3} \geq 0 \\
 \quad \quad i_{2,1}y_1 + i_{2,2}y_2 + i_{2,3} \geq 0 \\
 \quad \quad y_1 \geq y_2 + 1 \\
 \quad \quad y'_1 = y_1 - y_2 \\
 \quad \quad y'_2 = y_2 \\
 \hline
 \quad \quad c_1y_1 + c_2y_2 \geq c_1y'_1 + c_2y'_2 + \epsilon
 \end{array}
 \qquad
 \begin{array}{l}
 \mathbb{R}_2 : \quad i_{1,1}y_1 + i_{1,2}y_2 + i_{1,3} \geq 0 \\
 \quad \quad i_{2,1}y_1 + i_{2,2}y_2 + i_{2,3} \geq 0 \\
 \quad \quad y_2 \geq y_1 + 1 \\
 \quad \quad y'_2 = y_2 - y_1 \\
 \quad \quad y'_1 = y_1 \\
 \hline
 \quad \quad c_1y_1 + c_2y_2 \geq c_1y'_1 + c_2y'_2 + \epsilon
 \end{array}$$

Example: GCD Synthesis

$$\Theta : \{y_1 \geq 1, y_2 \geq 1\}$$

$$\tau_1 : \{y_1 \geq y_2 + 1\} \Rightarrow \{y'_1 = y_1 - y_2, y'_2 = y_2\}$$

$$\tau_2 : \{y_2 \geq y_1 + 1\} \Rightarrow \{y'_2 = y_2 - y_1, y'_1 = y_1\}$$

Solving the constraint system induced by

$$\mathbb{I} \wedge \mathbb{C}_1 \wedge \mathbb{C}_2 \wedge \mathbb{B}_1 \wedge \mathbb{B}_2 \wedge \mathbb{R}_1 \wedge \mathbb{R}_2$$

reveals ranking function

$$c_1 = c_2 = 1, c_3 = 0 \Rightarrow \boxed{y_1 + y_2}$$

with $\epsilon = 1$, supported by the invariants

$$i_{1,1} = 1, i_{1,2} = 0, i_{1,3} = -1 \Rightarrow$$

$$i_{2,1} = 0, i_{2,2} = 1, i_{2,3} = -1 \Rightarrow$$

$$\boxed{y_1 \geq 1}$$

$$\boxed{y_2 \geq 1}$$

which proves that GCD always terminates.

Synthesis of
Linear Lexicographic Ranking Functions
with Supporting Invariants

Farkas Lemma Rules: Invariant

(Initiation)

$$\text{II} : \frac{\Theta \mathbf{x} \geq \mathbf{0}}{\mathbf{I} \mathbf{x} \geq \mathbf{0}}$$

(Consecution)

$$\text{C}_i : \frac{\begin{array}{l} \mathbf{I} \mathbf{x} \geq \mathbf{0} \\ \tau_i(\mathbf{x} \mathbf{x}') \geq \mathbf{0} \end{array}}{\mathbf{I} \mathbf{x}' \geq \mathbf{0}}$$

(Disabled)

$$\text{D}_i : \frac{\begin{array}{l} \mathbf{I} \mathbf{x} \geq \mathbf{0} \\ \tau_i(\mathbf{x} \mathbf{x}') \geq \mathbf{0} \end{array}}{-1 \geq \mathbf{0}} \leftarrow \text{false}$$

Farkas Lemma Rules: Ranking Function

(Bounded)

$$\begin{array}{l} \mathbf{I}\mathbf{x} \geq \mathbf{0} \\ \mathbb{B}_{ij} : \frac{\tau_i(\mathbf{x}\mathbf{x}') \geq 0}{\mathbf{c}_j^T \mathbf{x} \geq 0} \end{array}$$

(Ranking)

$$\begin{array}{l} \mathbf{I}\mathbf{x} \geq \mathbf{0} \\ \mathbb{R}_{ij} : \frac{\tau_i(\mathbf{x}\mathbf{x}') \geq 0}{\mathbf{c}_j^T \mathbf{x} - \mathbf{c}_j^T \mathbf{x}' - \epsilon \geq 0} \end{array}$$

(Nonincreasing)

$$\begin{array}{l} \mathbf{I}\mathbf{x} \geq \mathbf{0} \\ \mathbb{N}_{ij} : \frac{\tau_i(\mathbf{x}\mathbf{x}') \geq 0}{\mathbf{c}_j^T \mathbf{x} - \mathbf{c}_j^T \mathbf{x}' \geq 0} \end{array}$$

Synthesis: Soundness and Completeness

Theorem (General Case)

Linear loop $L : \langle \mathcal{V}, \Theta, \mathcal{T} \rangle$ has a
 k -lexicographic linear ranking function
supported by an ℓ -conjunct linear invariant

\Leftrightarrow

the constraint system generated by

$$\mathbb{I} \wedge \bigwedge_{\tau_i \in \mathcal{T}} (\mathbb{D}_i \vee (\mathbb{C}_i \wedge \mathbb{B}_{i, \pi(i)} \wedge \mathbb{R}_{i, \pi(i)})) \wedge \bigwedge_{\tau_i \in \mathcal{T}, j < \pi(i)} (\mathbb{D}_i \vee \mathbb{N}_{ij})$$

is **satisfiable** for some $\pi : \mathcal{T} \rightarrow \{1, \dots, k\}$.

One Constraint System per Component

Given lexicographic template components

$$\{\mathbf{c}_1^T \mathbf{x}, \dots, \mathbf{c}_n^T \mathbf{x}\}$$

and **partial order** \prec over \mathcal{T} , solve n constraint systems induced by

$$\underbrace{\mathbb{I} \wedge \bigwedge_{\tau_i} (\mathbb{D}_i \vee \mathbb{C}_i)}_{\text{Invariant}} \quad \wedge \quad \underbrace{(\mathbb{D}_j \vee (\mathbb{B}_j \wedge \mathbb{R}_j))}_{\tau_j \text{ ranked by } \mathbf{c}_j^T}$$

$$\begin{array}{ccc} \wedge & & \langle \dots, \mathbf{c}_j^T \mathbf{x}, \dots, \mathbf{c}_i^T \mathbf{x}, \dots \rangle \\ & \bigwedge_{\tau_i \text{ s.t. } \tau_j \prec \tau_i} (\mathbb{D}_i \vee \mathbb{N}_{ij}) & \begin{array}{cc} \uparrow & \uparrow \\ \tau_j & \tau_i \end{array} \end{array}$$

τ_i does not increase \mathbf{c}_j^T if $\tau_j \prec \tau_i$

for $j \in \{1, \dots, n\}$.

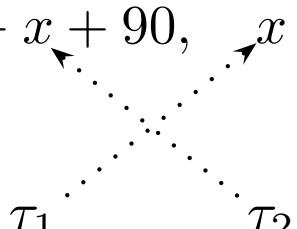
One Constraint System per Component

Advantages:

- Multiple smaller constraint systems are easier to solve in practice.
- Invariant template may be instantiated differently for each constraint system.

Synthesizing Lexicographic Ranking Functions

Goal: Find map π .

$$\langle 10s - x + 90, x \rangle$$


τ_1 τ_2

- τ_1 decreases component 2, does not increase component 1
- τ_2 decreases component 1

Synthesizing Lexicographic Ranking Functions

Overview:

- Initially propose one template component per transition

$$\{\mathbf{c}_1^T \mathbf{x}, \dots, \mathbf{c}_n^T \mathbf{x}\}$$

Inexpensive because template coefficients \mathbf{c}_i appear only **linearly** in generated constraints.

- Incrementally build linear order \prec over \mathcal{T} .

Linear order gives π .

- Use intermediate partial orders to generate and solve constraint systems to guide search.

Simplified Farkas Lemma Rules

One component per transition.

(Bounded)

$$\mathbb{B}_i : \begin{array}{r} \mathbf{I}\mathbf{x} \geq \mathbf{0} \\ \tau_i(\mathbf{x}\mathbf{x}') \geq \mathbf{0} \\ \hline \mathbf{c}_i^T \mathbf{x} \geq \mathbf{0} \end{array}$$

(Ranking)

$$\mathbb{R}_i : \begin{array}{r} \mathbf{I}\mathbf{x} \geq \mathbf{0} \\ \tau_i(\mathbf{x}\mathbf{x}') \geq \mathbf{0} \\ \hline \mathbf{c}_i^T \mathbf{x} - \mathbf{c}_i^T \mathbf{x}' - \epsilon \geq \mathbf{0} \end{array}$$

Synthesizing Lexicographic Ranking Functions

Incrementally build \prec over \mathcal{T} :

1. Guess $\tau_i \prec \tau_j$:

$$\langle \dots, \mathbf{c}_i^T \mathbf{x}, \dots, \mathbf{c}_j^T \mathbf{x}, \dots \rangle$$

2. Generate constraint system and solve.

- $\mathbf{I}, \mathbf{C}_i, \mathbf{D}_i, \mathbf{B}_i, \mathbf{R}_i$ are fixed.
- $\tau_i \prec \tau_j$ induces \mathbb{N}_{ji} : τ_j should not increase $\mathbf{c}_i^T \mathbf{x}$

Satisfiable \Rightarrow Continue search

Unsatisfiable \Rightarrow Try $\tau_i \succ \tau_j$

Satisfiable \Rightarrow Continue search

Unsatisfiable \Rightarrow Backtrack

3. Finished when order is **linear**

and generated constraint system is **satisfiable**.

(Otherwise, add conjunct to $\mathbf{I}\mathbf{x} \geq \mathbf{0}$.)

Example: McCarthy 91

$$\Theta : \{s = 1\}$$

$$\tau_1 : \{x \geq 101, s \leq 0\} \Rightarrow \{x' = x - 10, s' = s - 1\}$$

$$\tau_2 : \{x \geq 101, s \geq 2\} \Rightarrow \{x' = x - 10, s' = s - 1\}$$

$$\tau_3 : \{x \leq 100\} \Rightarrow \{x' = x + 11, s' = s + 1\}$$

} split of
s ≠ 1

Find

- a 3-component lexicographic linear ranking function

$$\langle \mathbf{r}_1^T \mathbf{x}, \mathbf{r}_2^T \mathbf{x}, \mathbf{r}_3^T \mathbf{x} \rangle$$

with $\pi : \mathcal{T} \rightarrow \{1, 2, 3\}$

- with a 1-conjunct supporting invariant $\mathbf{I}\mathbf{x} \geq 0$

Example: McCarthy 91

Iteration 1:

- One template per transition.

$$\{\mathbf{c}_1^T \mathbf{x}, \mathbf{c}_2^T \mathbf{x}, \mathbf{c}_3^T \mathbf{x}\}$$

- No order assumed between transitions.
- Three sets of conditions:

$$\mathbf{c}_1^T \mathbf{x} : I \wedge C_1 \wedge C_2 \wedge C_3 \wedge B_1 \wedge R_1$$

$$\mathbf{c}_2^T \mathbf{x} : I \wedge C_1 \wedge C_2 \wedge C_3 \wedge B_2 \wedge R_2$$

$$\mathbf{c}_3^T \mathbf{x} : I \wedge C_1 \wedge C_2 \wedge C_3 \wedge B_3 \wedge R_3$$

- Induced constraint systems are **satisfiable**.

Example: McCarthy 91

Constraints on $\mathbf{c}_1^T \mathbf{x}$:

Initiation

$$\text{II} : \frac{s = 1}{i_1 s + i_2 x + i_3 \geq 0}$$

Consecution

$\mathbb{C}_1 :$

$$i_1 s + i_2 x + i_3 \geq 0$$

$$x \geq 101$$

$$s \leq 0$$

$$x' = x - 10$$

$$s' = s - 1$$

$$\frac{i_1 s' + i_2 x' + i_3 \geq 0}{}$$

$\mathbb{C}_2 :$

$$i_1 s + i_2 x + i_3 \geq 0$$

$$x \geq 101$$

$$s \geq 2$$

$$x' = x - 10$$

$$s' = s - 1$$

$$\frac{i_1 s' + i_2 x' + i_3 \geq 0}{}$$

$\mathbb{C}_3 :$

$$i_1 s + i_2 x + i_3 \geq 0$$

$$x \leq 100$$

$$x' = x + 11$$

$$s' = s + 1$$

$$\frac{i_1 s' + i_2 x' + i_3 \geq 0}{}$$

Example: McCarthy 91

Bounded

$$\mathbb{B}_1 : \frac{\begin{array}{l} i_1 s + i_2 x + i_3 \geq 0 \\ x \geq 101 \\ s \leq 0 \end{array}}{c_{1,1} s + c_{1,2} x + c_{1,3} \geq 0}$$

Ranking

$$\mathbb{R}_1 : \frac{\begin{array}{l} i_1 s + i_2 x + i_3 \geq 0 \\ x \geq 101 \\ s \leq 0 \\ x' = x - 10 \\ s' = s - 1 \end{array}}{c_{1,1} s + c_{1,2} x \geq c_{1,1} s' + c_{1,2} x' + \epsilon}$$

Example: McCarthy 91

Iteration 2:

- Guess $\tau_3 \prec \tau_2$.

$$\langle \dots, \mathbf{c}_3^T \mathbf{x}, \dots, \mathbf{c}_2^T \mathbf{x}, \dots \rangle$$

τ_2 should not increase $\mathbf{c}_3^T \mathbf{x}$.

- Three sets of conditions:

$$\mathbf{c}_1^T \mathbf{x} : \quad \mathbb{I} \wedge \mathbb{C}_1 \wedge \mathbb{C}_2 \wedge \mathbb{C}_3 \wedge \mathbb{B}_1 \wedge \mathbb{R}_1$$

$$\mathbf{c}_2^T \mathbf{x} : \quad \mathbb{I} \wedge \mathbb{C}_1 \wedge \mathbb{C}_2 \wedge \mathbb{C}_3 \wedge \mathbb{B}_2 \wedge \mathbb{R}_2$$

$$\mathbf{c}_3^T \mathbf{x} : \quad \mathbb{I} \wedge \mathbb{C}_1 \wedge \mathbb{C}_2 \wedge \mathbb{C}_3 \wedge \mathbb{B}_3 \wedge \mathbb{R}_3 \wedge \mathbb{N}_{2,3}$$

- Induced constraint systems are **satisfiable**.

Example: McCarthy 91

Iteration 3:

- Guess $\tau_1 \prec \tau_3$.

$$\langle \dots, \mathbf{c}_1^T \mathbf{x}, \dots, \mathbf{c}_3^T \mathbf{x}, \dots \rangle$$

τ_3 should not increase $\mathbf{c}_1^T \mathbf{x}$.

- Three sets of conditions:

$$\mathbf{c}_1^T \mathbf{x} : \quad \mathbb{I} \wedge \mathbb{C}_1 \wedge \mathbb{C}_2 \wedge \mathbb{C}_3 \wedge \mathbb{B}_1 \wedge \mathbb{R}_1 \wedge \mathbb{N}_{3,1}$$

$$\mathbf{c}_2^T \mathbf{x} : \quad \mathbb{I} \wedge \mathbb{C}_1 \wedge \mathbb{C}_2 \wedge \mathbb{C}_3 \wedge \mathbb{B}_2 \wedge \mathbb{R}_2$$

$$\mathbf{c}_3^T \mathbf{x} : \quad \mathbb{I} \wedge \mathbb{C}_1 \wedge \mathbb{C}_2 \wedge \mathbb{C}_3 \wedge \mathbb{B}_3 \wedge \mathbb{R}_3 \wedge \mathbb{N}_{2,3}$$

- Induced constraint system for $\mathbf{c}_1^T \mathbf{x}$ is **unsatisfiable**.

Example: McCarthy 91

Iteration 3:

- Try $\tau_3 \prec \tau_1$ instead.

$$\langle \dots, \mathbf{c}_3^T \mathbf{x}, \dots, \mathbf{c}_1^T \mathbf{x}, \dots \rangle$$

τ_1 should not increase $\mathbf{c}_3^T \mathbf{x}$.

- Three sets of conditions:

$$\mathbf{c}_1^T \mathbf{x} : \quad \mathbb{I} \wedge \mathbb{C}_1 \wedge \mathbb{C}_2 \wedge \mathbb{C}_3 \wedge \mathbb{B}_1 \wedge \mathbb{R}_1$$

$$\mathbf{c}_2^T \mathbf{x} : \quad \mathbb{I} \wedge \mathbb{C}_1 \wedge \mathbb{C}_2 \wedge \mathbb{C}_3 \wedge \mathbb{B}_2 \wedge \mathbb{R}_2$$

$$\mathbf{c}_3^T \mathbf{x} : \quad \mathbb{I} \wedge \mathbb{C}_1 \wedge \mathbb{C}_2 \wedge \mathbb{C}_3 \wedge \mathbb{B}_3 \wedge \mathbb{R}_3 \wedge \mathbb{N}_{2,3} \wedge \mathbb{N}_{1,3}$$

- Induced constraint systems are **satisfiable**.

Example: McCarthy 91

Iteration 4:

- Guess $\tau_2 \prec \tau_1$ (but $\tau_1 \prec \tau_2$ works, too).

$$\langle \dots, \mathbf{c}_2^T \mathbf{x}, \dots, \mathbf{c}_1^T \mathbf{x}, \dots \rangle$$

τ_1 should not increase $\mathbf{c}_2^T \mathbf{x}$.

- Three sets of conditions:

$$\mathbf{c}_1^T \mathbf{x} : \quad \mathbb{I} \wedge \mathbb{C}_1 \wedge \mathbb{C}_2 \wedge \mathbb{C}_3 \wedge \mathbb{B}_1 \wedge \mathbb{R}_1$$

$$\mathbf{c}_2^T \mathbf{x} : \quad \mathbb{I} \wedge \mathbb{C}_1 \wedge \mathbb{C}_2 \wedge \mathbb{C}_3 \wedge \mathbb{B}_2 \wedge \mathbb{R}_2 \wedge \mathbb{N}_{1,2}$$

$$\mathbf{c}_3^T \mathbf{x} : \quad \mathbb{I} \wedge \mathbb{C}_1 \wedge \mathbb{C}_2 \wedge \mathbb{C}_3 \wedge \mathbb{B}_3 \wedge \mathbb{R}_3 \wedge \mathbb{N}_{2,3} \wedge \mathbb{N}_{1,3}$$

- Induced constraint systems are **satisfiable**.

Example: McCarthy 91

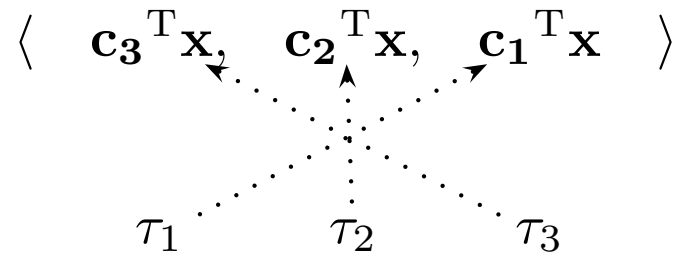
Iteration 4:

- \prec is a linear order:

$$\langle \mathbf{c}_3^T \mathbf{x}, \mathbf{c}_2^T \mathbf{x}, \mathbf{c}_1^T \mathbf{x} \rangle$$

$$\tau_3 \prec \tau_2 \prec \tau_1$$

- \prec gives π :



$$\pi(\tau_1) = 3, \pi(\tau_2) = 2, \pi(\tau_3) = 1$$

Example: McCarthy 91

$$\Theta : \{s = 1\}$$

$$\tau_1 : \{x \geq 101, s \leq 0\} \Rightarrow \{x' = x - 10, s' = s - 1\}$$

$$\tau_2 : \{x \geq 101, s \geq 2\} \Rightarrow \{x' = x - 10, s' = s - 1\}$$

$$\tau_3 : \{x \leq 100\} \Rightarrow \{x' = x + 11, s' = s + 1\}$$

Solving the final constraint systems reveals ranking function

$$\begin{cases} \langle 10s - x + 90, x, x \rangle \\ \pi(\tau_1) = 3, \pi(\tau_2) = 2, \pi(\tau_3) = 1 \end{cases} \Rightarrow \begin{cases} \langle 10s - x + 90, x \rangle \\ \pi(\tau_1) = \pi(\tau_2) = 2, \pi(\tau_3) = 1 \end{cases}$$

supported by the invariant

$$s \geq 1$$

which proves that McCarthy 91 always terminates.

In Practice

lines loops abstracted proved

Name	LOC	L	A	P	P/A	P/L	Sec
meschach	28K	911	778	758	97%	83%	64
gnuplot	50K	826	312	301	96%	36%	88
gaim	57K	594	54	52	96%	8%	94
ffmpeg	108K	2674	2115	2081	98%	78%	198

- Prototype **unsound** abstraction of C loops with CIL [Necula 02].

Why unsound?

- Overflows, **unsigned**, **doubles** abstracted as \mathbb{R} s, *etc.*
- Aliasing, globals changed by function calls, *etc.*

In principle, can be sound — an engineering task.

- Synthesis of lexicographic linear ranking functions. Lexicographic function needed for 10 loops.

Example of Failed Abstraction

```
List * iter = items;
while (iter != NULL) {
    ...
    iter = iter->next;
}
```

Proving termination requires:

- proving that `items` is noncircular (**nontrivial**);
- abstracting iteration to counting down (**trivial**).

Example of Failed Abstraction

```
char * ptr = input;
while (*ptr != '\0') {
    ...
    ptr++;
}
```

Proving termination requires:

- proving that `input` is a well-formed C string (**nontrivial**).
- abstracting pointer arithmetic to counting down (**trivial**).

Reasons for Failed Proofs

1. **Prototype** abstracter!

2. Need for invariants. Examples:

$i = 2 * i;$ $i \geq 0$ to deduce increase in i

$i = i + k;$ $k > 0$ to deduce increase in i

3. Need for **summarizing** embedded loops. Example:

```
while (i < n) {  
    while (...) { i++; }    ← summarize with  $i' \geq i$   
    i++;  
}
```

4. Need for function invariants. Example:

$i = i + \text{strlen}(\text{str});$ knowledge about str and strlen

5. Loop does not terminate.

What about Integer Operations?

So far, abstract loops have variables ranging over \mathbb{R} .

What if loop has integer division or modulo?

$$\{x \% 3 = 0, x \geq 0\} \Rightarrow \{x' = (x \text{ div } 3) - 2\}$$

$$\{x \% 3 \neq 0, x \geq 0\} \Rightarrow \{x' = x - 1\}$$

- [Bradley, Manna & Sipma 2005] (CONCUR)
- Complete for linear ranking functions and linear invariants of integer linear loops.
- Constraint-based, but does not use Farkas Lemma.

Appendix

Expanding a Farkas Lemma Rule

$$\mathbb{R}_{ij} : \begin{array}{r} \mathbf{I}\mathbf{x} \geq \mathbf{0} \\ \tau_i(\mathbf{x}\mathbf{x}') \geq \mathbf{0} \\ \mathbf{c}_j^T \mathbf{x} - \mathbf{c}_j^T \mathbf{x}' - \epsilon \geq \mathbf{0} \end{array}$$

$$\begin{aligned} \mathbf{x} &= (x_1, \dots, x_m, 1)^T \\ (\mathbf{x}\mathbf{x}') &= (x_1, \dots, x_m, x'_1, \dots, x'_m, 1)^T \end{aligned}$$

⇓

$$\begin{array}{r} \lambda_I \quad \mathbf{I}\mathbf{x} \quad + \quad \mathbf{i} \quad \geq \mathbf{0} \\ \lambda_G \quad \mathbf{G}_i \mathbf{x} \quad + \quad \mathbf{g}_i \quad \geq \mathbf{0} \\ \lambda_U \quad \mathbf{U}_i \mathbf{x} + \mathbf{V}_i \mathbf{x}' + \mathbf{u}_i \geq \mathbf{0} \\ \hline \mathbf{c}_j^T \mathbf{x} - \mathbf{c}_j^T \mathbf{x}' - \epsilon \geq \mathbf{0} \end{array}$$

Expand assertions:

$$\begin{aligned} \mathbf{x} &= (x_1, \dots, x_m)^T \\ \mathbf{x}' &= (x'_1, \dots, x'_m)^T \end{aligned}$$

\mathbf{I}, \mathbf{i} define Θ

$\mathbf{G}_i, \mathbf{g}_i, \mathbf{U}_i, \mathbf{V}_i, \mathbf{u}_i$ define τ_i

⇓

$$\begin{array}{r} \lambda_I^T \mathbf{I} + \lambda_G^T \mathbf{G}_i + \lambda_U^T \mathbf{U}_i = \mathbf{c}_j \\ \lambda_U^T \mathbf{V}_i = -\mathbf{c}_j \\ \lambda_I^T \mathbf{i} + \lambda_G^T \mathbf{g}_i + \lambda_U^T \mathbf{u}_i \leq -\epsilon \\ \lambda_I, \lambda_G, \lambda_U \geq 0 \\ \epsilon > 0 \end{array}$$

Constraints over $\{\lambda_I, \lambda_G, \lambda_U, \mathbf{c}_j, \epsilon\}$