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Combining Combination Properties, Part I: Nelson-Oppen and Politeness

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Abstract
This is the first part of an analysis of the interplay between multiple properties that are related to combination methodologies for theories in the field of satisfiability modulo theories. We here focus on Nelson-Oppen and polite theory combinations, leading to a total of five model-theoretic properties to be considered: stable infiniteness, smoothness, finite witnessability, strong finite witnessability, and convexity. Our first result is an improvement on polite theory combination, showing that it is possible when only assuming stable infiniteness and strong finite witnessability, and thus implying smoothness is not a prerequisite for this method. Second, we provide examples of Boolean combinations of the aforementioned 5 properties whenever they are possible (e.g., a theory that admits all the properties, a theory that admits none, etc.), sharp in the sense that no theories within simpler signatures may exhibit the exact same properties, and prove which combinations cannot occur. Among these examples, the most surprising one is that of a polite yet not strongly polite theory in one sort, a combination whose previous example in the literature was two-sorted.

Keywords Satisfiability modulo theories · Theory combination · Theory politeness · Nelson-Oppen

1 Introduction

Satisfiability modulo theories [2], often referred to as SMT, deals with algorithms for deciding the satisfiability of (quantifier-free) formulas given a (first-order, many-sorted) background theory; this generalizes the area of research known as SAT, which focuses on Boolean satisfiability. Within this broad topic of research, the motivation for theory combination becomes

A preliminary and concise version of this paper appears in [1]. The current version includes all the proofs and examples missing from that version, as well as more detailed constructions of theories.

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clear: suppose one has two theories, and two algorithms for deciding the satisfiability of formulas in each theory; if we could combine these two algorithms into one solver for the combined theory (meaning, the one whose models are the shared ones between the two original theories, when restricting the signatures accordingly), that would be more efficient than constructing from scratch a decision procedure for this new theory. This way, if the theories one studies are combinations and recombinations of a simple generating family of theories for which we already have SMT solvers, it would follow that modulo actually performing the combinations of the algorithms we already have solvers for all relevant theories.

Combination algorithms exist: the first was developed by Nelson and Oppen [3], and relied on a model-theoretic property, stable infiniteness, which, roughly speaking, requires that every quantifier-free formula is satisfied by an infinite model. They also studied methods involving convexity, a more abstract property of theories. Roughly, a theory is convex if whenever a cube (conjunction of literals) implies a disjunction of variable equalities, then the same cube implies one of the disjuncts. The Nelson-Oppen approach works only when the theories share no functions or predicates (except for equality symbols). Another critical limitation of this method arises from the restriction that the combined theories must be stably infinite: so, to use a common example, combining the theory of lists with that of bit-vectors to obtain a theory of lists of bit-vectors is not possible with this approach, as models of the theory of bit-vectors are always finite. Later competing methods include polite, shiny [4], flexible [5] or gentle [6] theory combination, and for those (polite an shiny) that do not require anything from one theory, the price to be paid is that they require more from the other.

We focus here on, besides Nelson-Oppen, the polite combination method. This method [7] requires one of the theories to be polite, that is, smooth and finitely witnessable, (notions that will be defined later) while the other theory need not have any other property other than decidability; the fact is, however, that the polite algorithm is not guaranteed to work on polite theories due to a bug in its definition. This was later corrected so that it works for all *strongly* polite theories [8, 9], combining smoothness with a strengthened form of finite witnessability now aptly called *strong* finite witnessability; of course, given the difficulty involved in defining the algorithm for polite theory combination that led to its first instance, it is still of interest to researchers the comparison between politeness and its strong version in order to understand its inner machinations.

In this paper, we make two main contributions. The first is a new polite combination theorem, with fewer prerequisites: we show that polite combination is possible if the smoothness assumption is replaced by stable infiniteness, a much simpler property. Our second contribution is a thorough analysis of all Boolean combinations of stable infiniteness, smoothness, finite witnessability, strong finite witnessability and convexity; whenever such a combination is not possible, we prove so, and whenever it is, we provide an example to show that that is true. These examples are sharp in the sense that we define them over the simplest possible signature. Perhaps most surprising among these specimens is a one-sorted theory that is polite without being strongly so: an example separating the two forms of politeness was known [10] but only for two-sorted signatures.

The paper is organized as follows: Section 2 introduces the necessary background, both in many-sorted, first-order logic, and the model-theoretic properties used when dealing with Nelson-Oppen and polite theory combination; in Section 3 we prove that polite theory combination may be done by assuming stable infiniteness instead of smoothness; Section 4.1 proves which combinations of properties are impossible; Section 4.2 provides examples for all combinations that are actually possible, dividing those by the signatures over which they are defined, and offering operators that simplify, given a theory, the process of obtaining

$$\psi^{\sigma}_{\geq n} = \exists\, \overrightarrow{x}.\ \bigwedge_{1\leq i<j\leq n} \neg(x_i = x_j), \quad \psi^{\sigma}_{\leq n} = \exists\, \overrightarrow{x}.\forall\, y.\ \bigvee_{i=1}^{n} y = x_i, \quad \psi^{\sigma}_{=n} = \psi^{\sigma}_{\geq n} \wedge \psi^{\sigma}_{\leq n}$$

Fig. 1 Cardinality Formulas. $\overrightarrow{x}$ stands for $x_1, \ldots, x_n$

theories in other signatures with comparable properties; and, finally, Section 5 concludes the paper, pointing to some possible further directions this work could take. In Appendix A we perform the painstaking task of proving that the examples provided in Section 4.2 do indeed have the properties we affirm that they have; and in Section B we do the same but for our theory operators, also defined in Section 4.2.

2 Preliminary Notions

2.1 First-Order Signatures and Structures

A many-sorted signature Σ is a triple formed by a countable and non-empty set $\mathcal{S}_{\Sigma}$ of *sorts*, a countable set of function symbols $\mathcal{F}_{\Sigma}$, and a countable set of predicate symbols $\mathcal{P}_{\Sigma}$ which contains, for every sort $\sigma \in \mathcal{S}_{\Sigma}$, an equality symbol $=_{\sigma}$ (often denoted by $=$); each function symbol has an arity $\sigma_1 \times \cdots \times \sigma_n \to \sigma$ and each predicate symbol an arity $\sigma_1 \times \cdots \times \sigma_n$, where $\sigma_1, \ldots, \sigma_n, \sigma \in \mathcal{S}_{\Sigma}$ and $n \in \mathbb{N}$. Each equality symbol $=_{\sigma}$ has arity $\sigma \times \sigma$. A signature with no function or predicate symbols other than equalities is called *empty*.

A many-sorted signature Σ is *one-sorted* if $\mathcal{S}_{\Sigma}$ has one element; we may refer to many-sorted signatures simply as signatures. Two signatures are said to be *disjoint* if they share at most sorts and equality symbols.

We assume for each sort in $\mathcal{S}_{\Sigma}$ a distinct countably infinite set of variables, and define terms, literals, cubes (conjunctions of literals), and formulas (atomic or not) in the usual way. If s is a function symbol of arity $\sigma \to \sigma$ and x is a variable of sort σ, we define recursively the term $s^k(x)$, for $k \in \mathbb{N}$, as follows: $s^0(x) = x$, and $s^{k+1}(x) = s(s^k(x))$. We denote the set of free variables of sort σ in a formula φ by $vars_{\sigma}(\varphi)$, and given $S \subseteq \mathcal{S}_{\Sigma}$, $vars_S(\varphi) = \bigcup_{\sigma \in S} vars_{\sigma}(\varphi)$ (we use $vars(\phi)$ as shorthand for $vars_{\mathcal{S}_{\Sigma}}(\phi)$). The set of quantifier-free formulas on a signature Σ may be denoted by $QF(\Sigma)$. If we replace each occurrence of a variable x in a formula φ by y, we denote the resulting formula by $\varphi[x/y]$; if we replace the variables x_1 through x_n of φ by, respectively, y_1 through y_n, the resulting formula is denoted by $\varphi[x_1/y_1, \cdots, x_n/y_n]$.

A Σ-*structure* $\mathcal{A}$ is composed of non-empty sets $\sigma^{\mathcal{A}}$ for each sort $\sigma \in \mathcal{S}_{\Sigma}$, called the *domain of* σ, equipped with interpretations $f^{\mathcal{A}}$ and $P^{\mathcal{A}}$ of the function and predicate symbols, in a way that respects their arities. Furthermore, $=^{\mathcal{A}}_{\sigma}$ must be the identity on $\sigma^{\mathcal{A}}$.

A Σ-*interpretation* $\mathcal{A}$ is an extension of a Σ-structure that also interprets variables, with the value of a variable x of sort σ being an element $x^{\mathcal{A}}$ of $\sigma^{\mathcal{A}}$. We write $\alpha^{\mathcal{A}}$ for the interpretation of the term α under $\mathcal{A}$; if Γ is a set of terms, we define $\Gamma^{\mathcal{A}} = \{\alpha^{\mathcal{A}} : \alpha \in \Gamma\}$. We write $\mathcal{A} \vDash \varphi$ if $\mathcal{A}$ satisfies φ. A formula φ is called *satisfiable* if it is satisfied by some interpretation $\mathcal{A}$.

We shall make use of standard cardinality formulas, given in Fig. 1. $\psi^{\sigma}_{\geq n}$ is only satisfied by a structure $\mathcal{A}$ if $|\sigma^{\mathcal{A}}|$ is at least n, $\psi^{\sigma}_{\leq n}$ is only satisfied by $\mathcal{A}$ if $|\sigma^{\mathcal{A}}|$ is at most n, and $\psi^{\sigma}_{=n}$ is only satisfied by $\mathcal{A}$ if $|\sigma^{\mathcal{A}}|$ is exactly n. In one-sorted signatures, we may drop σ from the formulas, giving us $\psi_{\geq n}$, $\psi_{\leq n}$ and $\psi_{=n}$.

The following lemmas are generalizations of the standard compactness and downward Skolem-Löwenheim theorems of first-order logic to the many sorted case. They are proved in [11].

Lemma 1 ([11]) *A set of formulas is satisfiable iff each of its finite subsets is satisfiable.*

Lemma 2 ([11]) *If a set of formulas is satisfiable, there exists an interpretation $\mathcal{A}$ which satisfies it and where $\sigma^{\mathcal{A}}$ is countable whenever it is infinite, for every sort σ.*

A *theory* $\mathcal{T}$ is a class of all Σ-structures that satisfy some set of closed formulas (formulas without free variables), called the *axiomatization* of $\mathcal{T}$, which we denote as $Ax(\mathcal{T})$; such structures will be called the *models* of $\mathcal{T}$. A Σ-interpretation $\mathcal{A}$ whose underlying structure is in $\mathcal{T}$ is called a $\mathcal{T}$-interpretation. A formula is said to be $\mathcal{T}$-*satisfiable* if there is a $\mathcal{T}$-interpretation that satisfies it; a set of formulas is $\mathcal{T}$-satisfiable if there is a $\mathcal{T}$-interpretation that satisfies all of its elements simultaneously. Two formulas are $\mathcal{T}$-*equivalent* when every $\mathcal{T}$-interpretation satisfies one if and only if it satisfies the other. We write $\vdash_{\mathcal{T}} \varphi$ and say that φ is $\mathcal{T}$-valid if $\mathcal{A} \vDash \varphi$ for every $\mathcal{T}$-interpretation $\mathcal{A}$. Let Σ_1 and Σ_2 be disjoint signatures; by $\Sigma = \Sigma_1 \cup \Sigma_2$, we mean the signature with the union of the sorts, function symbols, and predicate symbols of Σ_1 and Σ_2, all arities preserved. Given a Σ_1-theory $\mathcal{T}_1$ and a Σ_2-theory $\mathcal{T}_2$, the $\Sigma_1 \cup \Sigma_2$-theory $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2$ is the theory axiomatized by the union of the axiomatizations of $\mathcal{T}_1$ and $\mathcal{T}_2$.

2.2 Model-Theoretic Properties

Let Σ be a signature. A Σ-theory $\mathcal{T}$ is said to be *stably infinite* w.r.t. $S \subseteq \mathcal{S}_\Sigma$ if, for every $\mathcal{T}$-satisfiable quantifier-free formula ϕ, there exists a $\mathcal{T}$-interpretation $\mathcal{A}$ satisfying ϕ such that, for each $\sigma \in S$, $\sigma^{\mathcal{A}}$ is infinite. $\mathcal{T}$ is *smooth* w.r.t. $S \subseteq \mathcal{S}_\Sigma$ when, for every quantifier-free formula ϕ, $\mathcal{T}$-interpretation $\mathcal{A}$ satisfying ϕ, and function κ from S to the class of cardinals such that $\kappa(\sigma) \geq |\sigma^{\mathcal{A}}|$ for every $\sigma \in S$, there exists a $\mathcal{T}$-interpretation $\mathcal{B}$ satisfying ϕ with $|\sigma^{\mathcal{B}}| = \kappa(\sigma)$, for every $\sigma \in S$.

Clearly, we have the following result:

Theorem 3 *Let Σ be a signature, $S \subseteq \mathcal{S}_\Sigma$, and $\mathcal{T}$ a Σ-theory. If $\mathcal{T}$ is smooth w.r.t. S, then it is also stably infinite w.r.t. S.*

For a finite set of sorts S, finite sets of variables V_σ of sort σ for each $\sigma \in S$, and equivalence relations E_σ on V_σ, the arrangement on $V = \bigcup_{\sigma \in S} V_\sigma$ induced by $E = \bigcup_{\sigma \in S} E_\sigma$, denoted by δ_V or δ_V^E, is the quantifier-free formula given by

$$\delta_V = \bigwedge_{\sigma \in S} \Big[\bigwedge_{x E_\sigma y} (x = y) \wedge \bigwedge_{x \overline{E_\sigma} y} \neg(x = y) \Big],$$

where $\overline{E_\sigma}$ denotes the complement of the equivalence relation E_σ.

A theory $\mathcal{T}$ is said to be *finitely witnessable* w.r.t. the set of sorts $S \subseteq \mathcal{S}_\Sigma$ when there exists a function *wit*, called a *witness*, from the quantifier-free formulas into themselves that is computable and satisfies, for every quantifier-free formula ϕ: (i) ϕ and $\exists \overrightarrow{w}.\ wit(\phi)$ are $\mathcal{T}$-equivalent, where $\overrightarrow{w} = vars(wit(\phi)) \setminus vars(\phi)$; and (ii) if $wit(\phi)$ is $\mathcal{T}$-satisfiable, then there exists a $\mathcal{T}$-interpretation $\mathcal{A}$ satisfying $wit(\phi)$ such that $\sigma^{\mathcal{A}} = vars_\sigma(wit(\phi))^{\mathcal{A}}$ for each $\sigma \in S$. $\mathcal{T}$ is said to be *strongly finitely witnessable* if it has a strong witness *wit*, which has the properties of a witness with the exception of (ii), satisfying instead: (ii') given a finite set of variables V and an arrangement δ_V on V, if $wit(\phi) \wedge \delta_V$ is $\mathcal{T}$-satisfiable, then there exists a $\mathcal{T}$-interpretation $\mathcal{A}$ satisfying $wit(\phi) \wedge \delta_V$ such that $\sigma^{\mathcal{A}} = vars_\sigma(wit(\phi) \wedge \delta_V)^{\mathcal{A}}$ for all $\sigma \in S$.

From the definitions, the following theorem directly follows:

Theorem 4 *Let Σ be a signature, $S \subseteq \mathcal{S}_\Sigma$, and $\mathcal{T}$ a Σ-theory. If $\mathcal{T}$ is strongly finitely witnessable w.r.t. S, then it is also finitely witnessable w.r.t. S.*

A theory that is both smooth and finitely witnessable w.r.t. (a set of sorts) S is said to be *polite* w.r.t. S; a theory that is both smooth and strongly finitely witnessable w.r.t. S is called *strongly polite* w.r.t. S. For theories over one-sorted empty signatures, we have the following theorem from [10]:

Theorem 5 ([10]) *Every one-sorted theory over the empty signature that is polite w.r.t. its only sort is strongly polite w.r.t. that sort.*

A one-sorted theory $\mathcal{T}$ is said to be *convex* if, for any conjunction of literals ϕ and any finite set of variables $\{u_1, v_1, \ldots, u_n, v_n\}$, $\vdash_{\mathcal{T}} \phi \rightarrow \bigvee_{i=1}^{n} u_i = v_i$ implies $\vdash_{\mathcal{T}} \phi \rightarrow u_i = v_i$, for some $i \in [1, n]$.

Given a one-sorted theory $\mathcal{T}$, its *mincard* function takes a quantifier-free formula ϕ and returns the countable cardinal $\min\{|\sigma^{\mathcal{A}}| : \mathcal{A} \text{ is a } \mathcal{T}- \text{ interpretation that satisfies } \phi\}$.[1]

The following result is a consequence of the well-known fact that the theory of uninterpreted functions with equality is convex, but can also be derived as a corollary of exercise 10.5 in Section 10 of [12].

Theorem 6 *If $\mathcal{T}$ is a theory over a one-sorted signature with only one unary function, and if $\mathcal{T}$ is axiomatized by the empty set (that is, its function is* uninterpreted*), then $\mathcal{T}$ is convex.*

Throughout this paper, we will use **SI** as a shorthand for stably infinite, **SM** for smooth, **FW** for finitely witnessable, **SW** for strongly finitely witnessable, and **CV** for convex.

3 Polite Combination without Smoothness

Polite combination of theories was introduced in [7]. There, it was claimed that in order to combine a theory $\mathcal{T}$ with any other decidable theory using polite combination, it suffices for $\mathcal{T}$ to be smooth and finitely witnessable (that is, polite). Later, in [8], this condition was corrected, and it was shown that in fact a stronger requirement is needed from $\mathcal{T}$: in order for polite combination to work, $\mathcal{T}$ has to be smooth and strongly finitely witnessable (that is, strongly polite).

Given that weakening strong finite witnessability to finite witnessability results in a condition that does not suffice, it is natural to ask whether there is any other way to weaken the required conditions for polite combination. Rather than weakening strong finite witnessability to finite witnessability, here we consider another option: weakening the smoothness condition to stable infiniteness. Thus, the main result of this section is that polite combination can be done for theories that are stably infinite and strongly finitely witnessable.

Remark 1 We prove this result explicitly, although it follows from the fact proven in [13] that stable infiniteness and strong finite witnessability imply smoothness. We do this for two

[1] Note that this definition was generalized in two different ways to the many-sorted case in [9] and [7]. However, for our investigation, the single-sorted case is enough.

main reasons: firstly, the result in [13] crucially relies on Lemma 8 below (which was stated in [1] although its proof, for lack of space, was only available in the technical report [14]); secondly, the proof below is much more direct than the one that follows from [13], and sheds light on the way these properties are utilized for polite combination.

For the proof of the new combination theorem (Theorem 9), we rely on the following theorem, which may be found, with a proof, in [8] as Theorem 2.5: it is based, according to authors of that work, on Theorems 10 and 11 of [15].

Theorem 7 *Let Σ_1 and Σ_2 be disjoint signatures, $\mathcal{T}_1$ be a Σ_1-theory and $\mathcal{T}_2$ be a Σ_2-theory. Consider $\Sigma = \Sigma_1 \cup \Sigma_2$, $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2$ and S the set of sorts shared by Σ_1 and Σ_2. Let ϕ_1 and ϕ_2 be quantifier-free, respectively, Σ_1 and Σ_2-formulas, and let $U_\sigma = vars_\sigma(\phi_1) \cap vars_\sigma(\phi_2)$.*

If there exists a $\mathcal{T}_1$-interpretation $\mathcal{A}$ and a $\mathcal{T}_2$-interpretation $\mathcal{B}$, and an arrangement δ_U on U such that $\mathcal{A}$ satisfies $\phi_1 \wedge \delta_U$, $\mathcal{B}$ satisfies $\phi_2 \wedge \delta_U$ and, for all sorts $\sigma \in S$, $|\sigma^{\mathcal{A}}| = |\sigma^{\mathcal{B}}|$, then there exists a $\mathcal{T}$-interpretation $\mathcal{C}$ such that $\mathcal{C}$ satisfies $\phi_1 \wedge \phi_2 \wedge \delta_U$, $\sigma^{\mathcal{C}} = \sigma^{\mathcal{A}}$ for all $\sigma \in \mathcal{S}_{\Sigma_1}$, and $\sigma^{\mathcal{C}} = \sigma^{\mathcal{B}}$ for all $\sigma \in \mathcal{S}_{\Sigma_2} \setminus S$.

The key ingredient of our proof is Lemma 8 below, which relaxes the need for smoothness in polite theory combination by proving that stable infiniteness and strong finite witnessability imply an apparently weaker notion of smoothness. In this notion, uncountable domains in the original structure $\mathcal{A}$ are reduced to countable ones, and the function κ, that dictates the cardinalities of models, is assumed to never assign an uncountable cardinal to any of the sorts. This notion, however, was proven in [13] to be equivalent to smoothness, although that proof still relies on Lemma 8.

Lemma 8 *Let Σ be a signature with $S \subseteq \mathcal{S}_\Sigma$, and $\mathcal{T}$ a theory over Σ. If $\mathcal{T}$ is a stably-infinite and strongly finitely witnessable theory, both w.r.t. the set of sorts S, then: for every quantifier-free Σ-formula ϕ; $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies ϕ; and function κ from $S^{\mathcal{A}}_{\aleph_0} = \{\sigma \in S : |\sigma^{\mathcal{A}}| \leq \aleph_0\}$ to the class of cardinals such that $|\sigma^{\mathcal{A}}| \leq \kappa(\sigma) \leq \aleph_0$ for every $\sigma \in S^{\mathcal{A}}_{\aleph_0}$, there exists a $\mathcal{T}$-interpretation $\mathcal{B}$ that satisfies ϕ with $|\sigma^{\mathcal{B}}| = \kappa(\sigma)$ for every $\sigma \in S^{\mathcal{A}}_{\aleph_0}$, and $|\sigma^{\mathcal{B}}| = \aleph_0$ for every $\sigma \in S \setminus S^{\mathcal{A}}_{\aleph_0}$.*

Proof Suppose that $\mathcal{T}$ is stably-infinite and strongly finitely witnessable w.r.t. a set of sorts S; let ϕ be a quantifier-free formula, $\mathcal{A}$ a $\mathcal{T}$-interpretation that satisfies ϕ, and take a function κ from $S^{\mathcal{A}}_{\aleph_0}$ to the class of cardinals such that $|\sigma^{\mathcal{A}}| \leq \kappa(\sigma) \leq \aleph_0$ for every $\sigma \in S^{\mathcal{A}}_{\aleph_0}$. For simplicity, we also define $S^{-}_{\aleph_0}$ to be the set of sorts $\sigma \in S^{\mathcal{A}}_{\aleph_0}$ such that $\kappa(\sigma) < \aleph_0$, while $S^{+}_{\aleph_0} = S \setminus S^{-}_{\aleph_0}$ will be its complement in S, that is, those sorts in $S^{\mathcal{A}}_{\aleph_0}$ with $\kappa(\sigma) = \aleph_0$, and those sorts in S with $\sigma^{\mathcal{A}} > \aleph_0$.

Suppose wit is our strong witness: since $\mathcal{A}$ satisfies ϕ, it must also satisfy $\exists \overrightarrow{x}.\, wit(\phi)$, for $\overrightarrow{x} = vars(wit(\phi)) \setminus vars(\phi)$; by changing $\mathcal{A}$ at most on these variables, we obtain a second $\mathcal{T}$-interpretation $\mathcal{A}'$ that satisfies $wit(\phi)$. For each $\sigma \in S$, let $W_\sigma = vars_\sigma(wit(\phi))$, equipped with the equivalence relations F_σ such that $x F_\sigma y$ iff $x^{\mathcal{A}'} = y^{\mathcal{A}'}$; the corresponding arrangement will be

$$\delta_W = \bigwedge_{\sigma \in S} \Big[\bigwedge_{x F_\sigma y} x = y \wedge \bigwedge_{x \overline{F_\sigma} y} \neg(x = y) \Big],$$

where $\overline{F_\sigma}$ is the complement of F_σ.

Now, take a positive integer M and consider, for every $\sigma \in S$, a set of fresh variables U_σ of sort σ with:

$$|U_\sigma| = \begin{cases} \kappa(\sigma) - |W_\sigma/F_\sigma| & \text{if } \sigma \in S^-_{\aleph_0}; \\ M & \text{if } \sigma \in S^+_{\aleph_0} \end{cases}$$

(notice that $|W_\sigma/F_\sigma| \leq |\sigma^{\mathcal{A}'}|$, by definition of F_σ, and $|\sigma^{\mathcal{A}'}| = |\sigma^{\mathcal{A}}| \leq \kappa(\sigma)$, all for each $\sigma \in S$, meaning $\kappa(\sigma) - |W_\sigma/F_\sigma|$ is always non-negative for $\sigma \in S^-_{\aleph_0}$). We also define the relation E_σ on $V_\sigma = U_\sigma \cup W_\sigma$ such that $x E_\sigma y$ iff $x F_\sigma y$ or if $x = y$, and we will denote the corresponding arrangement by δ_V.

Now, because $\mathcal{T}$ is stably-infinite w.r.t. S, and $wit(\phi) \wedge \delta_W$ is quantifier-free (and satisfied by the $\mathcal{T}$-interpretation $\mathcal{A}'$), there must exist a $\mathcal{T}$-interpretation $\mathcal{B}$, with $\sigma^{\mathcal{B}}$ infinite for every sort $\sigma \in S$, that satisfies $wit(\phi) \wedge \delta_W$; since the variables in U_σ are fresh and thus not in $wit(\phi)$, we can change the value of $\mathcal{B}$ on U_σ in a way that different variables of this set are mapped to different elements without affecting the satisfaction of $wit(\phi) \wedge \delta_W$, thus obtaining a $\mathcal{T}$-interpretation $\mathcal{B}'$ such that $\mathcal{B}'$ satisfies $wit(\phi) \wedge \delta_V$.

Since we have now established that $wit(\phi) \wedge \delta_V$ is $\mathcal{T}$-satisfiable, and this theory is strongly finitely witnessable w.r.t. S, there must exist a $\mathcal{T}$-interpretation $\mathcal{C}_M$ that satisfies $wit(\phi) \wedge \delta_V$ (and, since $\mathcal{C}_M$ satisfies $wit(\phi)$, it must satisfy $\exists \overrightarrow{x}\,.\, wit(\phi)$ and thus ϕ as well) with $\sigma^{\mathcal{C}_M} = V_\sigma^{\mathcal{C}_M}$ for every $\sigma \in S$; but, since the definition of δ_V forces V_σ/E_σ to have $\kappa(\sigma)$ equivalence classes for $\sigma \in S^-_{\aleph_0}$, and $M + |W_\sigma/F_\sigma|$ equivalence classes for $\sigma \in S^+_{\aleph_0}$, we have that $\sigma^{\mathcal{C}_M}$ has $\kappa(\sigma)$ elements for $\sigma \in S^-_{\aleph_0}$, and $M + |W_\sigma/F_\sigma|$ elements for $\sigma \in S^+_{\aleph_0}$.

From the $\mathcal{T}$-interpretations $\mathcal{C}_M$ we constructed it is clear that

$$\Gamma_M = \{\phi\} \cup \{\psi^\sigma_{=\kappa(\sigma)} : \sigma \in S^-_{\aleph_0}\} \cup \{\psi^\sigma_{\geq M} : \sigma \in S^+_{\aleph_0}\}$$

is $\mathcal{T}$-satisfiable for all M. We now state that, through Lemma 1, this implies that $\Gamma = \bigcup_{M\in\mathbb{N}} \Gamma_M$ is also $\mathcal{T}$-satisfiable, what will finish proving our theorem. Indeed, if $\mathcal{C}$ is a $\mathcal{T}$-interpretation that satisfies Γ, each $\sigma^{\mathcal{C}}$, for $\sigma \in S^-_{\aleph_0}$, has cardinality $\kappa(\sigma)$; and each $\sigma^{\mathcal{C}}$, for $\sigma \in S^+_{\aleph_0}$, will be infinite, given Γ contains the formulas $\psi^\sigma_{\geq M}$ for all $M \in \mathbb{N}$. Using Lemma 2 with the union of Γ and $Ax(\mathcal{T})$ (which is satisfied by $\mathcal{C}$, given that that is a $\mathcal{T}$-interpretation), we obtain a $\mathcal{T}$-interpretation $\mathcal{D}$ that satisfies Γ, where $\sigma^{\mathcal{D}}$ has cardinality $\aleph_0$ whenever it is infinite, that is, whenever $\sigma \in S^+_{\aleph_0}$. Since $|\sigma^{\mathcal{D}}| = \kappa(\sigma)$ for $\sigma \in S^-_{\aleph_0}$, given that $\mathcal{D}$ satisfies Γ and therefore $\psi^\sigma_{=\kappa(\sigma)}$, and $|\sigma^{\mathcal{D}}| = \aleph_0 = \kappa(\sigma)$ for $\sigma \in S^+_{\aleph_0}$, we see that $\mathcal{D}$ is the interpretation we intended to build.

Now, to see that Γ is $\mathcal{T}$-satisfiable, suppose that it is not: by Lemma 1, there must exist finite sets $Ax_0 \subseteq Ax(\mathcal{T})$, $S^-_0 \subseteq S^-_{\aleph_0}$ and $S^+_0 \subseteq S^+_{\aleph_0} \times \mathbb{N}$ such that

$$\Gamma_0 = Ax_0 \cup \{\phi\} \cup \{\psi^\sigma_{=\kappa(\sigma)} : \sigma \in S^-_0\} \cup \{\psi^\sigma_{\geq k} : (\sigma, k) \in S^+_0\}$$

is unsatisfiable. But, by taking $M = \max\{k : (\sigma, k) \in S^+_0\}$, we see that $\mathcal{C}_M$ is a $\mathcal{T}$-interpretation that satisfies Γ_M, the latter set including ϕ, $\{\psi^\sigma_{=\kappa(\sigma)} : \sigma \in S^-_{\aleph_0}\}$ and $\{\psi^\sigma_{\geq M} : \sigma \in S^+_{\aleph_0}\}$. Of course, this last set implies $\{\psi^\sigma_{\geq k} : \sigma \in S^+_{\aleph_0}, k \leq M\}$, and therefore that $\mathcal{C}_M$ satisfies Γ_0, contradicting the fact that Γ_0 is supposed to be itself contradictory.
□

We can now prove the desired combination theorem.

Theorem 9 *Let Σ_1 and Σ_2 be disjoint signatures with sorts S_1 and S_2; let $\mathcal{T}_1$ be a Σ_1-theory, $\mathcal{T}_2$ be a Σ_2-theory, and $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2$; and let ϕ_1 be a quantifier-free Σ_1-formula and ϕ_2 a quantifier-free Σ_2-formula.*

Assume that $\mathcal{T}_2$ is stably-infinite and strongly finitely witnessable w.r.t. $S = S_1 \cap S_2$, with strong witness wit. Let $\psi = wit(\phi_2)$, $V_\sigma = vars_\sigma(\psi)$ for every $\sigma \in S$, and $V = \bigcup_{\sigma \in S} vars_\sigma(\psi)$. Then the following are equivalent:

1. *$\phi_1 \wedge \phi_2$ is $\mathcal{T}$-satisfiable;*
2. *there exists an arrangement δ_V over V such that $\phi_1 \wedge \delta_V$ is $\mathcal{T}_1$-satisfiable and $\psi \wedge \delta_V$ is $\mathcal{T}_2$-satisfiable.*

Proof Start by assuming that $\phi_1 \wedge \phi_2$ is $\mathcal{T}$-satisfiable and take $\overrightarrow{x} = vars(\psi) \setminus vars(\phi_2)$. Since ϕ_2 and $\exists \overrightarrow{x}.\, \psi$ are $\mathcal{T}_2$-equivalent, we have that $\phi_1 \wedge \psi$ is also $\mathcal{T}$-satisfiable; let $\mathcal{A}$ be a $\mathcal{T}$-interpretation which satisfies that formula. For each $\sigma \in S$, take the equivalence relation E_σ on V_σ such that $x E_\sigma y$ iff $x^{\mathcal{A}} = y^{\mathcal{A}}$, and set δ_V to be the corresponding arrangement. Then $\mathcal{A}$, when restricted to the signature Σ_1, is a $\mathcal{T}_1$-interpretation that satisfies $\phi_1 \wedge \delta_V$, and when restricted to the signature Σ_2, is a $\mathcal{T}_2$-interpretation that satisfies $\psi \wedge \delta_V$.

Now, for the reciprocal: let $\mathcal{A}$ be a $\mathcal{T}_1$-interpretation satisfying $\phi_1 \wedge \delta_V$, and let $\mathcal{B}$ be a $\mathcal{T}_2$-interpretation satisfying $\psi \wedge \delta_V$. Using Lemma 2, we can build a second $\mathcal{T}_1$-interpretation $\mathcal{A}'$ that satisfies $\phi_1 \wedge \delta_V$ and where $\sigma^{\mathcal{A}'}$ is at most countable, for every sort $\sigma \in S_2$; and, since $\mathcal{T}_2$ is strongly finitely witnessable and $\psi = wit(\phi_2)$, we can obtain a $\mathcal{T}_2$-interpretation $\mathcal{B}'$ which satisfies $\psi \wedge \delta_V$ and has $\sigma^{\mathcal{B}'} = vars_\sigma(\psi \wedge \delta_V)^{\mathcal{B}'} = V_\sigma^{\mathcal{B}'}$ for every $\sigma \in S$, where $vars_\sigma(\psi \wedge \delta_V) = V_\sigma$ because δ_V is a formula where all and only variables of V occur. Therefore,

$$|\sigma^{\mathcal{B}'}| = |V_\sigma^{\mathcal{B}'}| = |V_\sigma^{\mathcal{A}'}| \leq |\sigma^{\mathcal{A}'}| \leq \aleph_0,$$

where the second equality comes from the fact that both $\mathcal{A}'$ and $\mathcal{B}'$ satisfy δ_V. Using Lemma 8 for the theory $\mathcal{T}_2$ and: the fact that $\psi \wedge \delta_V$ is a quantifier-free formula; the second fact that $\mathcal{B}'$ is a $\mathcal{T}_2$-interpretation satisfying $\psi \wedge \delta_V$; and setting $\kappa(\sigma) = |\sigma^{\mathcal{A}'}|$ for every $\sigma \in S$, as a function from $\{\sigma \in S : |\sigma^{\mathcal{B}'}| \leq \aleph_0\} = S$ to the class of cardinals such that $|\sigma^{\mathcal{B}'}| \leq \kappa(\sigma) \leq \aleph_0$, we can obtain a $\mathcal{T}_2$-interpretation $\mathcal{C}$ that satisfies $\psi \wedge \delta_V$ and where $|\sigma^{\mathcal{C}}| = |\sigma^{\mathcal{A}'}|$ for every $\sigma \in S$.

In other words, $\mathcal{A}'$ is a $\mathcal{T}_1$-interpretation that satisfies $(\phi_1 \wedge \delta_V) \wedge \delta_V$, $\mathcal{C}$ is a $\mathcal{T}_2$-interpretation satisfying $(\psi \wedge \delta_V) \wedge \delta_V$, and for all sorts $\sigma \in S$ we have $|\sigma^{\mathcal{A}'}| = |\sigma^{\mathcal{C}}|$. We duplicate δ_V when writing the formulas $(\phi_1 \wedge \delta_V) \wedge \delta_V$ and $(\psi \wedge \delta_V) \wedge \delta_V$ (equivalent, more simply, to $\phi_1 \wedge \delta_V$ and $\psi \wedge \delta_V$, respectively) as we wish to meet the conditions of Theorem 7: if we try to apply the theorem to ϕ_1 and ψ, we would need an arrangement over the set of variables U, where $U_\sigma = vars_\sigma(\phi_1) \cap vars_\sigma(\psi)$, but instead we have one over $V_\sigma = vars_\sigma(\psi)$, which may contain other variables than those of sort σ shared by ϕ_1 and ψ; however, $V_\sigma = vars_\sigma(\phi_1 \wedge \delta_V) \cap vars_\sigma(\psi \wedge \delta_V)$, and we therefore can apply the theorem verbatim to the formulas $\phi_1 \wedge \delta_V$ and $\psi \wedge \delta_V$ instead.

After applying Theorem 7 we obtain a $\mathcal{T}$-interpretation $\mathcal{D}$ which satisfies $\phi_1 \wedge \psi \wedge \delta_V$. Since $\exists \overrightarrow{x}.\, \psi$ and ϕ_2 are $\mathcal{T}_2$-equivalent, it follows that $\mathcal{D}$ satisfies $\phi_1 \wedge \phi_2$, as we wished to show. □

So, we have shown in Theorem 9 that smoothness is not needed for polite theory combination. Does that mean that it is not needed at all for theory combination in general? The answer is negative: smoothness is also present in other theory combination methods, such as the shiny one [4]. And, in that method, smoothness cannot be replaced by stable infiniteness, as was shown in [16]. For this reason, we keep smoothness in our analysis of theory combination properties, presented in the next sections.

Table 1 Summary of all possible combinations of theory properties. Red cells represent impossible combinations. In row 26: $n > 1$; in row 28: $m > 1$, $n > 1$ and $|m - n| > 1$. The definitions of the theories can be found in Tables 3, 4 and 6, and the definitions of the operators in Definitions 3, 5 and 6, while their properties are in Theorems 16, 17 and 19

					Empty		Non-empty		
SI	**SM**	**FW**	**SW**	**CV**	One-sorted	Many-sorted	One-sorted	Many-sorted	$N^{\underline{o}}$
T	T	T	T	T	$\mathcal{T}_{\geq n}$	$(\mathcal{T}_{\geq n})^2$	$(\mathcal{T}_{\geq n})_s$	$((\mathcal{T}_{\geq n})^2)_s$	1
				F	Theorem 11		$(\mathcal{T}_{\geq n})_\vee$	$((\mathcal{T}_{\geq n})^2)_\vee$	2
			F	T	Theorem 5	$\mathcal{T}_{2,3}$	$\mathcal{T}_f$	$(\mathcal{T}_f)^2$	3
				F		Theorem 11	$\mathcal{T}_f^s$	$(\mathcal{T}_{2,3})_\vee$	4
		F	T	T	Theorem 4				5
				F					6
			F	T	$\mathcal{T}_\infty$	$(\mathcal{T}_\infty)^2$	$(\mathcal{T}_\infty)_s$	$((\mathcal{T}_\infty)^2)_s$	7
				F	Theorem 11		$(\mathcal{T}_\infty)_\vee$	$((\mathcal{T}_\infty)^2)_\vee$	8
	F	T	T	T	[13]				9
				F					10
			F	T	$\mathcal{T}^\infty_{even}$	$(\mathcal{T}^\infty_{even})^2$	$(\mathcal{T}^\infty_{even})_s$	$((\mathcal{T}^\infty_{even})^2)_s$	11
				F	Theorem 11		$(\mathcal{T}^\infty_{even})_\vee$	$((\mathcal{T}^\infty_{even})^2)_\vee$	12
		F	T	T	Theorem 4				13
				F					14
			F	T	$\mathcal{T}_{n,\infty}$	$(\mathcal{T}_{n,\infty})^2$	$(\mathcal{T}_{n,\infty})_s$	$((\mathcal{T}_{n,\infty})^2)_s$	15
				F	Theorem 11		$(\mathcal{T}_{n,\infty})_\vee$	$((\mathcal{T}_{n,\infty})^2)_\vee$	16
F	T	T	T	T	Theorem 3				17
				F					18
			F	T					19
				F					20
		F	T	T	Theorems 3 and 4				21
				F					22
			F	T	Theorem 3				23
				F					24
	F	T	T	T	$\mathcal{T}_{\leq 1}$	$(\mathcal{T}_{\leq 1})^2$	$(\mathcal{T}_{\leq 1})_s$	$((\mathcal{T}_{\leq 1})^2)_s$	25
				F	$\mathcal{T}_{\leq n}$	$(\mathcal{T}_{\leq n})^2$	$(\mathcal{T}_{\leq n})_s$	$((\mathcal{T}_{\leq n})^2)_s$	26
			F	T	Theorem 15	$\mathcal{T}_1^{odd}$	$\mathcal{T}^{\neq}_{odd}$	$(\mathcal{T}_1^{odd})_s$	27
				F	$\mathcal{T}_{\langle m,n\rangle}$	$(\mathcal{T}_{\langle m,n\rangle})^2$	$(\mathcal{T}_{\langle m,n\rangle})_s$	$((\mathcal{T}_{\langle m,n\rangle})^2)_s$	28
		F	T	T	Theorem 4				29
				F					30
			F	T	Theorem 14	$\mathcal{T}_1^\infty$	$\mathcal{T}^{\neq}_{1,\infty}$	$(\mathcal{T}_1^\infty)_s$	31
				F		$\mathcal{T}_2^\infty$	$\mathcal{T}^{\neq}_{2,\infty}$	$(\mathcal{T}_2^\infty)_s$	32

4 A table of examples and theorems

We now set out to accomplish the second main result of our paper: a taxonomic analysis of all combinations of stable-infiniteness, smoothness, finite witnessability, strong finite witnessability, and convexity, summarized in Table 1. Section 4.2 presents a menagerie of examples that will populate the table, while Appendices A and B will prove them correct.

If it were possible, we would present examples of every combination of the studied properties using only the one-sorted empty signature, which is the simplest signature imaginable, skipping directly to the aforementioned Section 4.2. Of course, this is not always feasible: smooth theories are necessarily stably infinite, and strongly finitely witnessable theories are necessarily finitely witnessable. But there are several other connections we proceed to show in Section 4.1, which further restrict the combinations of properties that are possible.

The results are summarized in Table 1. Each row corresponds to a possible combination of properties, as determined by the truth values in the first five columns. For example, in the first row, the entries in the first five columns are all true, indicating that in this row, all theory examples must be stably-infinite, smooth, finitely witnessable, strongly finitely witnessable, and convex. The rest of the columns correspond to different possibilities for the theory signatures: either empty or non-empty, and either one-sorted or many-sorted. Again, looking at the first row, we see four different theories listed, one for each of the signature possibilities.

Some entries in the table list theorems instead of providing example theories. The listed theorems tell us that there do not exist any example theories for these entries. For example, lines 3 and 4 cannot provide examples over a one-sorted empty signature because of Theorem 5. Some other cells, specifically all of those in rows 9 and 10, are marked with a [13], a paper that proves *unicorn theories* do not exist: these are stably infinite and strongly finitely witnessable theories that are not smooth which were previously conjectured to not exist, explaining our choice of nomenclature. Unicorn theories are also related to the result of Section 3; see Remark 1 above for details.

Definition 1 A theory $\mathcal{T}$ is said to be a unicorn theory *w.r.t.* a set of sorts $S \subseteq \mathcal{S}$ if it is stably infinite and strongly finitely witnessable without being smooth, all *w.r.t.* to S.

When an example is available, its name is given in the corresponding cell of the table. The theories themselves are defined in Sections 4.2.1 to 4.2.4.

4.1 Negative Results

We start with the negative results, saying a combination of properties is not possible. In Section 4.1.1, we show that, under reasonable conditions, a convex theory must be stably infinite, while the reciprocal is also true over empty signatures. In Section 4.1.2 we show some further results, such as the fact that, over the empty one-sorted signature, theories that are not stably infinite are necessarily finitely witnessable (a somewhat counter-intuitive result, since we usually look for theories that are, simultaneously, smooth and strongly finitely witnessable).

4.1.1 Stable-Infiniteness and Convexity

Convexity is typically defined over one-sorted signatures. Here we offer the following generalization to arbitrary signatures.

Definition 2 A theory $\mathcal{T}$ is said to be convex *w.r.t.* a set of sorts $S \subseteq \mathcal{S}_\Sigma$ if, for any conjunction of literals ϕ and any finite set of variables $\{u_1, v_1, \ldots, u_n, v_n\}$ with sorts in S, if $\vdash_\mathcal{T} \phi \rightarrow \bigvee_{i=1}^{n} u_i = v_i$ then $\vdash_\mathcal{T} \phi \rightarrow u_i = v_i$, for some $i \in [1, n]$.

If we assume, as it is often possible to do without losing crucial information, that our theories have no models with a domain of cardinality 1, then convexity implies stable infiniteness. This is true for the one-sorted case, as proved in [17], but also for the many-sorted case as we show here. The proof is similar, though here we need to account for several sorts at once. In particular, the proof relies on Lemma 1.

Theorem 10 *If a Σ-theory $\mathcal{T}$ is convex w.r.t. some set S of sorts and, for each $\sigma \in S$, $\vdash_\mathcal{T} \psi_{\geq 2}^{\sigma}$, then $\mathcal{T}$ is stably infinite w.r.t. S.*

Proof Suppose $\mathcal{T}$ is not stably-infinite *w.r.t.* S: then there exists a quantifier-free formula ϕ' that is $\mathcal{T}$-satisfiable, but every $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies ϕ' has $\sigma^\mathcal{A}$ finite for some $\sigma \in S$. Since every quantifier-free formula is equivalent to a disjunction of conjunctions of literals called its *Disjunctive Normal Form* (DNF), being a conjunction of literals called a *cube*, we state that a cube ϕ in the DNF of ϕ' also has this property: suppose that this is not actually true, and so every $\mathcal{T}$-satisfiable cube in the DNF of ϕ' is satisfied by a $\mathcal{T}$-interpretation $\mathcal{A}$ with $\sigma^{\mathcal{A}}$ infinite for every $\sigma \in S$; since ϕ', being equivalent to a disjunction of these cubes, must also be satisfied by each of these $\mathcal{T}$-interpretations $\mathcal{A}$, we reach a contradiction. So, to summarize, there is a conjunction of literals ϕ that is $\mathcal{T}$-satisfiable, but every $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies ϕ must have $\min\{|\sigma^{\mathcal{A}}| : \sigma \in S\}$ finite.

By taking $\overrightarrow{x} = vars(\phi)$, we have that any $\mathcal{T}'$-interpretations $\mathcal{B}$, for $\mathcal{T}'$ the theory with axiomatization $Ax' = Ax(\mathcal{T}) \cup \{\exists \overrightarrow{x}.\, \phi\}$, must have $\min\{|\sigma^{\mathcal{B}}| : \sigma \in S\}$ finite. Therefore, the set

$$Ax' \cup \{\psi^{\sigma}_{\geq k} : \sigma \in S, k \in \mathbb{N}\}$$

is unsatisfiable, since the latter set of formulas states that the domain of sort σ of an interpretation is infinite, for each $\sigma \in S$. By Lemma 1, there must exist some K_σ, for each $\sigma \in S$, and a finite subset of Ax' that, together with the set $\{\psi^{\sigma}_{\geq K_\sigma} : \sigma \in S\}$, is unsatisfiable (notice this is the case since the formula $\psi^{\sigma}_{\geq k'} \rightarrow \psi^{\sigma}_{\geq k}$ is satisfied if $k' \geq k$, being therefore enough to take K_σ as the maximum of these indices). Of course, this means $Ax' \cup \{\psi^{\sigma}_{\geq K_\sigma} : \sigma \in S\}$ is unsatisfiable, and so if $\mathcal{B}$ is a $\mathcal{T}'$-interpretation with $|\tau^{\mathcal{B}}| \geq K_\tau$ for every sort $\tau \in S \setminus \{\sigma\}$, then $|\sigma^{\mathcal{B}}| < K_\sigma$.

Let $z_{\sigma,1}$ through z_{σ,K_σ} be fresh variables of sort σ for every $\sigma \in S$: we wish now to show that

$$\vdash_{\mathcal{T}} \phi \rightarrow \bigvee_{\sigma \in S} \bigvee_{1 \leq i < j \leq K_\sigma} z_{\sigma,i} = z_{\sigma,j}$$

but $\nvdash_{\mathcal{T}} \phi \rightarrow z_{\sigma,i} = z_{\sigma,j}$ for any $\sigma \in S$ and $1 \leq i < j \leq K_\sigma$, contradicting the fact that $\mathcal{T}$ is supposed to be convex, and thus allowing us to reach the conclusion that $\mathcal{T}$ is stably-infinite.

So, suppose $\mathcal{C}$ is a $\mathcal{T}$-interpretation that satisfies ϕ (and so is a $\mathcal{T}'$-interpretation), and we show that it must also satisfy $\bigvee_{\sigma \in S} \bigvee_{1 \leq i < j \leq K_\sigma} z_{\sigma,i} = z_{\sigma,j}$: suppose that $\mathcal{C}$ is able to not satisfy $\bigvee_{1 \leq i < j \leq K_\tau} z_{\tau,i} = z_{\tau,j}$ for all $\tau \in S \setminus \{\sigma\}$, and so has at least K_τ elements of each of these sorts. From our analysis of $\mathcal{T}'$, this means that $\sigma^{\mathcal{C}}$ must have less than K_σ elements, and by the pigeonhole principle $\mathcal{C}$ satisfies $\bigvee_{1 \leq i < j \leq K_\sigma} z_{\sigma,i} = z_{\sigma,j}$, and thus $\bigvee_{\sigma \in S} \bigvee_{1 \leq i < j \leq K_\sigma} z_{\sigma,i} = z_{\sigma,j}$ as we had previously stated.

But, by one of the hypothesis of the lemma, any $\mathcal{T}$-interpretation has more than one element in the domain of sort σ, for each $\sigma \in S$; so, if $\mathcal{C}$ in addition satisfies ϕ, for each $\sigma \in S$ and pair of elements $1 \leq i < j \leq K_\sigma$, given that $z_{\sigma,i}$ and $z_{\sigma,j}$ are variables that do not occur in ϕ, we can construct a $\mathcal{T}$-interpretation $\mathcal{C}_{\sigma,i,j}$ that only differs from $\mathcal{C}$ in the values given to $z_{\sigma,i}$ and $z_{\sigma,j}$, where we have instead $z_{\sigma,i}^{\mathcal{C}_{\sigma,i,j}} \neq z_{\sigma,j}^{\mathcal{C}_{\sigma,i,j}}$. Since $\mathcal{C}$ and $\mathcal{C}_{\sigma,i,j}$ agree on the values given to the variables in ϕ, and ϕ is satisfied by $\mathcal{C}$, we have that $\mathcal{C}_{\sigma,i,j}$ satisfies ϕ but not $z_{\sigma,i} = z_{\sigma,j}$, meaning

$$\nvdash_{\mathcal{T}} \phi \rightarrow z_{\sigma,i} = z_{\sigma,j}$$

for each $\sigma \in S$ and $1 \leq i < j \leq K_\sigma$, thus finishing the proof. □

Reciprocally, we may also obtain convexity from stable infiniteness, but only over empty signatures.

Theorem 11 *Any theory over an empty signature that is stably infinite w.r.t. the set of all of its sorts is convex w.r.t. any set of sorts.*

Proof Fix a theory $\mathcal{T}$ over the empty signature that is stably-infinite w.r.t. the set of all of its sorts S. Suppose ϕ is a conjunction of literals, and $u_1, v_1, \ldots, u_n, v_n$ are variables with sorts in some subset of S (say, u_i and v_i both have sort σ_i), such that $\vdash_{\mathcal{T}} \phi \rightarrow \bigvee_{i=1}^{n} u_i = v_i$.

Now, it is easy to see we cannot have $\vdash_{\mathcal{T}} \phi \rightarrow (u_i = v_i)$ for u_i or v_i not in ϕ: in fact, take a $\mathcal{T}$-interpretation $\mathcal{A}$ which satisfies ϕ, and since $\mathcal{T}$ is stably-infinite, there exists an infinite $\mathcal{T}$-interpretation $\mathcal{A}'$ which also satisfies ϕ; by changing its value on u_i, respectively v_i, to a value different from $v_i^{\mathcal{A}'}$, respectively $u_i^{\mathcal{A}'}$, we obtain a third $\mathcal{T}$-interpretation $\mathcal{A}''$ that still satisfies ϕ but not $u_i = v_i$. So, we may restrict ourselves to the pairs $(x_1, y_1), \ldots, (x_m, y_m)$ among $\{(u_1, v_1), \ldots, (u_n, v_n)\}$ where both x_i and y_i are variables in ϕ, while still having $\vdash_{\mathcal{T}} \phi \rightarrow \bigvee_{i=1}^{m} x_i = y_i$; we will keep on denoting the sort of x_j and y_j by σ_j, for simplicity.

Now, assuming for contradiction that $\nvdash_{\mathcal{T}} \phi \rightarrow (x_i = y_i)$ for all $1 \leq i \leq m$, there are $\mathcal{T}$-interpretations $\mathcal{B}_i$ which satisfy ϕ but not $x_i = y_i$; without loss of generality, by the stable-infiniteness of $\mathcal{T}$, we may assume that all $\mathcal{B}_i$ are infinite on all their sorts, and by Lemma 2 we may assume as well that $\mathcal{B}_1$ through $\mathcal{B}_m$ have the same countable domain (say $\mathbb{N}$) for all sorts. That means, since we have no function or predicate symbols, that all interpretations $\mathcal{B}_i$ are over the same model $\mathcal{B}$ of $\mathcal{T}$ that is countable in each domain of sort in S.

Now, for any $\mathcal{T}$-interpretation $\mathcal{B}_\lambda$ on $\mathcal{B}$, we define an equivalence relation E_λ^σ on the variables of ϕ of sort σ by making $x E_\lambda^\sigma y$ iff $x^{\mathcal{B}_\lambda} = y^{\mathcal{B}_\lambda}$; we also define an equivalence relation $x E^\sigma y$ iff $x E_\lambda^\sigma y$ for all $\mathcal{T}$-interpretations $\mathcal{B}_\lambda$ that satisfy ϕ. Because of the interpretations $\mathcal{B}_i$, we have that $x_i \overline{E^{\sigma_i}} y_i$ for each i, where $\overline{E^{\sigma_i}}$ is the complement of the relation E^{σ_i}. Now, we state that it is possible to define an interpretation $\mathcal{B}'$ on $\mathcal{B}$ such that $x E^\sigma y$ if, and only if, $x^{\mathcal{B}'} = y^{\mathcal{B}'}$, for variables x and y in ϕ: in addition, $\mathcal{B}'$ is a $\mathcal{T}$-interpretation that satisfies ϕ, while not satisfying $\bigvee_{i=1}^{m} x_i = y_i$, what leads to a contradiction.

It is rather easy to define $\mathcal{B}'$: it only needs to map all variables in an equivalence class of E^σ to the same element, while mapping variables in a different equivalence class to a different element; this is clearly possible since $\mathcal{B}$ has countably many elements in each domain. Of course, then $x^{\mathcal{B}'} = y^{\mathcal{B}'}$ iff $x E^\sigma y$. Furthermore, since we are over the empty signature, ϕ is a conjunction of equalities and disequalities: if x and y are of sort σ and $x = y$ (respectively $\neg(x = y)$) is one of the literals of ϕ, then for any $\mathcal{T}$-interpretation $\mathcal{B}_\lambda$ that satisfies ϕ, we must have that it also satisfies $x = y$ (respectively $\neg(x = y)$), and therefore $x E^\sigma y$ (respectively $x \overline{E^\sigma} y$); this means $x^{\mathcal{B}'} = y^{\mathcal{B}'}$ ($x^{\mathcal{B}'} \neq y^{\mathcal{B}'}$), and so $\mathcal{B}'$ indeed satisfies ϕ.

Finally, one lands at a contradiction: since $\mathcal{B}'$ does not satisfy $x_i = y_i$ for any $1 \leq i \leq m$, it cannot possibly satisfy $\bigvee_{i=1}^{m} x_i = y_i$, although it does satisfy ϕ. The conclusion must be that, for some i between 1 and m, $\vdash_{\mathcal{T}} \phi \rightarrow x_i = y_i$. □

As we shall see in Section 4.2, this result is tight: there are theories over non-empty signatures that are stably infinite but not convex.

Notice that the examples on rows 25, 27 and 31 in Table 1 must have at least one structure with a domain of cardinality 1 because of Theorem 10.

4.1.2 More Connections

We next present more connections between the properties. First, over the one-sorted empty signature, a theory must be either stably infinite or finitely witnessable. To prove this, we first need the following two lemmas.

Lemma 12 *If Σ is an empty signature, $S \subseteq \mathcal{S}_\Sigma$, and $\mathcal{T}$ is a Σ-theory with a model $\mathcal{A}$ where all domains of sort in S are infinite, then $\mathcal{T}$ is stably-infinite w.r.t. S.*

Proof Let ϕ be a quantifier-free Σ-formula and $\mathcal{B}$ a $\mathcal{T}$-interpretation that satisfies ϕ. For each $\sigma \in S$, $vars_\sigma(\phi)$ is finite, and so is $vars_\sigma(\phi)^{\mathcal{B}}$, meaning there is a subset $C(\sigma)$ of $\sigma^{\mathcal{A}}$ with the same cardinality as $vars_\sigma(\phi)^{\mathcal{B}}$; let $h_\sigma : vars_\sigma(\phi)^{\mathcal{B}} \to C(\sigma)$ be bijections for each $\sigma \in S$.

We define an interpretation $\mathcal{A}'$ on $\mathcal{A}$ such that, for every $x \in vars_\sigma(\phi)$, $x^{\mathcal{A}'} = h_\sigma(x^{\mathcal{B}})$, and for every variable x not in ϕ we may define $x^{\mathcal{A}'}$ arbitrarily. Now, let $x = y$ be an atomic subformula of ϕ, with x and y of sort σ: since h_σ is a bijection, $x^{\mathcal{A}'} = y^{\mathcal{A}'}$ iff $x^{\mathcal{B}} = y^{\mathcal{B}}$; since all atomic subformulas of ϕ receive precisely the same truth-value in either $\mathcal{A}'$ or $\mathcal{B}$, and a quantifier-free formula's truth value is entirely determined by the truth-values of its atomic subformulas, we get that $\mathcal{A}'$ satisfies ϕ, and so $\mathcal{T}$ is stably-infinite w.r.t. S. □

Lemma 13 *If a one-sorted theory over the empty signature is not stably-infinite, it has a model of maximum finite cardinality.*

Proof Let $\mathcal{T}$ be a one-sorted, not stably-infinite theory over the empty signature; $\mathcal{T}$ cannot have infinite models because of Lemma 12; and it has finite models since, otherwise, it would be vacuously stably-infinite. So, suppose that $\mathcal{T}$ has models of arbitrarily finite size: since $\Gamma = \{\psi_{\geq n} : n \in \mathbb{N}\}$ is only satisfied by structures with an infinite domain, we have that $Ax(\mathcal{T}) \cup \Gamma$ is unsatisfiable. By Lemma 1, there must exist finite subsets $Ax_0 \subseteq Ax(\mathcal{T})$ and $\Gamma_0 \subseteq \Gamma$ such that $Ax_0 \cup \Gamma_0$ is unsatisfiable. Let N be the largest index of a formula $\psi_{\geq n}$ showing up in Γ_0, and we can derive from the fact that $\psi_{\geq j}$ implies $\psi_{\geq i}$ for $j > i$ that $Ax_0 \cup \{\psi_{\geq N}\}$ is unsatisfiable, meaning that there are no structures that satisfy Ax_0 with more than N elements in their domains. But models of $Ax(\mathcal{T})$ are also models of Ax_0, what implies that $Ax(\mathcal{T})$ has no models with more than N elements, contradicting our hypothesis. □

With the above lemmas in place, we are able now to prove the theorem.

Theorem 14 *Every one-sorted, non-stably-infinite theory $\mathcal{T}$ with an empty signature is finitely witnessable w.r.t. its only sort.*

Proof By Lemma 13, a one-sorted, not stably-infinite theory over the empty signature must have a model of maximum size. Let M be that finite cardinality; we can then prove that, for a quantifier-free formula ϕ and fresh variables x_1 through x_M, $wit(\phi) = \phi \wedge \bigwedge_{i=1}^{M} x_i = x_i$ is a witness. We begin by noticing that $\exists\, \overrightarrow{x}\,.\, wit(\phi)$ and ϕ are $\mathcal{T}$-equivalent, where $\overrightarrow{x} = vars(wit(\phi)) \setminus vars(\phi)$, since ϕ and $wit(\phi)$ are themselves equivalent, being $wit(\phi)$ the conjunction of ϕ and a tautology. Now, assume that the $\mathcal{T}$-interpretation $\mathcal{A}$ satisfies $wit(\phi)$: since the maximum size of a model of $\mathcal{T}$ is M, we have $|\sigma_1^{\mathcal{A}}| \leq M$. We define another $\mathcal{T}$-interpretation $\mathcal{A}'$ by changing the value of $\mathcal{A}$ only on the variables of $\overrightarrow{x}$ so that $x_i \mapsto x_i^{\mathcal{A}'}$ is surjective. $wit(\phi)$ remains valid in $\mathcal{A}'$ and, in addition, $vars(wit(\phi))^{\mathcal{A}'} = \sigma_1^{\mathcal{A}'}$. □

Finally, by combining previous results, we can also get the following theorem, which relates stable infiniteness, strong finite witnessability, and convexity.

Theorem 15 *A one-sorted theory $\mathcal{T}$ with an empty signature that is neither strongly finitely witnessable nor stably infinite w.r.t. its only sort cannot be convex.*

Proof Assume that $\mathcal{T}$ is one-sorted and convex without being strongly finitely witnessable nor being stably-infinite, over the empty signature. By Lemma 13, we know that $\mathcal{T}$ must have a model of maximum finite cardinality, say M. We can now state that $M > 1$, meaning the maximum model of $\mathcal{T}$ has more than one element, since otherwise $\mathcal{T}$ would consist only of the model with a single element, and one can easily prove in that case that $wit(\phi) = \phi$ is a

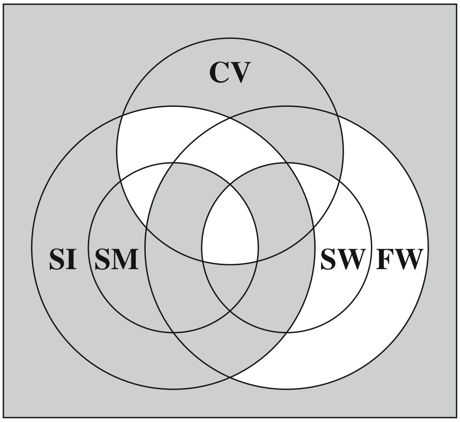

Fig. 2 A diagram of combinations over a one-sorted, empty signature: gray regions are empty [1]

strong witness: if $\phi \wedge \delta_V$ is $\mathcal{T}$-satisfiable, it is satisfied in the $\mathcal{T}$-interpretation $\mathcal{A}$ with only one element in its domain, and of course $vars(\phi \wedge \delta_V)^{\mathcal{A}} = \sigma_1^{\mathcal{A}}$.

But now we can reach a contradiction: suppose ϕ is a conjunction of literals that is tautological, such as $x = x$; we have that $\vdash_{\mathcal{T}} \phi \rightarrow \bigvee_{i=1}^{M} \bigvee_{j=i+1}^{M+1} y_i = y_j$ since, by the pigeonhole principle, the disjunction on the right of the implication must always be true in $\mathcal{T}$-interpretations. But we do not have $\vdash_{\mathcal{T}} \phi \rightarrow y_i = y_j$ for any $1 \leq i < j \leq M+1$, since ϕ is a tautology and there are $\mathcal{T}$-interpretations on a structure with $M > 1$ elements. Therefore $\mathcal{T}$ is not convex, against our hypothesis that it in fact was. □

To summarize, while Theorem 10 is restricted to structures with no domains of cardinality 1, the remaining theorems of this section are not restricted to such structures. Theorem 11 applies to empty signatures, and Theorems 14 and 15 apply to signatures that are both empty and one-sorted. Put together, we see that many combinations of properties for theories over a one-sorted empty signature are actually impossible. This is depicted in Fig. 2, in which all areas but the white ones are empty. For example, Theorem 14 shows that the area outside the SI and FW circles (representing theories that are neither stably infinite nor finitely witnessable) is empty, as every theory (over an empty one-sorted signature) must have one of these properties. Similarly, Theorem 15 further shows that within the CV (convex) circle, even more is empty, namely anything outside the SI and SW circles.

4.2 Positive Results

This section completes our analysis, providing the examples fpr when a combination of properties is possible. Before defining all the theories of Table 1, we introduce some signatures that we neatly summarize in Table 2: Σ_1 and Σ_s have one sort σ_1, while Σ_2 and $\Sigma_{2,s}$ have two, namely σ_1 and σ_2; both Σ_1 and Σ_2 are empty, while Σ_s and $\Sigma_{2,s}$ possess a single function symbol of arity $\sigma_1 \rightarrow \sigma_1$; none of these contain predicates.

Remark 2 For non-empty signatures, we chose to include functions rather than predicates. This is not essential as we can replace function symbols by predicate symbols: we do this by including the sort of the codomain of the function as the last component of the arity of the predicate, and then adding an axiom stating the predicate behaves as a function.

We now describe the theories: Section 4.2.1 defines the theories that are over the empty one-sorted signature Σ_1; Section 4.2.2 then continues to the next column of Table 1, describ-

Table 2 Signatures to be used below

Signature	Sorts	Function Symbols	Predicate Symbols
Σ_1	$\{\sigma_1\}$	$\emptyset$	$\emptyset$
Σ_2	$\{\sigma_1, \sigma_2\}$	$\emptyset$	$\emptyset$
Σ_s	$\{\sigma_1\}$	$\{s : \sigma_1 \to \sigma_1\}$	$\emptyset$
$\Sigma_{2,s}$	$\{\sigma_1, \sigma_2\}$	$\{s : \sigma_1 \to \sigma_1\}$	$\emptyset$

Table 3 Σ_1-theories

Name	Axiomatization
$\mathcal{T}_{\geq n}$	$\{\psi_{\geq n}\}$
$\mathcal{T}_{\infty}$	$\{\psi_{\geq k} : k \in \mathbb{N}\}$
$\mathcal{T}^{\infty}_{even}$	$\{\neg\psi_{=2k+1} : k \in \mathbb{N}\}$
$\mathcal{T}_{n,\infty}$	$\{\psi_{=n} \vee \psi_{\geq k} : k \in \mathbb{N}\}$
$\mathcal{T}_{\leq n}$	$\{\psi_{\leq n}\}$
$\mathcal{T}_{\langle m,n\rangle}$	$\{\psi_{=m} \vee \psi_{=n}\}$

Table 4 Σ_2-theories

Name	Axiomatization
$\mathcal{T}_{2,3}$	$\{(\psi^{\sigma_1}_{=2} \wedge \psi^{\sigma_2}_{\geq k}) \vee (\psi^{\sigma_1}_{\geq 3} \wedge \psi^{\sigma_2}_{\geq 3}) : k \in \mathbb{N}\}$
$\mathcal{T}^{odd}_1$	$\{\psi^{\sigma_1}_{=1}\} \cup \{\neg\psi^{\sigma_2}_{=2k} : k \in \mathbb{N}\}$
$\mathcal{T}^{\infty}_1$	$\{\psi^{\sigma_1}_{=1}\} \cup \{\psi^{\sigma_2}_{\geq k} : k \in \mathbb{N}\}$
$\mathcal{T}^{\infty}_2$	$\{\psi^{\sigma_1}_{=2}\} \cup \{\psi^{\sigma_2}_{\geq k} : k \in \mathbb{N}\}$

ing theories over the many-sorted empty signature Σ_2. Some build on the theories of the previous column, but some do not. Section 4.2.3 describes the next column of Table 1, presenting theories over the one-sorted non-empty signature Σ_s. Here, we use two constructions to generate new theories from previously introduced ones. One construction adds a function symbol to an empty signature (in a way that preserves all properties), and the second preserves all properties but convexity, making it possible to construct non-convex examples in a uniform way. We also present new theories when the constructions are not sufficient. Section 4.2.4 describes theories over the non-empty many-sorted signature $\Sigma_{2,s}$. One can find in Appendix A the proofs that the defined theories indeed have the properties we wish them to have.

4.2.1 Theories over the One-Sorted Empty Signature

The axiomatizations for theories over the one-sorted empty signature Σ_1 are given in Table 3. We briefly describe them here.

For each $n > 0$, $\mathcal{T}_{\geq n}$ includes all structures with domains of cardinality at least n, and only those; $\mathcal{T}_{\infty}$ is the theory composed of all structures whose domains are infinite; $\mathcal{T}^{\infty}_{even}$ has structures with either an even or an infinite number of elements in their domains (and only those), and was defined in [10], where it was proved to be finitely witnessable, but neither smooth nor strongly finitely witnessable. The proofs justifying Table 1 show additionally that it is stably infinite and convex. $\mathcal{T}_{n,\infty}$ contains those structures whose domains have either exactly n or an infinite number of elements, and no others; $\mathcal{T}_{\leq n}$ is composed of all structures with at most n elements in their domains; and for positive integers m and n, $\mathcal{T}_{\langle m,n\rangle}$ has structures whose domains have either precisely m elements, or precisely n elements, and no others. This completes the first column of theory examples.

Example 1 The theory $\mathcal{T}_{\geq n}$ admits all considered properties (this is proved in Lemmas 22 and 23 of Appendix A.1.1), while $\mathcal{T}_{\langle m,n\rangle}$ admits only finite witnessability (this is proved in Lemmas 40 to 43 of Appendix A.1.7).

4.2.2 Theories over the Two-Sorted Empty Signature

We next introduce the theories over empty two-sorted signatures. For many cases, we can simply add a trivial sort to one of the theories defined in Section 4.2.1. When this is not possible, we introduce new theories.

4.2.2.1 Adding a Sort to a Theory

Any Σ_1-theory can be used to generate a Σ_2-theory simply by adding the sort σ_2 to the signature (without changing the axiomatization). But the same works for Σ_s-theories too, leading to $\Sigma_{2,s}$-theories, which will be helpful for the last column of the table (see Table 2 for the definition of these signatures). All of this can be formalized as follows:

Definition 3 Let $\mathcal{T}$ be a Σ_1 or Σ_s-theory. $(\mathcal{T})^2$ is the Σ_2-theory, respectively $\Sigma_{2,s}$, axiomatized by $Ax(\mathcal{T})$.

Theorem 16 then shows that adding a sort to a theory in this way preserves all properties that we study. The proof of this result is formulaic and tedious, but can still be found in Appendix B.1. Notice that both directions of the theorem are equally important: to see why, suppose $\mathcal{T}$ is, say, stably infinite; from Theorem 16 we can derive that so is $(\mathcal{T})^2$. If, however, $\mathcal{T}$ is not stably infinite we can also conclude $(\mathcal{T})^2$ is not stably infinite: were that not the case we would be able to prove $\mathcal{T}$ is stably infinite, a contradiction, by again using Theorem 16.

Theorem 16 *A Σ_1 or Σ_s-theory $\mathcal{T}$ is stably infinite, smooth, finitely witnessable, strongly finitely witnessable, or convex w.r.t. $\{\sigma_1\}$ if and only if $(\mathcal{T})^2$ is, respectively, stably infinite, smooth, finitely witnessable, strongly finitely witnessable, or convex w.r.t. $\{\sigma_1, \sigma_2\}$.*

Using Definition 3 and Theorem 16, we can populate many rows in the second column of Table 1 by extending the corresponding theory from the previous column.

Example 2 $(\mathcal{T}_{\infty})^2$ is a theory over two sorts, σ_1 and σ_2, whose structures must have infinitely many elements in the domain of σ_1 (but have no restrictions on the size of the domain of σ_2). As seen in Table 1, $\mathcal{T}_{\geq n}$ admits all the considered properties but (strong) finite witnessability. By Theorem 16, so does $(\mathcal{T}_{\geq n})^2$.

4.2.2.2 Additional Theories over Σ_2

On some rows of Table 1 in page 10, e.g., row 3, there is no Σ_1-theory to extend; meaning, we have proved that no Σ_1-theory can have the specific combination of properties found in that particular row. In such cases, we cannot use Definition 3 to construct a many-sorted variant.

We introduce the theories shown in Table 4 to cover these cases. The theory $\mathcal{T}_{2,3}$ contains two kinds of structures, and only those: (i) structures whose domains both have at least 3 elements; and (ii) structures with exactly two elements in the domain of σ_1, and an infinite number of elements in the domain of σ_2. The theory $\mathcal{T}_1^{odd}$ is composed of structures with

Table 5 Some generalized signatures

Signature	Sorts	Function Symbols	Predicate Symbols
Σ_n	$\{\sigma_1, \ldots, \sigma_n\}$	$\emptyset$	$\emptyset$
$\Sigma_{n,s}$	$\{\sigma_1, \ldots, \sigma_n\}$	$\{s : \sigma_1 \to \sigma_1\}$	$\emptyset$

exactly one element in the domain of σ_1, and either an odd or an infinite number of elements in the domain of σ_2. The theory $\mathcal{T}_1^\infty$ is similar: its structures have exactly one element in the domain of σ_1, and an infinite number of elements in the domain of σ_2. Finally, $\mathcal{T}_2^\infty$ is also similar, this time to $\mathcal{T}_1^\infty$, except that its structures have exactly 2 elements in the domain of σ_1.

Example 3 The theory $\mathcal{T}_{2,3}$ was first defined in [9] and later used in [10], where it was proved to be polite (and therefore smooth, stably infinite, and finitely witnessable) without being strongly polite (and therefore not strongly finitely witnessable). The justification proofs for Table 1 (see Appendix A) show that $\mathcal{T}_{2,3}$ is convex as well.[2]

4.2.3 Theories over a One-Sorted Non-Empty Signature

We continue to the next column, with one-sorted non-empty signatures. Paragraph 4.2.3.1 shows how to construct non-empty theories from one-sorted theories over the empty signature, while preserving all their properties. In Paragraph 4.2.3.2, we provide a similar construction which generates non-convex theories from the theories in the first column of Table 1. And in Paragraph 4.2.3.3, we introduce additional theories not captured by the above constructions. Two of these theories are described in more detail in Paragraph 4.2.3.4..

4.2.3.1 Extending a Theory with a Function while Preserving Properties

Whenever we have a theory over an empty signature, we can construct a variant of it over a non-empty signature by introducing a function symbol and interpreting it as the identity function. This extension then preserves all the properties that we consider. Although we use this result for both one and two-sorted empty signatures, the use on the two-sorted case could be circumvented: the reader can check that, any time this operator is used on a two-sorted theory in Table 1, there is also a theory on a one-sorted, non-empty signature where we could have applied the operator from Paragraph 4.2.2.1 instead. We still choose to define the operator for any finite number of sorts for the simple reason that the result is still valid in this more general case, and its proof is exactly the same regardless of the number of sorts. We now proceed with the formalization of the operator.

Definition 4 We define two families of signatures, generalizing respectively Σ_1 and Σ_2, and Σ_s and $\Sigma_{2,s}$: for any $n > 0$, Σ_n is the empty signature with sorts $S = \{\sigma_1, \ldots, \sigma_n\}$; meanwhile $\Sigma_{n,s}$ is the signature with S as set of sorts, and a single unary function symbol s of arity $\sigma_1 \to \sigma_1$. These signatures are summarized in Table 5.

Definition 5 Given a Σ_n-theory $\mathcal{T}$, $(\mathcal{T})_s$ is the $\Sigma_{n,s}$-theory axiomatized by $Ax(\mathcal{T}) \cup \{\forall x. [s(x) = x]\}$, where x is a variable of sort σ_1.

[2] We thank Oded Padon for raising the question of whether there exists a theory that is polite and convex, but not strongly polite.

$$\psi_{\vee} = \forall\, x.\ \left[\left(s^2(x) = x\right) \vee \left(s^2(x) = s(x)\right)\right]$$

Fig. 3 The formula $\psi_{\vee}$ for non-convex theories

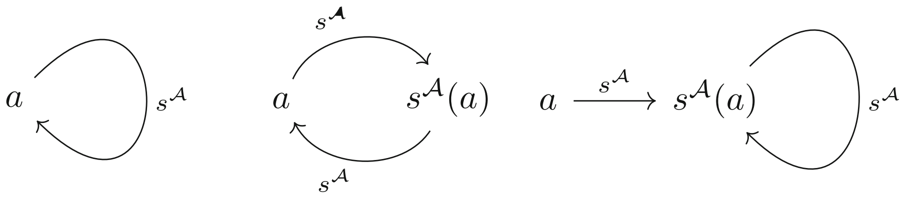

Fig. 4 Possible scenarios when $\psi_{\vee}$ holds [1]

We state in Theorem 17 that adding a function symbol in this way preserves all the studied properties; this proof can be found in Appendix B.2.

Theorem 17 *For every theory $\mathcal{T}$ over an empty signature Σ_n with sorts $S = \{\sigma_1, \ldots, \sigma_n\}$: $\mathcal{T}$ is stably infinite, smooth, finitely witnessable, strongly finitely witnessable, or convex w.r.t. S if and only if $(\mathcal{T})_s$ is, respectively, stably infinite, smooth, finitely witnessable, strongly finitely witnessable, or convex w.r.t. S.*

We use the operator $(\cdot)_s$ in various places in Table 1 in order to obtain examples in non-empty signatures from existing examples over Σ_1 and Σ_2.

Example 4 $(\mathcal{T}_{\geq n})_s$ is a one-sorted theory, whose structures have at least n elements and interpret the function symbol s as the identity. As seen above, $\mathcal{T}_{\geq n}$ admits all the considered properties. By Theorem 17, so does $(\mathcal{T}_{\geq n})_s$.

4.2.3.2 Making a Theory Non-convex

The last general construction that we present aims at taking a theory and creating a non-convex variant of it while preserving the other properties we consider. This can be done with the addition of a single unary function symbol s. To define such a theory, we make use of the formula $\psi_{\vee}$ from Fig. 3. Intuitively, $\psi_{\vee}$ states that in an interpretation $\mathcal{A}$ in which it holds, $s^{\mathcal{A}}(s^{\mathcal{A}}(a))$ must equal either $s^{\mathcal{A}}(a)$ or a itself; in other words, either $a = s^{\mathcal{A}}(a) = s^{\mathcal{A}}(s^{\mathcal{A}}(a))$, $a = s^{\mathcal{A}}(s^{\mathcal{A}}(a)) \neq s^{\mathcal{A}}(a)$, or $a \neq s^{\mathcal{A}}(a) = s^{\mathcal{A}}(s^{\mathcal{A}}(a))$, as shown in Fig. 4.

This is especially useful for defining non-convex theories, since $(s^2(x) = x) \vee (s^2(x) = s(x))$ is valid in the theory, but neither $s^2(x) = x$ nor $s^2(x) = s(x)$ is. Notice, of course, that non-convexity is only possible when there are at least two elements available in the domain – otherwise, all equalities are satisfied.

Definition 6 Let $\mathcal{T}$ be a theory over an empty signature with sorts $S = \{\sigma_1, \ldots, \sigma_n\}$. Then $(\mathcal{T})_{\vee}$ is the $\Sigma_{n,s}$-theory axiomatized by $Ax(\mathcal{T}) \cup \{\psi_{\vee}\}$.

We prove in Theorem 19 that this construction preserves all properties but convexity, and guarantees non-convexity; but we will need an important result before we can do so, and to better motivate it we must take a look at witnesses, specifically those for theories on an empty signature. In the most general case, the variables present in a witness for ϕ depend not only on the variables present in ϕ: say, in a signature with function symbols, it is often necessary to flatten ϕ in order to find a witness, and thus the number of variables we must add depends on the indexes with which these symbols appear in ϕ. Many times, however, one can find a witness for ϕ by considering the conjunction of ϕ and a formula stating there are at least some specific number of variables, different from each other and the variables of ϕ. This is actually so often the case that we provide a formal statement to this end in Definition 7, and prove in Lemma 18 that if we can find a witness for a theory on an empty signature, a more careful search can find one of these special witnesses. This is important because such a witness behaves much better under the action of our theory operators.

Definition 7 A witness wit is called *variable-dependent* if there is a function χ from sets of variables to quantifier-free formulas such that, for every formula ϕ, $wit(\phi) = \phi \wedge \chi(vars(\phi))$.

Lemma 18 *Every theory $\mathcal{T}$ defined over an empty signature Σ that is finitely witnessable (respectively strongly finitely witnessable) w.r.t. $S \subseteq \mathcal{S}_\Sigma$, has a witness (strong witness) that is variable-dependent.*

Theorem 19 *Let $\mathcal{T}$ be a theory over an empty signature Σ_n with sorts $S = \{\sigma_1, \ldots, \sigma_n\}$. Then: $(\mathcal{T})_\vee$ is stably infinite, smooth, finitely witnessable, or strongly finitely witnessable w.r.t. S if and only if $\mathcal{T}$ is, respectively, stably infinite, smooth, finitely witnessable, or strongly finitely witnessable w.r.t. S. In addition, if $\mathcal{T}$ has a model $\mathcal{A}$ with $|\sigma_1^{\mathcal{A}}| \geq 2$, $(\mathcal{T})_\vee$ is not convex with respect to S.*

The proof for these two results can be found in Appendix B.3.

Example 5 The theory $(\mathcal{T}_{\geq n})_\vee$ is one-sorted, and its structures have at least n elements. they interpret the symbol s in a way that satisfies $\psi_\vee$. In particular, for each element a of the domain, one of the scenarios from Fig. 4 holds. According to Theorem 19, since $\mathcal{T}_{\geq n}$ admits all properties, $(\mathcal{T}_{\geq n})_\vee$ admits all properties but convexity.

4.2.3.3 Additional Theories over Σ_s

Whenever there is a Σ_1-theory with some properties (among stable infiniteness, smoothness, finite witnessability, strong finite witnessability and convexity), we can obtain a Σ_s-theory with the same properties using one of the techniques above. To cover cases for which there is no corresponding Σ_1-theory, we use the theories presented in Table 6 and described below.

We start with $\mathcal{T}^{\neq}_{odd}$, $\mathcal{T}^{\neq}_{1,\infty}$, and $\mathcal{T}^{\neq}_{2,\infty}$, deferring the discussion on $\mathcal{T}_f$ and $\mathcal{T}^{\vee}_f$ to Paragraph 4.2.3.4. The theory $\mathcal{T}^{\neq}_{odd}$ is composed of structures $\mathcal{A}$ with either an infinite or an odd number of elements, and with the property that if $|\sigma_1^{\mathcal{A}}| \neq 1$, then $s^{\mathcal{A}}(a) \neq a$ for all $a \in \sigma_1^{\mathcal{A}}$. The theory $\mathcal{T}^{\neq}_{1,\infty}$ has all structures (and only those) $\mathcal{A}$ that either: (i) have $|\sigma_1^{\mathcal{A}}| = 1$; or (ii) have infinitely many elements, and for which $s^{\mathcal{A}}(a) \neq a$ for each $a \in \sigma_1^{\mathcal{A}}$. Similarly, $\mathcal{T}^{\neq}_{2,\infty}$ is composed of the structures $\mathcal{A}$ that either: (i) have exactly two elements and interpret s as the identity; or (ii) have infinitely many elements, and interpret s in such a way that $s^{\mathcal{A}}(a) \neq a$ for all $a \in \sigma_1^{\mathcal{A}}$.

4.2.3.4 On the Theories $\mathcal{T}_f$ and $\mathcal{T}^{\vee}_f$

We now introduce the theories $\mathcal{T}_f$ and $\mathcal{T}^{\vee}_f$. The importance of these theories is that both of them are *one-sorted* theories, with a single unary function symbol and no predicate symbols other than equality, that are polite but not strongly polite (the first is also convex and the second is not). Their existence improves on the result of [10], which introduced a *two-sorted* theory that is polite but not strongly polite (namely $\mathcal{T}_{2,3}$). More recently we have used them in [18] to show that polite theory combination is not possible for not-strongly polite theories, that is, there are two theories over disjoint signatures, both decidable and one polite (specifically, $\mathcal{T}_f$), such that their combination is not decidable.

Table 6 Σ_s-theories

Name	Axiomatization
$\mathcal{T}_f$	$\{[\psi^{=}_{\geq f_1(k)} \wedge \psi^{\neq}_{\geq f_0(k)}] \vee \bigvee_{i=1}^{k}[\psi^{=}_{=f_1(i)} \wedge \psi^{\neq}_{=f_0(i)}] : k \in \mathbb{N} \setminus \{0\}\}$
$\mathcal{T}_f^{\vee}$	$Ax(\mathcal{T}_f) \cup \{\psi_{\vee}\}$
$\mathcal{T}^{\neq}_{odd}$	$\{\psi_{=1} \vee [\neg\psi_{=2k} \wedge \forall x. \neg(s(x) = x)] : k \in \mathbb{N}\}$
$\mathcal{T}^{\neq}_{1,\infty}$	$\{\psi_{=1} \vee [\psi_{\geq k} \wedge \forall x. \neg(s(x) = s)] : k \in \mathbb{N}\}$
$\mathcal{T}^{\neq}_{2,\infty}$	$\{[\psi_{=2} \wedge \forall x. (s(x) = x)] \vee [\psi_{\geq k} \wedge \forall x. \neg(s(x) = x)] : k \in \mathbb{N}\}$

$$\Psi^{=s}_{\geq n} = \exists \overrightarrow{x}. [\delta_n \wedge \bigwedge_{i=1}^{n} p(x_i)], \qquad \Psi^{\neq s}_{\geq n} = \exists \overrightarrow{x}. [\delta_n \wedge \bigwedge_{i=1}^{n} \neg p(x_i)],$$

$$\Psi^{=s}_{=n} = \exists \overrightarrow{x}. [\delta_n \wedge \bigwedge_{i=1}^{n} p(x_i) \wedge \forall x. [p(x) \rightarrow \bigvee_{i=1}^{n} x = x_i]],$$

$$\Psi^{\neq s}_{=n} = \exists \overrightarrow{x}. [\delta_n \wedge \bigwedge_{i=1}^{n} \neg p(x_i) \wedge \forall x. [\neg p(x) \rightarrow \bigvee_{i=1}^{n} x = x_i]].$$

Fig. 5 Cardinality formulas for signatures with a unary function symbol s. $\overrightarrow{x}$ stands for $x_1, \ldots, x_n$, $p(x)$ for $s(x) = x$, and δ_n for $\bigwedge_{1 \leq i < j \leq n} \neg(x_i = x_j)$

For their axiomatizations, we use the formulas from Fig. 5, in which s is a unary function symbol. $\Psi^{=s}_{\geq n}$ ($\Psi^{=s}_{=n}$) states that a structure $\mathcal{A}$ has at least (exactly) n elements a satisfying $s^{\mathcal{A}}(a) = a$; similarly, $\Psi^{\neq s}_{\geq n}$ ($\Psi^{\neq s}_{=n}$) states that a structure $\mathcal{A}$ has at least (exactly) n elements a satisfying $s^{\mathcal{A}}(a) \neq a$.

Further, the axiomatization requires a function f from $\mathbb{N}$ to $\{0, 1\}$ that is not computable, but with the property that for $k > 0$, f maps half of the numbers in the interval $[1, 2^k]$ to 1, and the other half to 0. The existence of such a function is stated below. We start by defining counting functions f_0 and f_1.

Definition 8 Let $f : \mathbb{N} \setminus \{0\} \rightarrow \{0, 1\}$. For $i \in \{0, 1\}$ and $n \in \mathbb{N} \setminus \{0\}$, $f_i(n)$ is defined by: $f_i(n) = |f^{-1}(i) \cap [1, n]|$.

Intuitively, $f_0(n)$ counts how many numbers between 1 and n (inclusive) are mapped by f to 0, and $f_1(n)$ counts how many are mapped to 1. Because $f(n)$ always equals 0 or 1, it is easy to see that for every $n > 0$, $n = f_1(n) + f_0(n)$.

Lemma 20 *There exists a function $f : \mathbb{N} \setminus \{0\} \rightarrow \{0, 1\}$ such that $f(1) = 1$ with the properties that: f is not computable; and, for every $k \in \mathbb{N} \setminus \{0\}$, $f_0(2^k) = f_1(2^k)$.*

Example 6 The constant function that assigns 0 to all positive integers satisfies neither the first nor the second condition of Lemma 20. The function that assigns 0 to even numbers and 1 to odd numbers satisfies the second condition, but not the first. Of course, any non-computable function satisfies the first condition: an example could be found by a function T that, under some encoding of Turing machines, returns 1 if the Turing machine that is encoded by the given number halts, and 0 if it doesn't. Finding a function that admits both conditions is more challenging.

Let f be some function with the properties listed in Lemma 20. We can now define $\mathcal{T}_f$ over Σ_s, the one-sorted signature with a single unary function s and no predicate symbols (note that f itself is not a part of the signature, but is rather used to help define the axioms of $\mathcal{T}_f$). $\mathcal{T}_f$ (its axiomatization can be found in Table 6) consists of those structures $\mathcal{A}$ that either (i) have a finite cardinality n, with $f_1(n)$ elements satisfying $s^{\mathcal{A}}(a) = a$, and $f_0(n)$ elements satisfying $s^{\mathcal{A}}(a) \neq a$ (and thus $\mathcal{A}$ satisfies $\psi^{=}_{\geq f_1(k)} \wedge \psi^{\neq}_{\geq f_0(k)}$, for $k \leq n$, and $\psi^{=}_{=f_1(n)} \wedge \psi^{\neq}_{=f_0(n)}$, and hence $\bigvee_{i=1}^{k}[\psi^{=}_{=f_1(i)} \wedge \psi^{\neq}_{=f_0(i)}]$ for all $k \geq n$); or (ii) have infinitely many elements, with infinitely many elements satisfying each condition, $s^{\mathcal{A}}(a) = a$ and $s^{\mathcal{A}}(a) \neq a$ (and thus $\mathcal{A}$ satisfies $\psi^{=}_{\geq f_1(k)} \wedge \psi^{\neq}_{\geq f_0(k)}$ for all $k \in \mathbb{N}$). Note that the description is well-defined because an element must always satisfy either $s^{\mathcal{A}}(a) = a$ or $s^{\mathcal{A}}(a) \neq a$, but never both or neither of these. The theory $\mathcal{T}^{\vee}_f$ is similar to $\mathcal{T}_f$, being defined over the same signature Σ_s, but in addition to $Ax(\mathcal{T}_f)$ its structures must also satisfy $\psi_{\vee}$; see Table 6 for its axiomatization.

Lemma 21 *1. $\mathcal{T}_f$ is stably-infinite, smooth, finitely witnessable and convex, but not strongly finitely witnessable, with respect to its only sort.*

2. $\mathcal{T}^{\vee}_f$ is stably-infinite, smooth and finitely witnessable, but neither strongly finitely witnessable nor convex, with respect to its only sort.

The proof of the first item in Lemma 21 can be found in Appendix A.3.2, and that for the second one is in Appendix A.3.3.

The following example constructs some standard models of $\mathcal{T}_f$ and $\mathcal{T}^{\vee}_f$, which we hope can help in understanding them at an intuitive level.

Example 7 Let f be a function meeting the conditions of Lemma 20, and suppose $f(1) = 1$, $f(2) = 0$ and $f(3) = 0$ as in Fig. 7. Take the Σ_s-model $\mathcal{A}$ with $\sigma_1^{\mathcal{A}} = \{a_1, a_2, a_3\}$, $s^{\mathcal{A}}(a_1) = a_2$, $s^{\mathcal{A}}(a_2) = a_3$ and $s^{\mathcal{A}}(a_3) = a_3$. It is a model of $\mathcal{T}_f$, but not of $\mathcal{T}^{\vee}_f$, as $(s^{\mathcal{A}})^2(a_1) = a_3$ does not equal a_1 or $s^{\mathcal{A}}(a_1) = a_2$.

More generally, let $\mathcal{A}_n$ be a Σ_s-model with domain $\{a_1, \ldots, a_n\}$ such that: $s^{\mathcal{A}_n}(a_i)$ equals a_i if $1 \leq i \leq f_1(n)$, so there are $f_1(n)$ elements where $s^{\mathcal{A}}$ is the identity; and a_1 if $f_1(n) < i \leq n$, so there are $n - f_1(n) = f_0(n)$ elements where $s^{\mathcal{A}}$ is not the identity (the second condition is void when $n = 1$). Then $\mathcal{A}_n$ is a model of both $\mathcal{T}_f$ and $\mathcal{T}^{\vee}_f$.

If κ is an infinite cardinal, let $\mathcal{A}_\kappa$ be a Σ_s-model with domain $A \cup \{a_n : n \in \mathbb{N} \setminus \{0\}\}$ (where A is a set of cardinality κ disjoint from $\{a_n : n \in \mathbb{N} \setminus \{0\}\}$) such that $s^{\mathcal{A}_\kappa}(a_i) = a_i$ for each $i \in \mathbb{N} \setminus \{0\}$, and $s^{\mathcal{A}_\kappa}(a) = a_1$ for each $a \in A$. Then $\mathcal{A}_\kappa$ is a model of both $\mathcal{T}_f$ and $\mathcal{T}^{\vee}_f$.

Remark 3 The construction of $\mathcal{T}^{\vee}_f$ from $\mathcal{T}_f$ is very similar to the general construction outlined in Definition 6, but for that definition the signature is required to be empty. The result associated to Definition 6, Theorem 19 (according to which all properties but convexity are preserved by this operation), is therefore only proven to hold for theories over the empty signature, which is not the case of $\mathcal{T}_f$. Obtaining $\mathcal{T}^{\vee}_f$ from $\mathcal{T}_f$ is not done by adding a function symbol, but rather by changing the axiomatization of the already existing function symbol. While we do prove that $\mathcal{T}^{\vee}_f$ has the required properties, a more general result in the style of Theorem 19, with the ability to preserve an existing function symbol instead of adding a new one, is left for future work, and should not be expected to hold for all theories.

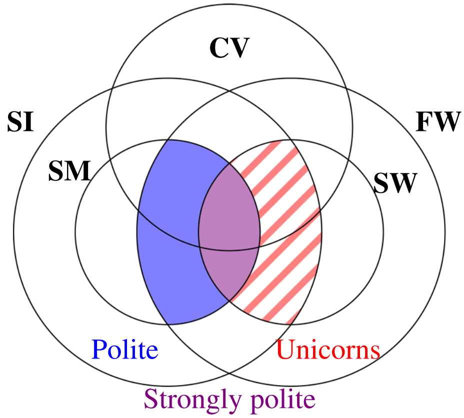

Fig. 6 A diagram of the various notions studied in this paper [1]

4.2.4 Theories Over Many-Sorted Non-empty Signatures

For the last column of Table 1, all possible theories can be obtained from theories that were already defined, using a combination of Definitions 3, 5 and 6, so there is no need to present additional theories specifically for many-sorted non-empty signatures.

Example 8 Line 1 includes the theory $((\mathcal{T}_{\geq n})^2)_s$, obtained from $(\mathcal{T}_{\geq n})^2$ using Definition 5, where the latter theory is obtained from $\mathcal{T}_{\geq n}$ using Definition 3. This theory admits all properties, including convexity. To obtain a non-convex variant, the theory $((\mathcal{T}_{\geq n})^2)_\vee$ is constructed in a similar fashion, using Definition 6 instead of Definition 5.

With many-sorted non-empty signatures, we can always find an example for each combination of properties, except for those that are trivially impossible due to Theorems 3 and 4 (i.e., theories that are strongly finitely witnessable but not finitely witnessable, and theories that are smooth but not stably infinite), as well as unicorn theories, thanks to [13]. This is nicely depicted by Fig. 6. Theorems 3 and 4 are represented in this figure by the location of the circles: the circle for smooth theories is entirely inside the circle for stably infinite theories, and similarly for strongly finitely witnessable and finitely witnessable theories. For every region in this figure, the right-most column of Table 1 has then an example, the sole exception being the hatched region that represents unicorn theories (as they do not exist). Strongly polite theories are represented in purple, and polite theories that are not strongly polite show up in blue.

5 Conclusion

As mentioned, there are two main contributions offered in this paper, both associated with the theme of theory combination. Section 3 presents a new combination theorem, according to which checking for stable infiniteness, instead of smoothness, when performing polite theory combination is already enough. Section 4.2 provides examples for all combinations of stable infiniteness, smoothness, convexity, finite witnessability, and strong finite witnessabillity known to be possible; Section 4.1 provides theorems proving the sharpness of the examples provided.

Many ideas born from this paper led to further studies in theory combination: the question of whether unicorn theories exist was answered negatively in [13]; adding shininess to the analysis here presented was carried out in two steps, first in [19] and later in [16] (whose proofs may be found in the technical reports [20, 21], respectively). Further directions for future work include adding decidability and gentleness into the mix of combination properties, as well as better studying the behavior of non-computable functions on which we heavily relied, such as the f in $\mathcal{T}_f$.

A Proofs for the Theories in Table 1

In this section we prove the necessary results to correctly place the theories defined in Section 4.2.1 and paragraphs 4.2.2.2 and 4.2.3.3 on Table 1; these results are separated by theory, in order to make the section more understandable, and follow the order in which the theories appear in Table 1, from top-to-bottom and then left-to-right.

A.1 Theories over the One-Sorted Empty Signature

A.1.1 $\mathcal{T}_{\geq n}$

$$\{\psi_{\geq n}\} \qquad \textbf{(Axiomatization:)}$$

$\mathcal{T}_{\geq n}$ is defined by a single axiom, which has the form $\exists \overrightarrow{x}\,.\,\psi$ for a quantifier-free formula ψ. Such theories are called *existential* in [23] and are proven there to be strongly polite. Thus we obtain the next lemma:

Lemma 22 *$\mathcal{T}_{\geq n}$ is smooth, and thus stably-infinite. It is also strongly finitely witnessable, and thus finitely witnessable.*

It is left to show that it is convex:

Lemma 23 *$\mathcal{T}_{\geq n}$ is convex.*

Proof Since $\mathcal{T}_{\geq n}$ is stably infinite from Lemma 22, Theorem 11 guarantees it is also convex. □

A.1.2 $\mathcal{T}_{\infty}$

$$\{\psi_{\geq k} : k \in \mathbb{N}\} \qquad \textbf{(Axiomatization:)}$$

Lemma 24 *$\mathcal{T}_{\infty}$ is smooth, and thus stably-infinite.*

Proof Given any $\mathcal{T}_{\infty}$-interpretation $\mathcal{A}$, the theory is seen to be smooth since every larger interpretation $\mathcal{B}$ must necessarily be a $\mathcal{T}_{\infty}$-interpretation as well. □

Lemma 25 *$\mathcal{T}_{\infty}$ is not finitely witnessable, and thus not strongly finitely witnessable.*

Proof Suppose wit is a witness. For any variable x, $x = x$ is satisfied by any $\mathcal{T}_{\infty}$-interpretations $\mathcal{A}$, following that $\mathcal{A}$ satisfies $\exists \overrightarrow{x}\,.\,wit(x = x)$ (for $\overrightarrow{x} = vars(wit(x = x)) \setminus vars(x = x)$), and therefore $wit(x = x)$ is satisfied by some $\mathcal{A}'$ differing from $\mathcal{A}$ at most on the value assigned to $\overrightarrow{x}$. There must then exist a $\mathcal{T}_{\infty}$-interpretation $\mathcal{B}$ that satisfies

$wit(x = x)$, where $\sigma_1^{\mathcal{B}} = vars(wit(x = x))^{\mathcal{B}}$. Of course, this is impossible: $vars(wit(x = x))$ must necessarily be finite, and therefore so is $vars(wit(x = x))^{\mathcal{B}}$, while $\mathcal{B}$ is a model of $\mathcal{T}_{\infty}$ if and only if its domain is infinite. □

Lemma 26 *$\mathcal{T}_{\infty}$ is convex.*

Proof A corollary of Theorem 11 and Lemma 24. □

A.1.3 $\mathcal{T}^{\infty}_{even}$

$$\{\neg\psi_{=2k+1} : k \in \mathbb{N}\} \qquad \textbf{(Axiomatization:)}$$

Lemma 27 *$\mathcal{T}^{\infty}_{even}$ is stably-infinite.*

Proof Follows from Lemma 12, given that $\mathcal{T}^{\infty}_{even}$ is defined on a signature with only one sort and has infinite models. □

Lemma 28 *$\mathcal{T}^{\infty}_{even}$ is finitely witnessable, but neither strongly finitely witnessable nor smooth.*

Proof See Section 3.4 of [10]. □

Lemma 29 *$\mathcal{T}^{\infty}_{even}$ is convex.*

Proof Since $\mathcal{T}^{\infty}_{even}$ is stably-infinite, from Lemma 27, Theorem 11 guarantees $\mathcal{T}^{\infty}_{even}$ is convex. □

A.1.4 $\mathcal{T}_{n,\infty}$

$$\{\psi_{=n} \vee \psi_{\geq k} : k \in \mathbb{N}\} \qquad \textbf{(Axiomatization:)}$$

Lemma 30 *$\mathcal{T}_{n,\infty}$ is stably-infinite.*

Proof Given Lemma 12, and the facts that $\mathcal{T}_{n,\infty}$ is a Σ_1-theory with infinite models, the result follows. □

Lemma 31 *$\mathcal{T}_{n,\infty}$ is not smooth.*

Proof Notice that $\mathcal{T}_{n,\infty}$ has models with n elements in their domains, but no models with m elements, for $n < m < \aleph_0$. □

Lemma 32 *. $\mathcal{T}_{n,\infty}$ is not finitely witnessable, and thus not strongly finitely witnessable.*

Proof Suppose we have a witness wit, and we shall use the quantifier-free formula

$$\phi = \bigwedge_{1 \leq i < j \leq n+1} \neg(x_i = x_j);$$

since ϕ is satisfied by some infinite $\mathcal{T}_{n,\infty}$-interpretations, so is $wit(\phi)$. There must then exist a $\mathcal{T}_{n,\infty}$-interpretation $\mathcal{A}$ that satisfies $wit(\phi)$ (and so ϕ) with $\sigma_1^{\mathcal{A}} = vars(wit(\phi))^{\mathcal{A}}$. This is, of course, absurd: if $\mathcal{A}$ satisfies ϕ, it has at least $n+1$ elements in its domain, while any finite $\mathcal{T}_{n,\infty}$-interpretation must have precisely n elements in its domain (recall that $\mathcal{A}$ is finite). □

Lemma 33 *$\mathcal{T}_{n,\infty}$ is convex.*

Proof Combine Theorem 11 with the fact that $\mathcal{T}_{n,\infty}$ is stably-infinite, present in Lemma 30. □

A.1.5 $\mathcal{T}_{\leq 1}$

$$\{\psi_{\leq 1}\} \qquad \textbf{(Axiomatization:)}$$

Lemma 34 $\mathcal{T}_{\leq 1}$ *is not stably-infinite, and thus not smooth.*

Proof Obvious, given it has finite models, but no infinite ones. □

Lemma 35 $\mathcal{T}_{\leq 1}$ *is strongly finitely witnessable, and thus finitely witnessable.*

Proof Trivially, $wit(\phi) = \phi$ is a strong witness, given that: wit is certainly computable; ϕ and $\exists\,\overrightarrow{x}\,.\,wit(\phi) = \phi$ are $\mathcal{T}_{\leq 1}$-equivalent, for $\overrightarrow{x} = vars(wit(\phi)) \setminus vars(\phi) = \emptyset$; and, for a set of variables V and an arrangement δ_V on V, if $\phi \wedge \delta_V$ is satisfied by a $\mathcal{T}_{\leq 1}$-interpretation $\mathcal{A}$, $\mathcal{A}$ has necessarily $|\sigma_1^{\mathcal{A}}|$ equal to 1, and we already have $vars(\phi \wedge \delta_V)^{\mathcal{A}} = \sigma_1^{\mathcal{A}}$. □

Lemma 36 $\mathcal{T}_{\leq 1}$ *is convex.*

Proof For any pair of variables x and y, trivially one finds that $\vdash_{\mathcal{T}_{\leq 1}} x = y$. So, whenever $\vdash_{\mathcal{T}_{\leq 1}} \phi \rightarrow \bigvee_{i=1}^{n} x_i = y_i$, for ϕ a conjunction of literals, we have $\vdash_{\mathcal{T}_{\leq 1}} x_i = y_i$, for any $1 \leq i \leq n$. □

A.1.6 $\mathcal{T}_{\leq n}$

$$\{\psi_{\leq n}\} \qquad \textbf{(Axiomatization:)}$$

Lemma 37 $\mathcal{T}_{\leq n}$ *is not stably-infinite, and thus not smooth.*

Proof Fairly obvious, since it has finite models, but no infinite ones. □

Lemma 38 $\mathcal{T}_{\leq n}$ *is strongly finitely witnessable, and thus finitely witnessable.*

Proof Consider the function wit from quantifier-free formulas into themselves such that $wit(\phi) = \phi$, which is obviously computable. Since $\overrightarrow{x} = vars(wit(\phi)) \setminus vars(\phi)$ is empty, trivially ϕ and $\exists\,\overrightarrow{x}\,.wit(\phi) = wit(\phi) = \phi$ are $\mathcal{T}_{\leq n}$-equivalent. Now, given a set of variables V and an arrangement δ_V on V, suppose that $\mathcal{A}$ is a $\mathcal{T}_{\leq n}$-interpretation that satisfies $wit(\phi) \wedge \delta_V$. Let $W = vars(wit(\phi) \wedge \delta_V)$ and take the $\mathcal{T}_{\leq n}$-interpretation $\mathcal{B}$ with domain $W^{\mathcal{A}}$, and $x^{\mathcal{B}} = x^{\mathcal{A}}$ for every $x \in W$ (and arbitrary otherwise). Of course $\mathcal{B}$ is indeed a $\mathcal{T}_{\leq n}$-interpretation, since $|\sigma_1^{\mathcal{A}}| \leq n$, and $\sigma_1^{\mathcal{B}} \subseteq \sigma_1^{\mathcal{A}}$, with $\sigma_1^{\mathcal{B}} = W^{\mathcal{B}} = vars(wit(\phi) \wedge \delta_V)^{\mathcal{B}}$. Furthermore, since all atomic subformulas of $wit(\phi) \wedge \delta_V$ are necessarily equalities $x = y$ with both x and y in W, and $x^{\mathcal{B}} = x^{\mathcal{A}}$ and $y^{\mathcal{B}} = y^{\mathcal{A}}$, $wit(\phi) \wedge \delta_V$ receives the same truth value in $\mathcal{A}$ and $\mathcal{B}$. □

Lemma 39 *If* $n > 1$, $\mathcal{T}_{\leq n}$ *is not convex.*

Proof Given variables x_1 through x_{n+1}, for any conjunction of literals ϕ which is a tautology (such as $x = x$) one has

$$\vdash_{\mathcal{T}_{\leq n}} \phi \rightarrow \bigvee_{1 \leq i < j \leq n+1} x_i = x_j$$

by the pigeonhole principle. But, since $n > 1$, we cannot have $\vdash_{\mathcal{T}_{\leq n}} \phi \rightarrow x_i = x_j$ for any pair $1 \leq i < j \leq n + 1$, since we can always set, in a $\mathcal{T}_{\leq n}$-interpretation $\mathcal{A}$ with at least 2 elements, $x_i^{\mathcal{A}} \neq x_j^{\mathcal{A}}$. □

A.1.7 $\mathcal{T}_{\langle m,n\rangle}$

$$\{\psi_{=m} \vee \psi_{=n}\} \qquad \textbf{(Axiomatization:)}$$

Lemma 40 *$\mathcal{T}_{\langle m,n\rangle}$ is not stably-infinite, and thus not smooth.*

Proof $\mathcal{T}_{\langle m,n\rangle}$ has finite models, but no infinite ones, so it cannot be stably-infinite. □

Lemma 41 *If $m > 1$ and $n > 1$, $\mathcal{T}_{\langle m,n\rangle}$ is finitely witnessable.*

Proof Without loss of generality, assume that $m \geq n$. Consider a quantifier-free formula ϕ, fresh variables x_1 through x_m, and define the witness $wit(\phi) = \phi \wedge \bigwedge_{i=1}^{m} x_i = x_i$, obviously computable. Since ϕ and $wit(\phi)$ are equivalent, given that $\bigwedge_{i=1}^{m} x_i = x_i$ is a tautology, we have that ϕ and $\exists\, \overrightarrow{x}.wit(\phi)$ are $\mathcal{T}_{\langle m,n\rangle}$-equivalent, for $\overrightarrow{x} = \{x_1, \ldots, x_m\} = vars(wit(\phi)) \backslash vars(\phi)$.

So suppose that $wit(\phi)$ is satisfied by a $\mathcal{T}_{\langle m,n\rangle}$-interpretation $\mathcal{A}$: let $W = vars(wit(\phi))$, $V = vars(\phi)$ and $k = m - |V^{\mathcal{A}}|$. We define a $\mathcal{T}_{\langle m,n\rangle}$-interpretation $\mathcal{B}$ with: domain $V^{\mathcal{A}} \cup \{a_1, \ldots, a_k\}$, for a_i not in $\sigma_1^{\mathcal{A}}$ (so that $|\sigma_1^{\mathcal{B}}| = m$); $x^{\mathcal{B}} = x^{\mathcal{A}}$ for $x \in V$; and $\{x^{\mathcal{B}} : x \in \overrightarrow{x}\} = \sigma_1^{\mathcal{B}}$ (what is possible, given $\overrightarrow{x}$ and $\sigma_1^{\mathcal{B}}$ both have m elements). Since $\mathcal{A}$ and $\mathcal{B}$ coincide on the variables of ϕ, the latter satisfies ϕ, and therefore $wit(\phi)$. Finally, given that $W^{\mathcal{B}} = \sigma_1^{\mathcal{B}}$, we obtain that wit is indeed a witness, and $\mathcal{T}_{\langle m,n\rangle}$ is finitely witnessable. □

Lemma 42 *If $|m - n| > 1$, $T_{m,n}$ is not strongly finitely witnessable.*

Proof Without loss of generality, assume $m > n$. Suppose wit is a strong witness, let x be a variable, and take the model $\mathbb{A}'$ of $\mathcal{T}_{\langle m,n\rangle}$ with n elements in its domain: since $x = x$ is satisfied by any interpretation over this structure, there must be an interpretation $\mathcal{A}$ over $\mathbb{A}'$ that satisfies $wit(x = x)$. Let $V = vars(wit(x = x))$ and take the equivalence E on V such that xEy iff $x^{\mathcal{A}} = y^{\mathcal{A}}$, with corresponding arrangement δ_V, and we have two cases to consider.

1. If $|V/E| < n$: since $\mathcal{A}$ clearly satisfies $wit(x = x) \wedge \delta_V$, we must have a $\mathcal{T}_{\langle m,n\rangle}$-interpretation $\mathcal{B}$ that satisfies $wit(x = x) \wedge \delta_V$ with $\sigma_1^{\mathcal{B}} = V^{\mathcal{B}}$. Hence $|\sigma_1^{\mathcal{B}}| < n$, and therefore $\mathcal{B}$ cannot be a $\mathcal{T}_{\langle m,n\rangle}$-interpretation, leading to a contradiction.
2. If $|V/E| = n$, take a variable $y \notin vars(wit(x = x))$, define $W = V \cup \{y\}$ and consider the equivalence relation F on W such that xFy iff xEy or $x = y$, and let the corresponding arrangement on W be δ_W. We state that $wit(x = x) \wedge \delta_V$ is then satisfied by $\mathcal{B}$, a $\mathcal{T}_{\langle m,n\rangle}$-interpretation with $\sigma_1^{\mathcal{B}} = \sigma_1^{\mathcal{A}} \cup \{a_1, \ldots, a_{m-n}\}$ (where $\{a_1, \ldots, a_{m-n}\} \cap \sigma_1^{\mathcal{A}} = \emptyset$, what implies $\mathcal{B}$ has m elements), $x^{\mathcal{B}} = x^{\mathcal{A}}$ for every $x \in V$, and $y^{\mathcal{B}} = a_1$.
   We see that $\mathcal{B}$ satisfies $wit(x = x) \wedge \delta_V$, since this formula is true in $\mathcal{A}$, all its atomic subformulas are equalities of variables, and $\mathcal{A}$ and $\mathcal{B}$ coincide on $vars(wit(x = x))$; and $\mathcal{B}$ must also satisfy $wit(\phi) \wedge \delta_W$, since $y^{\mathcal{B}} \neq x^{\mathcal{B}}$ for every $x \in V$. Since we are under the assumption that wit is a strong witness, there must exist a $\mathcal{T}_{\langle m,n\rangle}$-interpretation $\mathcal{C}$ that satisfies $wit(x = x) \wedge \delta_W$ with $\sigma_1^{\mathcal{C}} = W^{\mathcal{C}}$, and this is impossible: since $|V/E| = n$, $|W/F| = n + 1$, meaning that if $\mathcal{C}$ satisfies δ_W, $\sigma_1^{\mathcal{C}} = W^{\mathcal{C}}$ must have exactly $n + 1$ elements, what contradicts the fact that $\mathcal{C}$ is a model of $\mathcal{T}_{\langle m,n\rangle}$ and therefore should have either n or $m \geq n + 2$ elements in its domain.

□

Lemma 43 *If $m > 1$ and $n > 1$, $\mathcal{T}_{\langle m,n\rangle}$ is not convex.*

Proof Under the assumption that $m > 1$ and $n > 1$, $\mathcal{T}_{\langle m,n\rangle}$ has no models of cardinality 1; since it is not stably-infinite by Lemma 40, by Theorem 10 it cannot be convex. □

A.2 Theories over the Two-Sorted Empty Signature

A.2.1 $\mathcal{T}_{2,3}$

$$\{(\psi^{\sigma_1}_{=2} \wedge \psi^{\sigma_2}_{\geq k}) \vee (\psi^{\sigma_1}_{\geq 3} \wedge \psi^{\sigma_2}_{\geq 3}) : k \in \mathbb{N}\} \qquad \textbf{(Axiomatization:)}$$

The following was proven in [10]:

Lemma 44 *$\mathcal{T}_{2,3}$ is smooth, finitely witnessable, but not strongly finitely witnessable w.r.t. $\{\sigma_1, \sigma_2\}$.*

Lemma 45 *$\mathcal{T}_{2,3}$ is convex w.r.t. $\{\sigma_1, \sigma_2\}$.*

Proof Follows from Theorem 11. □

A.2.2 $\mathcal{T}^{odd}_1$

$$\{\psi^{\sigma_1}_{=n}\} \cup \{\neg\psi^{\sigma_2}_{=2k} : k \in \mathbb{N}\} \qquad \textbf{(Axiomatization:)}$$

Lemma 46 *$\mathcal{T}^{odd}_1$ is not stably-infinite, and thus not smooth, w.r.t. $\{\sigma_1, \sigma_2\}$.*

Proof $\mathcal{T}^{odd}_1$ has a model $\mathcal{A}$ where $|\sigma_1^{\mathcal{A}}| = 1$ and $|\sigma_2^{\mathcal{A}}| = \aleph_0$, but no models $\mathcal{B}$ where both $\sigma_1^{\mathcal{B}}$ and $\sigma_2^{\mathcal{B}}$ are infinite. □

Lemma 47 *$\mathcal{T}^{odd}_1$ is finitely witnessable w.r.t. $\{\sigma_1, \sigma_2\}$.*

Proof So, take a quantifier-free Σ_2-formula ϕ, and we wish to show

$$wit(\phi) = \phi \wedge (x = x) \wedge (y = y),$$

where x is a fresh variable of sort σ_1 and y is a fresh variable of sort σ_2, is a witness for $\mathcal{T}^{odd}_1$. Of course, if $\overrightarrow{x} = vars(wit(\phi)) \setminus vars(\phi) = \{x, y\}$, ϕ and $\exists\, \overrightarrow{x}\,.\, wit(\phi)$ are $\mathcal{T}^{odd}_1$-equivalent since ϕ and $wit(\phi)$ are, themselves, equivalent.

So assume that a $\mathcal{T}^{odd}_1$-interpretation $\mathcal{A}$ satisfies $wit(\phi)$, let $V = vars_{\sigma_2}(\phi)$ and E be the equivalence on V such that $y_1 E y_2$ iff $y_1^{\mathcal{A}} = y_2^{\mathcal{A}}$. There are then two cases to consider.

1. If $|V/E|$ is odd, we take the $\mathcal{T}^{odd}_1$-interpretation $\mathcal{B}$ with $\sigma_1^{\mathcal{B}} = \sigma_1^{\mathcal{A}}$, $\sigma_2^{\mathcal{B}} = V^{\mathcal{A}}$ and $z^{\mathcal{B}} = z^{\mathcal{A}}$ for every variable z. Not only $\mathcal{B}$ satisfies $wit(\phi)$, but $\sigma_1^{\mathcal{B}} = vars_{\sigma_1}(wit(\phi))^{\mathcal{B}}$ and $\sigma_2^{\mathcal{B}} = vars_{\sigma_2}(wit(\phi))^{\mathcal{B}}$.
2. If $|V/E|$ is even, we take the $\mathcal{T}^{odd}_1$-interpretation $\mathcal{B}$ with: $\sigma_1^{\mathcal{B}} = \sigma_1^{\mathcal{A}}$; $\sigma_2^{\mathcal{B}} = V^{\mathcal{A}} \cup \{a\}$, for an element $a \notin \sigma_2^{\mathcal{A}}$, being therefore $|\sigma_2^{\mathcal{B}}|$ odd; and $z^{\mathcal{B}} = z^{\mathcal{A}}$ for every variable z except y, where $y^{\mathcal{B}} = a$. Then $\mathcal{B}$ satisfies $wit(\phi)$, and $\sigma_1^{\mathcal{B}} = vars_{\sigma_1}(wit(\phi))^{\mathcal{B}}$ and $\sigma_2^{\mathcal{B}} = vars_{\sigma_2}(wit(\phi))^{\mathcal{B}}$.

□

Lemma 48 *$\mathcal{T}^{odd}_1$ is not strongly finitely witnessable w.r.t. $\{\sigma_1, \sigma_2\}$.*

Proof Suppose, for a proof by contradiction, that $\mathcal{T}_1^{odd}$ does have a strong witness *wit*. Take the $\mathcal{T}_1^{odd}$-interpretation $\mathcal{A}$ with one element of sort σ_1 and one element of sort σ_2 (so $|\sigma_2^{\mathcal{A}}| = 1$ is odd, as expected), and a variable y of sort σ_2: $\mathcal{A}$ satisfies $y = y$, and thus $\exists\, \overrightarrow{x}\,.\,wit(y = y)$, for $\overrightarrow{x} = vars(wit(y = y)) \setminus vars(y = y)$, as *wit* is a strong witness. Of course, there is an interpretation $\mathcal{A}'$, differing from $\mathcal{A}$ at most on the value assigned to the variables in $\overrightarrow{x}$, that satisfies $wit(y = y)$; of course $\mathcal{A}'$ also satisfies $y = y$, so there must then exist a $\mathcal{T}_1^{odd}$-interpretation $\mathcal{B}$ that satisfies $wit(y = y) \wedge (y = y)$, $\sigma_1^{\mathcal{B}} = vars_{\sigma_1}(wit(y = y) \wedge (y = y))^{\mathcal{B}}$ and $\sigma_2^{\mathcal{B}} = vars_{\sigma_2}(wit(y = y) \wedge (y = y))^{\mathcal{B}}$ (so $\sigma_2^{\mathcal{B}}$ is finite, and therefore has an odd cardinality).

Let V be $vars_{\sigma_2}(wit(y = y) \wedge (y = y))$, and E be the equivalence relation on V such that $y_1 E y_2$ iff $y_1^{\mathcal{B}} = y_2^{\mathcal{B}}$, with corresponding arrangement δ_V. Clearly $\mathcal{B}$ satisfies δ_V, and V/E has an odd number of equivalence classes since $|V/E| = |\sigma_2^{\mathcal{B}}|$; let y_0 be a fresh variable of sort σ_2, and take the equivalence F on $W = V \cup \{y_0\}$ such that $y_1 F y_2$ iff $y_1 = y_2$ or $y_1 E y_2$, with corresponding arrangement δ_W. Notice $|W/F| = |V/E| + 1$. We state, now, that $wit(y = y) \wedge \delta_W$ is $\mathcal{T}_1^{odd}$-satisfiable.

In fact, take the interpretation $\mathcal{C}$ with: $\sigma_1^{\mathcal{C}} = \sigma_1^{\mathcal{B}}$, both of then with cardinality 1; $\sigma_2^{\mathcal{C}} = \sigma_2^{\mathcal{B}} \cup \{a, b\}$, where $a, b \notin \sigma_2^{\mathcal{B}}$, and thus $|\sigma_2^{\mathcal{C}}| = |\sigma_2^{\mathcal{B}}| + 2$, an odd number; and $x^{\mathcal{C}} = x^{\mathcal{B}}$ for all variables x (of any sort) except y_0, where we use instead $y_0^{\mathcal{C}} = a$. Not only $\mathcal{C}$ is a $\mathcal{T}_1^{odd}$-interpretation, but one easily sees it satisfies $wit(y = y) \wedge \delta_W$, meaning we should be able, thanks to the fact again that *wit* is a strong witness, to find a $\mathcal{T}_1^{odd}$-interpretation $\mathcal{D}$ that satisfies $wit(y = y) \wedge \delta_W$ with $\sigma_1^{\mathcal{D}} = vars_{\sigma_1}(wit(y = y) \wedge \delta_W)^{\mathcal{D}}$ and $\sigma_2^{\mathcal{D}} = vars_{\sigma_2}(wit(y = y) \wedge \delta_W)^{\mathcal{D}}$, but this is impossible: $vars_{\sigma_2}(wit(y = y) \wedge \delta_W) = W$, but $|W/F|$ is an even number, what would force $\sigma_2^{\mathcal{D}}$ to have an even number of elements. □

Lemma 49 *$\mathcal{T}_1^{odd}$ is convex.*

Proof Suppose ϕ is a conjunction of literals, and that $\vdash_{\mathcal{T}_1^{odd}} \phi \rightarrow \bigvee_{i=1}^{n} x_i = y_i$; if some pair x_i and y_i is of sort σ_1, since all models of $\mathcal{T}_1^{odd}$ have exactly one element of sort σ_1, it follows that $\vdash_{\mathcal{T}_1^{odd}} \phi \rightarrow x_i = y_i$. So we can assume that all x_i and y_i are of sort σ_2; in addition, we may assume that ϕ is not a contradiction, given that in that case $\vdash_{\mathcal{T}_1^{odd}} \phi \rightarrow x_i = y_i$ for any $1 \leq i \leq n$.

Then consider the formula ϕ', obtained from ϕ by removing the literals with variables of sort σ_1. These are necessarily of the form $x = y$, since: we have no functions or predicates; and $\sigma_1^{\mathcal{A}}$ has always cardinality 1 for $\mathcal{A}$ a model of $\mathcal{T}_1^{odd}$, meaning it cannot satisfy $\neg(x = y)$ if ϕ is not a contradiction. Then, we have that $\vdash_{\mathcal{T}_{odd}} \phi' \rightarrow \bigvee_{i=1}^{n} x_i = y_i$, where $\mathcal{T}_{odd}$ is the theory over the one-sorted empty signature (let, in this case, the sort be σ_2 for clarity) whose models are all structures with an infinite or odd number of elements.

$\mathcal{T}_{odd}$ is obviously stably-infinite, and from Theorem 11, it follows it is convex, meaning $\vdash_{\mathcal{T}_{odd}} \phi' \rightarrow x_i = y_i$ for some $1 \leq i \leq n$. Of course, it follows that $\vdash_{\mathcal{T}_1^{odd}} \phi \rightarrow x_i = y_i$. □

A.2.3 $\mathcal{T}_1^{\infty}$

$$\{\psi_{=1}^{\sigma_1}\} \cup \{\psi_{\geq k}^{\sigma_2} : k \in \mathbb{N}\} \qquad \textbf{(Axiomatization:)}$$

Lemma 50 *$\mathcal{T}_1^{\infty}$ is not stably-infinite, and thus not smooth, w.r.t.* $\{\sigma_1, \sigma_2\}$.

Proof $\mathcal{T}_1^{\infty}$ has a model $\mathcal{A}$ where $|\sigma_1^{\mathcal{A}}| = 1$ and $|\sigma_2^{\mathcal{A}}| = \aleph_0$, but no models $\mathcal{B}$ where both $\sigma_1^{\mathcal{B}}$ and $\sigma_2^{\mathcal{B}}$ are infinite. □

Lemma 51 *$\mathcal{T}_1^\infty$ is not finitely witnessable, and thus not strongly finitely witnessable, w.r.t. $\{\sigma_1, \sigma_2\}$.*

Proof Suppose that *wit* is a witness for $\mathcal{T}_1^\infty$, and take a variable x (the sort is not important): ϕ equal to $x = x$ is then a tautology, and is satisfied by any $\mathcal{T}_1^\infty$-interpretation $\mathcal{A}$ (and there is one, such as the one with $|\sigma_1^\mathcal{A}| = 1$ and $|\sigma_2^\mathcal{A}| = \aleph_0$). Since $\mathcal{A}$ satisfies ϕ, it also satisfies $\exists\, \overrightarrow{x}\,.\, wit(\phi)$, for $\overrightarrow{x} = vars(wit(\phi)) \setminus vars(\phi)$, and so we can change $\mathcal{A}$ into a $\mathcal{T}_1^\infty$-interpretation $\mathcal{A}'$ that satisfies $wit(\phi)$; there must then exist a $\mathcal{T}_1^\infty$-interpretation $\mathcal{B}$ that satisfies $wit(\phi)$ with $\sigma_1^\mathcal{B} = vars_{\sigma_1}(wit(\phi))^\mathcal{B}$ and $\sigma_2^\mathcal{B} = vars_{\sigma_2}(wit(\phi))^\mathcal{B}$, contradicting the fact that all $\mathcal{T}_1^\infty$-interpretations have an infinite domain of sort σ_2. □

Lemma 52 *$\mathcal{T}_1^\infty$ is convex.*

Proof Suppose that ϕ is a cube, and that $\vdash_{\mathcal{T}_1^\infty} \phi \rightarrow \bigvee_{i=1}^n x_i = y_i$. If some pair (x_i, y_i) is of variables of sort σ_1, we already have $\vdash_{\mathcal{T}_1^\infty} \phi \rightarrow x_i = y_i$, so assume all x_i and y_i are of sort σ_2, and remove from ϕ any literals involving variables of sort σ_1 in order to obtain a formula ϕ'. We then have that $\vdash_\mathcal{T} \phi' \rightarrow \bigvee_{i=1}^n x_i = y_i$, where the antecedent and the consequent are formulas in an empty signature with only one sort σ_2, and $\mathcal{T}$ is the theory on this signature with only infinite models, which is stably- infinite. Theorem 11 can then be used to obtain that $\vdash_\mathcal{T} \phi' \rightarrow x_i = y_i$ for some $1 \leq i \leq n$, and it of course follows that $\vdash_{\mathcal{T}_1^\infty} \phi \rightarrow x_i = y_i$: indeed, suppose this is not true, and so there exists a $\mathcal{T}_1^\infty$-interpretation $\mathcal{A}$ that satisfies ϕ, and thus ϕ', but not $x_i = y_i$; taking the $\mathcal{T}$-interpretation $\mathcal{B}$ with $\sigma_2^\mathcal{B} = \sigma_2^\mathcal{A}$ and $x^\mathcal{B} = x^\mathcal{A}$ for every variable of sort σ_2, we see that $\mathcal{B}$ must satisfy ϕ', but not $x_i = y_i$, contradicting the fact that $\vdash_\mathcal{T} \phi' \rightarrow x_i = y_i$ for some $1 \leq i \leq n$. □

A.2.4 $\mathcal{T}_2^\infty$

$$\{\psi_{=2}^{\sigma_1}\} \cup \{\psi_{\geq k}^{\sigma_2} : k \in \mathbb{N}\} \qquad \textbf{(Axiomatization:)}$$

Lemma 53 *$\mathcal{T}_1^\infty$ is not stably-infinite, and thus not smooth, w.r.t. $\{\sigma_1, \sigma_2\}$.*

Proof $\mathcal{T}_2^\infty$ has a model $\mathcal{A}$ where $|\sigma_1^\mathcal{A}| = 2$ and $|\sigma_2^\mathcal{A}| = \aleph_0$, but no models $\mathcal{B}$ where both $\sigma_1^\mathcal{B}$ and $\sigma_2^\mathcal{B}$ are infinite. □

Lemma 54 *$\mathcal{T}_2^\infty$ is not finitely witnessable, and thus not strongly finitely witnessable, w.r.t. $\{\sigma_1, \sigma_2\}$.*

Proof The same as the one for Lemma 51, this time relying on the fact $\mathcal{T}_2^\infty$ has no models $\mathcal{A}$ where both $\sigma_1^\mathcal{A}$ and $\sigma_2^\mathcal{A}$ are finite. □

Lemma 55 *$\mathcal{T}_2^\infty$ is not convex.*

Proof If it were, Theorem 10 would guarantee that $\mathcal{T}_2^\infty$ is also stably-infinite, since it has no models $\mathcal{A}$ where either $|\sigma_1^\mathcal{A}|$ or $|\sigma_2^\mathcal{A}|$ equals 1. □

A.3 Theories over a One-Sorted Non-Empty Signature

A.3.1 Proofs for Paragraph 4.2.3.4.

We must define several auxiliary notions.

Definition 9 Let $n > 2$. $\kappa(n)$ is defined to be the unique natural number k such that $2^{k+1}+1 \leq n \leq 2^{k+2}$.

Remark 4 It is possible, although not very useful, to prove that $\kappa(n)$ is simply $\lceil log_2 n \rceil - 2$.

We start by proving κ has the properties that we need it to have.

Lemma 56 *κ is a well-defined function from $\mathbb{N} \setminus \{0, 1\}$ to $\mathbb{N}$.*

Proof We prove that for each $n > 2$ there exists a unique k such that $2^{k+1} + 1 \leq n \leq 2^{k+2}$.

Existence: by induction on n. For $n = 3$, take $k = 0$ and then $2^1 + 1 \leq 3 \leq 2^2$. For $n > 3$, by the induction hypothesis, there exists a unique k' such that $2^{k'+1} + 1 \leq \lfloor \frac{n}{2} \rfloor \leq 2^{k'+2}$. In particular, $2^{k'+1} < \lfloor \frac{n}{2} \rfloor$, and so $2^{k+1} = 2^{k'+2} < 2 \cdot \lfloor \frac{n}{2} \rfloor \leq n$ for $k = k' + 1$, which means $2^{k+1} + 1 \leq n$. Now, if n is even, then $n = 2 \cdot \lfloor \frac{n}{2} \rfloor$ and then $n = 2 \cdot \lfloor \frac{n}{2} \rfloor \leq 2 \cdot 2^{k'+2} = 2^{k'+3} = 2^{k+2}$. If n is odd, then $n = 2 \cdot \lfloor \frac{n}{2} \rfloor + 1$, and then $2 \cdot \lfloor \frac{n}{2} \rfloor \leq 2 \cdot 2^{k'+2} = 2^{k'+3} = 2^{k+2}$, so assume for contradiction that $2^{k+2} < n$: we therefore have $2^{k+2} < 2 \cdot \lfloor \frac{n}{2} \rfloor + 1$, which means that $2^{k+2} \leq 2 \cdot \lfloor \frac{n}{2} \rfloor$; but we also have $2 \cdot \lfloor \frac{n}{2} \rfloor \leq 2^{k+2}$, and this means that $2 \cdot \lfloor \frac{n}{2} \rfloor = 2^{k+2}$, and in particular $\lfloor \frac{n}{2} \rfloor = 2^{k+1}$; but $\lfloor \frac{n}{2} \rfloor \geq 2^{k+1} + 1$, leading to a contradiction.

Uniqueness: if there are k, k' such that $2^{k+1}+1 \leq n \leq 2^{k+2}$ and $2^{k'+1}+1 \leq n \leq 2^{k'+2}$, and $k \neq k'$, we obtain a contradiction as follows. W.l.g. assume $k < k'$, and so $k+1 \leq k'$, which means that $2^{k+2} = 2 \cdot 2^{k+1} \leq 2 \cdot 2^{k'} = 2^{k'+1}$. Thus we obtain: $2^{k+1} + 1 \leq n \leq 2^{k+2} \leq 2^{k'+1} < 2^{k'+1} + 1 \leq n$, which is a contradiction.

□

Definition 10 Given a function $F : \mathbb{N} \setminus \{0\} \to \{0, 1\}$ with $F(1) = 1$, the function $FF : \mathbb{N} \setminus \{0\} \to \{0, 1\}$ is defined by: $FF(1) = F(1) = 1$, $FF(2) = 0$, and, for every $n \in \mathbb{N} \setminus \{0, 1, 2\}$,

$$FF(n) = \begin{cases} F(n - 2^{\kappa(n)}) & \text{for } 2^{\kappa(n)+1} + 1 \leq n \leq 2^{\kappa(n)+1} + 2^{\kappa(n)} \\ 1 & \text{for } 2^{\kappa(n)+1} + 2^{\kappa(n)} + 1 \leq n \leq 2^{\kappa(n)+2} + 2^{\kappa(n)} - FF_1(2^{\kappa(n)+1} + 2^{\kappa(n)}); \\ 0 & \text{for } 2^{\kappa(n)+2} + 2^{\kappa(n)} - FF_1(2^{\kappa(n)+1} + 2^{\kappa(n)}) + 1 \leq n \leq 2^{\kappa(n)+2}. \end{cases}$$

We must also prove FF is well-defined.

Lemma 57 *For every $F : \mathbb{N} \setminus \{0\} \to \{0, 1\}$ with $F(1) = 1$, we have that FF is a (well-defined) function from $\mathbb{N} \setminus \{0\} \to \{0, 1\}$.*

Proof For each $n \in \mathbb{N} \setminus \{0\}$, if $n \leq 2$, then $FF(n)$ is clearly well-defined. For $n > 2$, $2^{\kappa(n)+1} + 1 \leq n \leq 2^{\kappa(n)+2}$ for the unique $\kappa(n)$. This holds by Definition 9 and Lemma 56. And clearly once $\kappa(n)$ is fixed, the definition of the function distinguishes 3 distinct cases that exactly cover that range. It is left to make sure that in the first of these cases, F is defined on $n - 2^{\kappa}(n)$. This holds as $n \geq 2^{\kappa(n)+1} + 1 > 2^{\kappa(n)}$, so $n - 2^{\kappa(n)} \geq 1$. □

Intuitively, suppose FF has been defined for n up to a certain power of two 2^{k+1}. Because we want FF to be equal to F (not necessarily given the same arguments, but only in terms of the sequence of 0's and 1's they output) as often as possible, we define $FF(n) = F(n - 2^k)$

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$
$F(n)$	1	0	0	0	0	1	0	1	$\cdots$								
$FF(n)$	1	0	0	1	0	0	1	1	0	1	0	1	1	1	0	0	$\cdots$
$FF_1(n)$	1	1	1	2	2	2	3	4	4	5	5	6	7	8	8	8	$\cdots$

Fig. 7 A diagram for the function FF [1]

for the following quarter of the next power of two, that is $2^{k+1} + 1 \leq n \leq 2^{k+1} + 2^k$. Notice that we are continuing from where we stopped last: that is, the last value for which we defined FF as a function of F was $m = 2^{(k-1)+1} + 2^{k-1}$, when $FF(m)$ is equal to $F(2^k + 2^{k-1} - 2^{k-1}) = F(2^k)$, while the next value of n is $n = 2^{k+1} + 1$, when we get $FF(n)$ equal to $F(2^{k+1} + 1 - 2^k) = F(2^k + 1)$.

However, after having done that, we must address that, at the same time, we want the number of times that FF equals one is, more or less, the same as the number of times it equals zero; we do that by demanding that FF equals 1 for numbers up to 2^{k+2} exactly 2^{k+1} times, that is, $FF_1(2^{k+2}) = 2^{k+1}$.[3] In order to accomplish that, we make $FF(n)$ equal to 1 for $n > 2^{k+1} + 2^k$ until $FF_1(n)$ becomes 2^{k+1}; after that, we let FF simply equal zero until the next power of two.

For an example, take a certain $F : \mathbb{N} \setminus \{0\} \to \{0, 1\}$ with $F(1) = 1$; say, with $F(1) = 1$, $F(2) = 0$, $F(3) = 0$, $F(4) = 0$, $F(5) = 0$, $F(6) = 1$, $F(7) = 0$ and $F(8) = 1$. We can then draw the elucidating diagram in Fig. 7.

Lemma 58 *For every* $F : \mathbb{N} \setminus \{0\} \to \{0, 1\}$ *with* $F(1) = 1$, $FF_1(2^{k+1}) = 2^k$ *for every* $k \in \mathbb{N}$.

Proof By induction on k. For $k = 0$, we have that $FF(2^0) = FF(1) = 1$ and $FF(2^1) = FF(2) = 0$. Thus $FF_1(2^1) = FF_1(2) = 1 = 2^0$.

For $k = 1$, we show that $FF(4) = 2$. Recall that $FF(2^0) = FF(1) = 1$ and $FF(2^1) = FF(2) = 0$. It is left to compute $FF(3)$, and $FF(4)$ (which depends on $FF(3)$). Clearly, $\kappa(3) = 0$, and $2^1 + 1 \leq 3 \leq 2^1 + 2^0$, thus $FF(3) = F(3-1) = F(2)$. We distinguish two cases:

1. If $F(2) = 1$, then $FF(3) = 1$, and so $FF_1(3) = 2$. In this case, $2^2 + 2^0 - FF_1(2^1 + 2^0) + 1 = 4 + 1 - FF_1(3) + 1 = 4 \leq 4 \leq 2^2$, and so $FF(4) = 0$, thus $FF_1(4) = 2$.
2. If $F(2) = 0$, then $FF(3) = 0$, and so $FF_1(3) = 1$. In this case, $2^1 + 2^0 + 1 \leq 4 \leq 4 = 4 + 1 - 1 = 4 + 1 - FF_1(3) = 2^2 + 2^0 - FF_1(2^1 + 2^0)$. Therefore, $FF(4) = 1$, and thus $FF_1(4) = 2$.

Now assume $FF_1(2^{k+1}) = 2^k$ $(*)$. We prove that $FF_1(2^{k+2}) = 2^{k+1}$. Denote $|\{2^{k+1} + 1 \leq n \leq 2^{k+2} \mid f(n) = 1\}|$ by M. Clearly, $FF_1(2^{k+2}) = FF_1(2^{k+1}) + M$. By $(*)$, we have $FF_1(2^{k+2}) = 2^k + M$. Since we would like to prove that $FF_1(2^{k+2}) = 2^{k+1}$, it suffices to show that $M = 2^k$.

Let m be the number of numbers n from $2^{k+1} + 1$ to $2^{k+1} + 2^k$ (inclusive) such that $F(n - 2^k) = 1$. Notice that for each such n, we have that $\kappa(n) = k$. Thus, $FF_1(2^{k+1} + 2^k) = FF_1(2^{k+1}) + m = 2^k + m$. Hence by the definition of FF, the number of numbers n from

[3] FF_1 is the result of applying Definition 8 to the function FF.

$2^{k+1}+2^k+1$ to $2^{k+2}+2^k-FF_1(2^{k+1}+2^k)$ where $FF(n)=1$ (that is, all of them) is

$$(2^{k+2}+2^k-FF_1(2^{k+1}+2^k))-(2^{k+1}+2^k+1)+1=$$
$$(2^{k+2}+2^k-2^k-m)-(2^{k+1}+2^k+1)+1=2^{k+2}-m-2^{k+1}-2^k-1+1=$$
$$2^k(4-2-1)-m=2^k\cdot 1-m=2^k-m.$$

Also, from the definition of FF, we have that the number of numbers n from $2^{k+2}+2^k-FF_1(2^{k+1}+2^k)+1$ to 2^{k+2} (inclusive) where $FF(n)=1$ is 0. In total, we get that $M=m+2^k-m+0=2^k$. □

Lemma 59 *Given a function $F:\mathbb{N}\setminus\{0\}\to\{0,1\}$ with $F(1)=1$, if we define f to be FF, then for every $n\geq 2$ it is true that $F(n)=f(n+2^{\kappa(n)}+1)$.*

Proof Since $2^{\kappa(n)+1}+1\leq n\leq 2^{\kappa(n)+2}$, we also have

$$2^{(\kappa(n)+1)}+1\leq n+2^{\kappa(n)+1}\leq 2^{(\kappa(n)+1)+1}+2^{\kappa(n)+1}\leq 2^{(\kappa(n)+1)+2}.$$

Hence $\kappa(n+2^{\kappa(n)+1})=\kappa(n)+1$. By the definition of FF, we therefore have $F(n)=F(n+2^{\kappa(n)+1}-2^{\kappa(n)+1})=FF(n+2^{\kappa(n)+1})=f(n+2^{\kappa(n)}+1)$.

□

Lemma 20 *There exists a function $f:\mathbb{N}\setminus\{0\}\to\{0,1\}$ such that $f(1)=1$ with the properties that: f is not computable; and, for every $k\in\mathbb{N}\setminus\{0\}$, $f_0(2^k)=f_1(2^k)$.*

Proof We start by taking a non-computable function $F:\mathbb{N}\setminus\{0\}\to\{0,1\}$ with $F(1)=1$.

Using Definition 10, we can make $f:\mathbb{N}\setminus\{0\}\to\{0,1\}$ equal to FF. This function is not computable since otherwise, by Lemma 59, F would be computable (as κ is computable as well). And, from Definition 10, if we make $k\geq 2$ equal to $k'+1$, for $k'\in\mathbb{N}\setminus\{0\}$, $f_1(2^k)=FF_1(2^{k'+1})=2^{k'}$ (from Lemma 58), and $f_0(2^k)=2^k-f_1(2^k)=2^{k'}$; for $k=1$, we know that $f(1)=1$ and $f(2)=0$, by Definition 10 of FF (and thus f), hence implying $f_0(2^0)=f_1(2^0)$. To summarize, for any $k\in\mathbb{N}\setminus\{0\}$, $f_0(2^k)=f_1(2^k)$. □

Example 9 To give a concrete example of a function satisfying the conditions of Lemma 20, take the function T from Example 6; then, take $F=T$ (if the first Turing machine does not halt, just change the value of $T(1)$), and FF has the properties we want.

A.3.2 $\mathcal{T}_f$

$$\{[\psi^{=}_{\geq f_1(k)}\wedge\psi^{\neq}_{\geq f_0(k)}]\vee\bigvee_{i=1}^{k}[\psi^{=}_{=f_1(i)}\wedge\psi^{\neq}_{=f_0(i)}]:k\in\mathbb{N}\setminus\{0\}\}\qquad\textbf{(Axiomatization:)}$$

To show that $\mathcal{T}_f$ is not strongly finitely witnessable, we use the following lemmas, which are interesting in their own right. First, in Lemma 60, we show that the *mincard* function of $\mathcal{T}_f$ indeed always return a finite value, as any quantifier-free, $\mathcal{T}_f$-satisfiable formula has at least one finite model; this way we can prove in Lemma 61 that this function is also not computable. The following result, Lemma 62, is quite surprising: as it turns out, for quantifier-free formulas, the set of $\mathcal{T}_f$-satisfiable formulas coincides with the set of satisfiable formulas. That is, even though the definition of $\mathcal{T}_f$ is very complex, it induces the same satisfiability relation, over quantifier-free formulas, as the simplest theory possible – the theory axiomatized by the empty set (or, equivalently, all valid first-order sentences). The analogue of Lemma 62 for $\mathcal{T}_f^{\vee}$, Theorem 67, is similar, although its proof is more tiresome. Using Lemmas 61 and 62, it is then possible to finally show that $\mathcal{T}_f$ is not strongly finitely witnessable, what will be done in Lemma 65; this proof will also serve verbatim for $\mathcal{T}_f^{\vee}$. Results about stable infiniteness, smoothness, finite witnessability and convexity are proved in a standard way.

Lemma 60 *Every quantifier-free $\mathcal{T}_f$-satisfiable formula is satisfied by a finite $\mathcal{T}_f$-interpretation.*

Proof Suppose ϕ is a quantifier-free formula, and let $\mathcal{A}$ be a $\mathcal{T}_f$-interpretation that satisfies ϕ: we may assume that it is infinite. The set $\{\alpha : \alpha \text{ is a term in } \phi\}$ is finite, and therefore so is $A = \{\alpha : \alpha \text{ is a term in } \phi\}^{\mathcal{A}}$. Let

$$m_0 = |\{a \in A : s^{\mathcal{A}}(a) \neq a\}| \quad \text{and} \quad m_1 = |\{a \in A : s^{\mathcal{A}}(a) = a\}|,$$

and take a $k \in \mathbb{N} \setminus \{0\}$ such that $2^k > \max\{m_0, m_1\}$. Take as well sets B and C with, respectively, $2^k - m_1$ and $2^k - m_0$ elements, disjoint from A, and we define a $\mathcal{T}_f$-interpretation $\mathcal{B}$ with: $\sigma_1^{\mathcal{B}} = A \cup B \cup C$;

$$s^{\mathcal{B}}(a) = \begin{cases} a & \text{if } a \in A \text{ and } s^{\mathcal{A}}(a) = a, \text{ or } a \in B; \\ s^{\mathcal{A}}(a) & \text{if } s^{\mathcal{A}}(a) \neq a \text{ but } s^{\mathcal{A}}(a) \in A; \\ \text{any element in } B & \text{if } s^{\mathcal{A}}(a) \neq a \text{ and } s^{\mathcal{A}}(a) \notin A, \text{ or } a \in C; \end{cases}$$

and $x^{\mathcal{B}} = x^{\mathcal{A}}$ for any variable x in ϕ, and arbitrary otherwise. $\mathcal{B}$ has m_1 elements in A, plus all $2^k - m_1$ of those in B, satisfying $s^{\mathcal{B}}(a) = a$, adding to a total of 2^k; and m_0 elements in A, plus all $2^k - m_0$ of the elements in C, satisfying $s^{\mathcal{B}}(a) \neq a$, to a grand total of 2^k, meaning $\mathcal{B}$ is a $\mathcal{T}_f$-interpretation with 2^{k+1} elements, and thus finite.

Furthermore, let $s^j(x)$ be a term in ϕ: if $j = 0$, $(s^j(x))^{\mathcal{B}} = x^{\mathcal{B}} = x^{\mathcal{A}} = (s^j(x))^{\mathcal{A}}$, so assume $(s^j(x))^{\mathcal{B}} = (s^j(x))^{\mathcal{A}}$ holds for an arbitrary j; in that case, if $s^{j+1}(x)$ is still a term in ϕ,

$$(s^{j+1}(x))^{\mathcal{B}} = s^{\mathcal{B}}\big((s^j(x)^{\mathcal{B}}\big) = s^{\mathcal{A}}\big((s^j(x)^{\mathcal{A}}\big) = (s^{j+1}(x))^{\mathcal{A}},$$

proving that for any terms α in ϕ, $\alpha^{\mathcal{B}} = \alpha^{\mathcal{A}}$; since ϕ is a quantifier-free formula in a signature without predicates, all of whose terms receive the same value in either $\mathcal{A}$ or $\mathcal{B}$, and $\mathcal{A}$ satisfies ϕ, we have that $\mathcal{B}$ also satisfies ϕ, finishing the proof. □

Lemma 61 *The mincard function of $\mathcal{T}_f$ is not computable.*

Proof We start by proving that $f(n+1) = 1$ iff $mincard(\phi_n) = n + 1$, for

$$\phi_n = \bigwedge_{i=1}^{f_1(n)+1} \big(s(x_i) = x_i\big) \wedge \bigwedge_{1 \leq i < j \leq f_1(n)+1} \neg(x_i = x_j)$$

a quantifier-free formula that is only true in a model $\mathcal{B}$ when there exist at least $f_1(n) + 1$ distinct elements in $\mathcal{B}$ satisfying $s^{\mathcal{B}}(a) = a$.

Notice that the theory $\mathcal{T}_f$ has models of all finite, non-zero cardinalities: indeed, given an $n \in \mathbb{N} \setminus \{0\}$, one such model $\mathcal{A}$ has domain $\{a_1, \ldots, a_n\}$, with $s^{\mathcal{A}}(a_i) = a_i$ for each $1 \leq i \leq f_1(n)$ (remember $f_1(n) \geq 1$ for all $n \in \mathbb{N} \setminus \{0\}$), and $s^{\mathcal{A}}(a_j) = a_1$ for each $f_1(n) < j \leq n$, if there are any such j (and as long as $n > 1$ one has $f_1(n) < n$). Notice as well that, for all $p \geq q$, $\psi^{=}_{=p} \rightarrow \psi^{=}_{\geq q}$ and $\psi^{\neq}_{=p} \rightarrow \psi^{\neq}_{\geq q}$ and so a model $\mathcal{A}$ of $\mathcal{T}_f$ must satisfy either $\psi^{=}_{=f_1(k)} \wedge \psi^{\neq}_{=f_0(k)}$, for some $k \in \mathbb{N} \setminus \{0\}$, or $\psi^{=}_{\geq f_1(k)} \wedge \psi^{\neq}_{\geq f_0(k)}$ for all $k \in \mathbb{N} \setminus \{0\}$, in which case $\mathcal{A}$ is infinite; if our model $\mathcal{A}$ is finite, it must then satisfy $\psi^{=}_{=f_1(k)} \wedge \psi^{\neq}_{=f_0(k)}$, for some $k \in \mathbb{N} \setminus \{0\}$, and therefore have exactly $f_1(k)$ elements satisfying $s^{\mathcal{A}}(a) = a$, and $f_0(k)$ elements satisfying $s^{\mathcal{A}}(a) \neq a$. Since an element a of $\mathcal{A}$ must satisfy either $s^{\mathcal{A}}(a) = a$ or $s^{\mathcal{A}}(a) \neq a$ and never both of them, we have that $\mathcal{A}$ must have precisely $f_1(k) + f_0(k) = k$ elements, and so a model $\mathcal{A}$ of $\mathcal{T}_f$ has k elements in its domain iff it contain $f_1(k)$ elements satisfying $s^{\mathcal{A}}(a) = a$, and $f_0(k)$ elements satisfying $s^{\mathcal{A}}(a) \neq a$.

So, let us prove the implication from left to right in the biconditional $f(n+1) = 1 \Leftrightarrow$ *mincard*$(\phi_n) = n+1$. If $\mathcal{A}$ is a model of $\mathcal{T}_f$ that satisfies ϕ_n and has minimal (finite, because of Lemma 60) cardinality *mincard*(ϕ_n) among the models of $\mathcal{T}_f$ that satisfy this formula, we have that $\mathcal{A}$ has at least $f_1(n)+1$ elements a that satisfy $s^{\mathcal{A}}(a) = a$ (because $\mathcal{A}$ satisfies ϕ_n), meaning $f_1(\mathit{mincard}(\phi_n)) \geq f_1(n)+1$. Since we are assuming $f(n+1) = 1$, $f_1(n+1) = f_1(n)+1$, and since any model of $\mathcal{T}_f$ with $n+1$ elements must have $f_1(n+1) = f_1(n)+1$ elements satisfying $s^{\mathcal{A}}(a) = a$, and thus actually satisfy ϕ_n, we have $n+1 \geq \mathit{mincard}(\phi_n)$. If *mincard*$(\phi_n)$ were strictly less than $n+1$, we would get $n \geq \mathit{mincard}(\phi_n)$, and since f_1 is non-decreasing, $f_1(n) \geq f_1(\mathit{mincard}(\phi_n)) \geq f_1(n)+1$, what is absurd: we must have instead *mincard*$(\phi_n) = n+1$.

Reciprocally, assume *mincard*$(\phi_n) = n+1$, and we know that some model $\mathcal{A}$ of $\mathcal{T}_f$ with $n+1$ elements satisfies ϕ_n, and therefore has at least $f_1(n)+1$ elements that satisfy $s^{\mathcal{A}}(a) = a$, from what follows that $f_1(n+1) \geq f_1(n)+1$, and thus the two values are equal (since $f_1(n)$ and $f_1(n+1)$ can only differ by 0 or 1). So $f(n+1) = 1$.

To summarize, were $\mathcal{T}_f$ to have a computable *mincard* function, by knowing the values of $f(1), \ldots, f(n)$ (and therefore of $f_0(n)$ and $f_1(n)$), we would be able to calculate $f(n+1)$ algorithmically as well, what is absurd. □

Lemma 62 *Every quantifier-free Σ_s-formula that is satisfiable is $\mathcal{T}_f$-satisfiable.*

Proof If the quantifier-free formula ϕ is satisfiable, then it must be satisfied by some $\mathcal{T}$-interpretation $\mathcal{A}$, where $\mathcal{T}$ is the theory with all Σ_s-structures as models, axiomatized by the empty set. Take then enumerable sets A and B disjoint from $\sigma_1^{\mathcal{A}}$ and each other, and build the interpretation $\mathcal{B}$ with: $\sigma_1^{\mathcal{B}} = \sigma_1^{\mathcal{A}} \cup A \cup B$;

$$\sigma_1^{\mathcal{B}}(a) = \begin{cases} s^{\mathcal{A}}(a) & \text{if } a \in \sigma_1^{\mathcal{A}}; \\ a & \text{if } a \in A; \\ \text{any element of } A & \text{if } a \in B; \end{cases}$$

and $x^{\mathcal{B}} = x^{\mathcal{A}}$ for all variables x. It is obvious $\mathcal{B}$ is a $\mathcal{T}_f$-interpretation, since it has infinite elements satisfying each condition, $s^{\mathcal{B}}(a) = a$ or $s^{\mathcal{B}}(a) \neq a$ (respectively, all of those in A and B). Furthermore, for any term α in ϕ, $\alpha^{\mathcal{B}} = \alpha^{\mathcal{A}}$, since $x^{\mathcal{B}} = x^{\mathcal{A}}$ for every variable x, and, assuming as induction hypothesis that $(s^j(x))^{\mathcal{B}} = (s^j(x))^{\mathcal{A}}$ for some j,

$$(s^{j+1}(x))^{\mathcal{B}} = s^{\mathcal{B}}\big((s^j(x))^{\mathcal{B}}\big) = s^{\mathcal{A}}\big((s^j(x))^{\mathcal{A}}\big) = (s^{j+1}(x))^{\mathcal{A}}.$$

Since ϕ is a quantifier-free formula in a signature without predicates, we get that ϕ is satisfied by the $\mathcal{T}_f$-interpretation $\mathcal{B}$ (given that is satisfied by $\mathcal{A}$). □

Note that Lemma 62 does not hold for quantified formulas in general. For example, the formula $\forall x.\, s(x) \neq x$ is satisfiable but not $\mathcal{T}_f$-satisfiable: because $f(1) = 1$, every $\mathcal{T}_f$-interpretation $\mathcal{A}$ must have at least one element a with $s^{\mathcal{A}}(a) = a$.

Lemma 63 *$\mathcal{T}_f$ is smooth, and thus stably-infinite.*

Proof Take a quantifier-free formula ϕ, a $\mathcal{T}_f$-interpretation $\mathcal{A}$ that satisfies ϕ, and a cardinal $\kappa \geq |\sigma_1^{\mathcal{A}}|$. If κ is infinite, we take two sets A and B with κ elements, disjoint from each other and $\sigma_1^{\mathcal{A}}$, and define an interpretation $\mathcal{B}$ as follows: $\sigma_1^{\mathcal{B}} = \sigma_1^{\mathcal{A}} \cup A \cup B$; $s^{\mathcal{B}}$ equal to $s^{\mathcal{A}}$ when restricted to $\sigma_1^{\mathcal{A}}$, equal to the identity when restricted to A, and for $a \in B$ we only require that $s^{\mathcal{B}}(a) \neq a$, the specific value of the function at this element being irrelevant; and, for all variables x, $x^{\mathcal{B}} = x^{\mathcal{A}}$.

It is clear that $\mathcal{B}$ is a $\mathcal{T}_f$-interpretation since it has infinitely many elements satisfying $s^{\mathcal{B}}(a) = a$ (namely, all those in A) and infinitely many elements that satisfy $s^{\mathcal{B}}(a) \neq a$ (those in B); furthermore, $|\sigma_1^{\mathcal{B}}| = |\sigma_1^{\mathcal{A}}| + |A| + |B| = \kappa$, as $|A| = |B| = \kappa$ and κ is infinite. Finally, $\mathcal{B}$ validates ϕ since the atomic formulas in the signature Σ_s are of the form $s^i(x) = s^j(y)$, for $i, j \in \mathbb{N}$, and these maintain their truth value in $\mathcal{B}$ if $x, y \in vars(\phi)$ since $s^{\mathcal{B}}$ equals $s^{\mathcal{A}}$ when restricted to $\sigma_1^{\mathcal{A}}$.

Now, assume that κ, and so $|\sigma_1^{\mathcal{A}}|$ as well, is finite, and let $\kappa = m$ and $|\sigma_1^{\mathcal{A}}| = n$. We consider two sets A and B disjoint from $\sigma_1^{\mathcal{A}}$ with, respectively, $f_1(m) - f_1(n)$ and $f_0(m) - f_0(n)$ elements, and define an interpretation $\mathcal{B}$ as follows: $\sigma_1^{\mathcal{B}} = \sigma_1^{\mathcal{A}} \cup A \cup B$; $s^{\mathcal{B}}$ as equal to $s^{\mathcal{A}}$ when restricted to $\sigma_1^{\mathcal{A}}$, as equal to the identity when restricted to A, and for $a \in B$ we only request $s^{\mathcal{B}}(a) \neq a$; and $x^{\mathcal{B}} = x^{\mathcal{A}}$ for every variable x. It is easy to see that $\mathcal{B}$ is then a $\mathcal{T}_f$-interpretation: it has

$$n + (f_1(m) - f_1(n)) + (f_0(m) - f_0(n)) = m$$

elements in its domain; $f_1(n)$ elements in $\sigma_1^{\mathcal{A}}$, and all $f_1(m) - f_1(n)$ elements in A (remember A and $\sigma_1^{\mathcal{A}}$ are disjoint), satisfy $s^{\mathcal{B}}(a) = a$, to a total of $f_1(m)$; and $f_0(n)$ elements in $\sigma_1^{\mathcal{A}}$, plus all $f_0(m) - f_0(n)$ elements in B (remember B and $\sigma_1^{\mathcal{A}}$ are disjoint), satisfy $s^{\mathcal{B}}(a) \neq a$, to a total of $f_0(m)$. Furthermore, since $\mathcal{B}$ and $\mathcal{A}$ agree on the interpretation of the variables in $vars(\phi)$ and the value given by s to the elements that may occur in ϕ, ϕ must be true in $\mathcal{B}$. □

Lemma 64 *$\mathcal{T}_f$ is finitely witnessable.*

Proof For a quantifier-free formula ϕ, consider the witness

$$wit(\phi) = \phi \wedge \bigwedge_{i=1}^{n} \bigwedge_{j=0}^{M_i+1} y_{i,j} = s^j(z_i) \wedge \bigwedge_{i=1}^{2^{k+1}} x_i = x_i,$$

for: $vars(\phi) = \{z_1, \ldots, z_n\}$; M_i the maximum of the indexes j such that the term $s^j(z_i)$ appears in ϕ; 2^k the smallest power of two equal to or larger than $2M$, where $M = \sum_{i=1}^{n}(M_i + 2)$; and x_i and $y_{i,j}$ fresh variables.[4] One can easily convince themselves that $wit(\phi)$ is computable, and it is obvious that ϕ and $\exists \overrightarrow{x}.\, wit(\phi)$ are $\mathcal{T}_f$-equivalent, for $\overrightarrow{x} = vars(wit(\phi)) \setminus vars(\phi)$, since: if ϕ is true in a $\mathcal{T}_f$-interpretation $\mathcal{A}$, and we change the assignment on $\mathcal{A}$ so that $y_{i,j}^{\mathcal{A}'} = (s^{\mathcal{A}})^j(z_i^{\mathcal{A}})$, it is clear that $wit(\phi)$ is satisfied by $\mathcal{A}'$; and if $\exists \overrightarrow{x}.\, wit(\phi)$ is satisfied by a $\mathcal{T}_f$-interpretation $\mathcal{A}$, given that the variables in $\overrightarrow{x}$ do not occur in ϕ,

$$\exists \overrightarrow{x}.\, wit(\phi) \quad \text{and} \quad \phi \wedge \exists \overrightarrow{x}.\, \Big[\bigwedge_{i=1}^{n} \bigwedge_{j=0}^{M_i+1} y_{i,j} = s^j(z_i) \wedge \bigwedge_{i=1}^{2^{k+1}} x_i = x_i\Big]$$

are equivalent, and thus $\mathcal{A}$ satisfies ϕ.

[4] Notice the double conjunction on $wit(\phi)$ is the flattening of ϕ.

So, assume that the $\mathcal{T}_f$-interpretation $\mathcal{A}$ satisfies $wit(\phi)$. Let: $m_1 < M$ be the number of terms $s^j(z_i)$ (for $1 \leq i \leq n$ and $0 \leq j \leq M_i$) appearing in ϕ with $(s^{\mathcal{A}})^{j+1}(z_i^{\mathcal{A}}) = (s^{\mathcal{A}})^j(z_i^{\mathcal{A}})$; $m_0 < M$ be the number of such terms $s^j(z_i)$ satisfying instead $(s^{\mathcal{A}})^{j+1}(z_i^{\mathcal{A}}) \neq (s^{\mathcal{A}})^j(z_i^{\mathcal{A}})$; and $m_1^* \leq n$ be the number of elements $(s^{\mathcal{A}})^{M_i+1}(z_i^{\mathcal{A}})$ that are not in the set $\{s^j(z_i) : 1 \leq i \leq n, 0 \leq j \leq M_i\}^{\mathcal{A}}$. We then have that $m_0 + m_1 + m_1^* \leq M < 2^k$, and so we can take sets A and B disjoint from $\sigma_1^{\mathcal{A}}$ and each other with, respectively, $2^k - m_1 - m_1^*$ and $2^k - m_0$ elements. Finally, we define a $\mathcal{T}_f$-interpretation $\mathcal{B}$, starting by setting its domain to

$$\sigma_1^{\mathcal{B}} = \{s^j(z_i) : 1 \leq i \leq n, 0 \leq j \leq M_i + 1\}^{\mathcal{A}} \cup A \cup B.$$

To define $s^{\mathcal{B}}$, we take an element $a_0 \in \sigma_1^{\mathcal{A}}$ (and there is at least one), and make

$$s^{\mathcal{B}}(a) = \begin{cases} s^{\mathcal{A}}(a) & \text{if } a = (s^{\mathcal{A}})^j(z_i^{\mathcal{A}}) \text{ for } 1 \leq i \leq n \text{ and } 0 \leq j \leq M_i; \\ a & \text{if } a = (s^{\mathcal{A}})^{M_i+1}(z_i^{\mathcal{A}}), \text{ for } 1 \leq i \leq n, \\ & \text{and } a \neq (s^{\mathcal{A}})^j(z_l^{\mathcal{A}}) \text{ for any } 1 \leq l \leq n \text{ and } 0 \leq j \leq M_i; \\ a & \text{if } a \in A; \\ a_0 & \text{if } a \in B; \end{cases}$$

notice that the first two cases are disjoint and cover all of $\{s^j(z_i) : 1 \leq i \leq n, 0 \leq j \leq M_i + 1\}^{\mathcal{A}}$, while the two last are disjoint from each other and from the previous ones (since A and B are disjoint from each other and from $\sigma_1^{\mathcal{A}}$), and finish covering $\sigma_1^{\mathcal{B}}$. To finish defining $\mathcal{B}$, we make $x^{\mathcal{B}} = x^{\mathcal{A}}$ for every variable in ϕ, $y_{i,j}^{\mathcal{B}} = (s^{\mathcal{B}})^j(z_i^{\mathcal{B}})$, and $x \mapsto x^{\mathcal{B}}$ a bijection between $\{x_1, \ldots, x_{2^{k+1}}\}$ and $\sigma_1^{\mathcal{B}}$, what is possible given both sets have 2^{k+1} elements (and arbitrarily for other variables). To see that indeed $|\sigma_1^{\mathcal{B}}| = 2^{k+1}$, notice the set $\alpha(\phi)^{\mathcal{A}}$, for $\alpha(\phi) = \{s^j(z_i) : 1 \leq i \leq n, 0 \leq j \leq M_i\}$ has m_0+m_1 elements, from the definition of m_0 as the number of elements a in $\alpha(\phi)^{\mathcal{A}}$ satisfying $s^{\mathcal{A}}(a) = a$, and of m_1 as the number of elements a in $\alpha(\phi)^{\mathcal{A}}$ satisfying instead $s^{\mathcal{A}}(a) \neq a$; therefore, $\{s^j(z_i) : 1 \leq i \leq n, 0 \leq j \leq M_i + 1\}^{\mathcal{A}}$ has $m_0 + m_1 + m_1^*$ elements, from the definition of m_1^* as being the number of elements of the form $(s^{\mathcal{A}})^{M_i+1}(z_i^{\mathcal{A}})$ which are not in $\alpha(\phi)^{\mathcal{A}}$. Using A has $2^k - m_1 - m_1^*$ elements, and B has $2^k - m_0$, we obtain the aforementioned total of 2^{k+1}.

$\mathcal{B}$ is a $\mathcal{T}_f$-interpretation since: it has 2^{k+1} elements in its domain; half of these elements satisfy $s^{\mathcal{B}}(a) = a$, explicitly m_1 of those in $\alpha(\phi)^{\mathcal{A}}$ that satisfy $s^{\mathcal{A}}(a) = a$, all m_1^* elements $(s^{\mathcal{A}})^{M_i+1}(z_i^{\mathcal{A}})$ that are not in $\alpha(\phi)^{\mathcal{A}}$, and all those $2^k - m_1 - m_1^*$ elements in A; and half satisfying $s^{\mathcal{A}}(a) \neq a$, explicitly m_0 of those in $\alpha(\phi)^{\mathcal{A}}$, and $2^k - m_0$ more elements in B. Since f is defined so that $f_1(2^k) = f_0(2^k)$ and $f(m) = f_0(m) + f_1(m)$ for all $k, m \in \mathbb{N} \setminus \{0\}$, it is then clear that $\mathcal{B}$ is indeed a $\mathcal{T}_f$-interpretation.

Furthermore, any term α (necessarily of the form $s^j(z_i)$ with $1 \leq i \leq n$ and $0 \leq j \leq M_i$) that appears in ϕ receives the same value in either $\mathcal{A}$ or $\mathcal{B}$, what we now prove by induction. Indeed, for any variable z_i that appears in ϕ, $z_i^{\mathcal{B}} = z_i^{\mathcal{A}}$ (and thus $(s^{\mathcal{B}})^0(z_i^{\mathcal{B}}) = (s^{\mathcal{A}})^0(z_i^{\mathcal{A}})$); and then, for each $0 \leq j < M_i$, assuming as induction hypothesis that $(s^{\mathcal{B}})^j(z_i^{\mathcal{B}}) = (s^{\mathcal{A}})^j(z_i^{\mathcal{A}})$ (call that element a, for convenience), by the definition of $s^{\mathcal{B}}$ we have $s^{\mathcal{B}}(a) = s^{\mathcal{A}}(a)$, and so

$$(s^{\mathcal{B}})^{j+1}(z_i^{\mathcal{B}}) = s^{\mathcal{B}}\big((s^{\mathcal{B}})^j(z_i^{\mathcal{B}})\big) = s^{\mathcal{A}}\big((s^{\mathcal{A}})^j(z_i^{\mathcal{A}})\big) = (s^{\mathcal{A}})^{j+1}(z_i^{\mathcal{A}}).$$

Since the underlying signature has no predicates, we have that all atomic formulas of ϕ receive the same truth-value in either $\mathcal{A}$ or $\mathcal{B}$; since ϕ has no quantifiers and is satisfied by $\mathcal{A}$, this means that ϕ is also satisfied by $\mathcal{B}$. We also have that $\mathcal{B}$ satisfies $wit(\phi)$, since $y_{i,j}^{\mathcal{B}} = (s^{\mathcal{B}})^j(z_i^{\mathcal{B}})$. Finally, $vars(wit(\phi))^{\mathcal{B}} = \sigma_1^{\mathcal{B}}$, since $\{x_1, \ldots, x_{2^{k+1}}\}^{\mathcal{B}} \subseteq vars(wit(\phi))^{\mathcal{B}}$ and, given that $x \mapsto x^{\mathcal{B}}$ is a bijection between $\{x_1, \ldots, x_{2^{k+1}}\}$ and $\sigma_1^{\mathcal{B}}$, $\{x_1, \ldots, x_{2^{k+1}}\}^{\mathcal{B}} = \sigma_1^{\mathcal{B}}$, proving that *wit* is indeed a witness. □

Lemma 65 *$\mathcal{T}_f$ is not strongly finitely witnessable.*

Proof Suppose *wit* is a strong witness; we start by proving that, given a quantifier-free formula ϕ,

$$mincard(\phi) = \min\{|V/E| : E \in Eq(V) \text{ and } wit(\phi) \wedge \delta_V^E \text{ is } \mathcal{T}_f - \text{satisfiable}\},$$

where $Eq(V)$ is the set of all equivalence relations E on $V = vars(wit(\phi))$, being the corresponding arrangements denoted by δ_V^E. To prove this identity, suppose $\mathcal{A}$ is a $\mathcal{T}_f$-interpretation that satisfies ϕ with minimal cardinality of the domain, and so $|\sigma_1^{\mathcal{A}}| = mincard(\phi)$; because ϕ and $\exists \overrightarrow{x}\,.\, wit(\phi)$ are $\mathcal{T}_f$-equivalent (for $\overrightarrow{x} = vars(wit(\phi)) \setminus vars(\phi)$), $\mathcal{A}$ also satisfies $\exists \overrightarrow{x}\,.\, wit(\phi)$, and by changing the value given by $\mathcal{A}$ to the variables in $\overrightarrow{x}$, we obtain a new $\mathcal{T}_f$-interpretation $\mathcal{A}'$ that satisfies $wit(\phi)$ and has the same underlying structure as $\mathcal{A}$.

Let E be the equivalence on V such that xEy iff $x^{\mathcal{A}'} = y^{\mathcal{A}'}$, and we have that $\mathcal{A}'$ satisfies $wit(\phi) \wedge \delta_V^E$ and $|V/E| \leq |\sigma_1^{\mathcal{A}'}|$; since *wit* is supposed to be a strong witness, there must then exist a $\mathcal{T}_f$-interpretation $\mathcal{B}$ that satisfies $wit(\phi) \wedge \delta_V^E$ with $\sigma_1^{\mathcal{B}} = V^{\mathcal{B}}$, and so $|\sigma_1^{\mathcal{B}}| = |V/E|$. But, again since ϕ and $\exists \overrightarrow{x}\,.\, wit(\phi)$ are $\mathcal{T}_f$-equivalent, $\mathcal{B}$ also satisfies $\exists \overrightarrow{x}\,.\, wit(\phi)$ and therefore ϕ, meaning that $|\sigma_1^{\mathcal{B}}| \geq |\sigma_1^{\mathcal{A}}| = |\sigma_1^{\mathcal{A}'}|$. With all of that, $|\sigma_1^{\mathcal{B}}| \geq |\sigma_1^{\mathcal{A}'}| \geq |V/E| = |\sigma_1^{\mathcal{B}}|$, implying all are equal and thus $|V/E| = |\sigma_1^{\mathcal{A}}| = mincard(\phi)$. Of course, we then get

$$mincard(\phi) \in \{|V/E| : E \in Eq(V) \text{ and } wit(\phi) \wedge \delta_V^E \text{ is } \mathcal{T}_f - \text{satisfiable}\},$$

so now suppose that there exists an equivalence E on V such that $wit(\phi) \wedge \delta_V^E$ is $\mathcal{T}_f$-satisfiable, but $|V/E| < mincard(\phi)$, and we shall reach a contradiction. Since $wit(\phi) \wedge \delta_V^E$ is $\mathcal{T}_f$-satisfiable and *wit* is a strong witness, there exists a $\mathcal{T}_f$-interpretation $\mathcal{A}$ that satisfies $wit(\phi) \wedge \delta_V^E$ with $\sigma_1^{\mathcal{A}} = V^{\mathcal{A}}$, and so $|\sigma_1^{\mathcal{A}}| = |V/E|$. But, since $\mathcal{A}$ also satisfies $\exists \overrightarrow{x}\,.\, wit(\phi)$, and ϕ and $\exists \overrightarrow{x}\,.\, wit(\phi)$ are $\mathcal{T}_f$-equivalent, $\mathcal{A}$ satisfies ϕ, contradicting the fact that the smallest $\mathcal{T}_f$-interpretation to satisfy ϕ has domain of cardinality $mincard(\phi) > |V/E| = |\sigma_1^{\mathcal{A}}|$. So our identity for $mincard(\phi)$ is true.

But the right side of the identity is indeed computable: in fact, finding the set $Eq(V)$ is trivial; testing whether $wit(\phi) \wedge \delta_V^E$ is $\mathcal{T}_f$-satisfiable is also decidable, since it is equivalent to testing whether the same formula is not contradictory according to Lemma 62; and finding the number of equivalence classes of E is also straightforward. Of course, this contradicts the fact that $\mathcal{T}_f$ does not have a computable *mincard* function, as proven in Lemma 61, proving that this theory is not strongly finitely witnessable. □

Lemma 66 *$\mathcal{T}_f$ is convex.*

Proof Assume that ϕ is a cube and $\vdash_{\mathcal{T}_f} \phi \rightarrow \bigvee_{i=1}^{n} x_i = y_i$, but at the same time $\mathcal{T}_f$ is not convex and so $\nvdash_{\mathcal{T}_f} \phi \rightarrow x_i = y_i$ for every $1 \leq i \leq n$, meaning we can find $\mathcal{T}_f$-interpretations $\mathcal{A}_i$ that satisfy ϕ and $\neg(x_i = y_i)$. We can conclude, then, that $\vdash_{\mathcal{T}} \phi \rightarrow \bigvee_{i=1}^{n} x_i = y_i$, where $\mathcal{T}$ is the theory with all Σ_s-structures as models, axiomatized by the empty set: if this were not true, then there would exist a $\mathcal{T}$-interpretation $\mathcal{A}$ satisfying ϕ but not $\bigvee_{i=1}^{n} x_i = y_i$, therefore implying $\phi \wedge \neg \bigvee_{i=1}^{n} x_i = y_i$ is $\mathcal{T}$-satisfiable, and thus non-contradictory and $\mathcal{T}_f$-satisfiable, given Lemma 62, something absurd given our assumptions. However, given that Theorem 6 states $\mathcal{T}$ is convex, we must have $\vdash_{\mathcal{T}} \phi \rightarrow x_j = y_j$, for some $1 \leq j \leq n$, and so $\phi \wedge \neg(x_j = y_j)$ is $\mathcal{T}$-unsatisfiable, itself contradicting the fact that $\mathcal{A}_j$ is a $\mathcal{T}_f$-interpretation (and so a $\mathcal{T}$-interpretation as well) that satisfies ϕ and $\neg(x_j = y_j)$. □

A.3.3 $\mathcal{T}_f^{\vee}$

We start by proving the analogue of Lemma 62 for $\mathcal{T}_f^{\vee}$, using some of the same notations found in that proof.

Theorem 67 *Given a quantifier-free formula ϕ in the signature Σ_s with $vars(\phi) = \{w_1, \ldots, w_n\}$, if*

$$\overline{\phi} = \phi \wedge \bigwedge_{i=1}^{n} s^2(w_i) = w_i \vee s^2(w_i) = s(w_i)$$

is satisfiable, then ϕ is $\mathcal{T}_f^{\vee}$-satisfiable.

Proof Suppose $\overline{\phi}$ is satisfiable, and from Lemma 62 there exists a $\mathcal{T}_f$-interpretation $\mathcal{A}$ that satisfies $\overline{\phi}$. We produce a $\mathcal{T}_f^{\vee}$-interpretation $\mathcal{B}$ as follows: $\sigma_1^{\mathcal{B}} = \sigma_1^{\mathcal{A}}$; if a is neither in $vars(\phi)^{\mathcal{A}}$ nor in the image of this set under $s^{\mathcal{A}}$ and $s^{\mathcal{A}}(a) \neq a$, we make $s^{\mathcal{B}}(a)$ equal to any element $b \in \sigma_1^{\mathcal{A}}$ such that $s^{\mathcal{A}}(b) = b$ (there must be one given $\mathcal{A}$ is a $\mathcal{T}_f$-interpretation), and otherwise $s^{\mathcal{B}}$ equals $s^{\mathcal{A}}$; and, for all variables x, $x^{\mathcal{B}} = x^{\mathcal{A}}$. Since the interpretation of all variables in $\mathcal{B}$ is the same as in $\mathcal{A}$, for every variable w_i of $\overline{\phi}$ one has $w_i^{\mathcal{B}} = w_i^{\mathcal{A}}$; and since $s^{\mathcal{B}}$ agrees with $s^{\mathcal{A}}$ in

$$vars(\phi)^{\mathcal{A}} \cup \{s^{\mathcal{A}}(a) : a \in vars(\phi)^{\mathcal{A}}\},$$

and for every $j \in \mathbb{N}$ one finds that $(s^{\mathcal{A}})^j(w_i^{\mathcal{A}})$ equals either $w_i^{\mathcal{A}}$ or $s^{\mathcal{A}}(w_i^{\mathcal{A}})$ because $\mathcal{A}$ satisfies $\bigwedge_{i=1}^{n} s^2(w_i) = w_i \vee s^2(w_i) = s(w_i)$, we get that for any term α in $\overline{\phi}$, $\alpha^{\mathcal{B}} = \alpha^{\mathcal{A}}$. Since Σ_s does not have predicates (other than equality) and $\overline{\phi}$ is quantifier-free, we reach the conclusion that $\mathcal{B}$ clearly satisfies $\overline{\phi}$, and so ϕ.

It is less clear, however, that $\mathcal{B}$ is indeed a $\mathcal{T}_f^{\vee}$-interpretation. We start by noticing $\mathcal{B}$ is at least a $\mathcal{T}_f$- interpretation, since $s^{\mathcal{B}}(a) = a$ iff $s^{\mathcal{A}}(a) = a$: indeed, begin by assuming that $s^{\mathcal{B}}(a) = a$. We cannot have that a is neither in $vars(\phi)^{\mathcal{A}}$ nor in $\{s^{\mathcal{A}}(a) : a \in vars(\phi)^{\mathcal{A}}\}$ and satisfies $s^{\mathcal{A}}(a) \neq a$, since in that case $s^{\mathcal{B}}(a) = b$ for an element b such that $s^{\mathcal{A}}(b) = b$ (which must necessarily be different from a, since $s^{\mathcal{A}}(a) \neq a$); so $s^{\mathcal{B}}$ must coincide with $s^{\mathcal{A}}$, implying that $s^{\mathcal{A}}(a) = a$. Reciprocally, assume $s^{\mathcal{A}}(a) = a$: then we are not in the case that $s^{\mathcal{A}}(a) \neq a$ and so $s^{\mathcal{B}}$ must coincide with $s^{\mathcal{A}}$, meaning $s^{\mathcal{B}}(a) = a$, as we wished to show.

Furthermore:

1. if $a \in vars(\phi)^{\mathcal{A}}$ or $a \in \{s^{\mathcal{A}}(a) : a \in vars(\phi)^{\mathcal{A}}\}$, by the fact that $\mathcal{B}$ satisfies $\overline{\phi}$ we get that either $s^{\mathcal{B}}(s^{\mathcal{B}}(a)) = a$ or $s^{\mathcal{B}}(s^{\mathcal{B}}(a)) = s^{\mathcal{B}}(a)$;
2. if a is neither in $vars(\phi)^{\mathcal{A}}$ nor in $\{s^{\mathcal{A}}(a) : a \in vars(\phi)^{\mathcal{A}}\}$ and $s^{\mathcal{A}}(a) = a$, $s^{\mathcal{B}}(a) = s^{\mathcal{A}}(a) = a$;
3. and if a is neither in $vars(\phi)^{\mathcal{A}}$ nor in $\{s^{\mathcal{A}}(a) : a \in vars(\phi)^{\mathcal{A}}\}$ and $s^{\mathcal{A}}(a) \neq a$, there is a $b \in \sigma_1^{\mathcal{A}}$ with $s^{\mathcal{A}}(b) = b$ such that $s^{\mathcal{B}}(a) = b$, and therefore $s^{\mathcal{B}}(s^{\mathcal{B}}(a)) = s^{\mathcal{B}}(a)$,

proving $\mathcal{B}$ is indeed a $\mathcal{T}_f^{\vee}$-interpretation. □

$$Ax(\mathcal{T}_f) \cup \{\psi_{\vee}\} \qquad \textbf{(Axiomatization:)}$$

Lemma 68 *The mincard function of $\mathcal{T}_f^{\vee}$ is not computable.*

Proof The proof is the same as the one of Lemma 61. □

Lemma 69 *$\mathcal{T}_f^{\vee}$ is smooth, and thus stably-infinite.*

Proof We can slightly adapt the proof of Lemma 63: the only difference in the proofs is that now one must require that $s^{\mathcal{B}}$ maps elements of B into those of A, so that $s^{\mathcal{B}}(s^{\mathcal{B}}(a)) = s^{\mathcal{B}}(a)$. □

Lemma 70 *$\mathcal{T}_f^{\vee}$ is finitely witnessable.*

Proof Take a quantifier-free formula ϕ and consider the witness

$$wit(\phi) = \phi \wedge \bigwedge_{i=1}^{n} y_i = s(w_i) \wedge z_i = s(y_i) \wedge \bigwedge_{i=1}^{2^{k+1}} x_i = x_i,$$

where $vars(\phi) = \{w_1, \ldots, w_n\}$, 2^k is the smallest power of two greater than $2n$, and x_i, y_i and z_i are fresh variables. If a $\mathcal{T}_f^{\vee}$-interpretation $\mathcal{A}$ satisfies ϕ, by making $y_i^{\mathcal{A}'} = s^{\mathcal{A}}(w_i^{\mathcal{A}})$ and $z_i^{\mathcal{A}'} = s^{\mathcal{A}}(y_i^{\mathcal{A}'})$, we obtain a second $\mathcal{T}_f^{\vee}$-interpretation $\mathcal{A}'$ that satisfies $wit(\phi)$, thus implying that, for $\overrightarrow{x} = vars(wit(\phi)) \setminus vars(\phi)$, ϕ and $\exists \overrightarrow{x}\,.\, wit(\phi)$ are $\mathcal{T}_f^{\vee}$-equivalent.

Now, suppose that the $\mathcal{T}_f^{\vee}$-interpretation $\mathcal{A}$ satisfies $wit(\phi)$, and let

$$V = \{w_i : 1 \le i \le n\} \cup \{y_i : 1 \le i \le n\} \cup \{z_i : 1 \le i \le n\}.$$

Let m_1 be the cardinality of $\{a \in V^{\mathcal{A}} : s^{\mathcal{A}}(a) = a\}$ and $m_0 = |V^{\mathcal{A}}| - m_1$, and then $2^k > \max\{m_0, m_1\}$: this is because either $z_i^{\mathcal{A}} = w_i^{\mathcal{A}}$ or $z_i^{\mathcal{A}} = y_i^{\mathcal{A}}$ for each $1 \le i \le n$, implying $|V^{\mathcal{A}}| \le 2n$, and thus $m_0, m_1 \le 2n$. Given sets A and B, disjoint from $\sigma_1^{\mathcal{A}}$, with respectively $2^k - m_1$ and $2^k - m_0$ elements, we define a new $\mathcal{T}_f^{\vee}$-interpretation $\mathcal{B}$ by making:

i. $\sigma_1^{\mathcal{B}} = V^{\mathcal{A}} \cup A \cup B$ (which has 2^{k+1} elements);
ii. $s^{\mathcal{B}}$ equal to $s^{\mathcal{A}}$ when restricted to $V^{\mathcal{A}}$ (what is well-defined, since $s^{\mathcal{A}}(w_i^{\mathcal{A}}) = y_i^{\mathcal{A}}$, $s^{\mathcal{A}}(y_i^{\mathcal{A}}) = z_i^{\mathcal{A}}$, and $z_i^{\mathcal{A}}$ equals either $w_i^{\mathcal{A}}$ or $y_i^{\mathcal{A}}$), equal to the identity when restricted to A, and equal to any function from B to A when restricted to B (this way, all elements of A satisfy $s^{\mathcal{B}}(a) = a$, while all of B satisfy $s^{\mathcal{B}}(a) \neq a$ and $s^{\mathcal{B}}(s^{\mathcal{B}}(a)) = s^{\mathcal{B}}(a)$);
iii. and $x^{\mathcal{B}} = x^{\mathcal{A}}$ for any variable in V, $x_i \mapsto x_i^{\mathcal{B}}$ a bijection between $\{x_1, \ldots, x_{2^{k+1}}\}$ and $\sigma_1^{\mathcal{B}}$, and arbitrarily otherwise.

Now we prove that $\mathcal{B}$ is a $\mathcal{T}_f^{\vee}$-interpretation: it has 2^{k+1} elements; 2^k of them, specifically m_1 in $\{a \in V^{\mathcal{A}} : s^{\mathcal{A}}(a) = a\}$ and another $2^k - m_1$ in A, satisfy $s^{\mathcal{B}}(a) = a$; and another 2^k, specifically m_0 in $V^{\mathcal{A}} \setminus \{a \in V^{\mathcal{A}} : s^{\mathcal{A}}(a) = a\}$ and $2^k - m_1$ in B, satisfy $s^{\mathcal{B}}(a) \neq a$ instead; so $\mathcal{B}$ is at least a $\mathcal{T}_f$-interpretation. Furthermore: for any element a of $V^{\mathcal{A}}$, $s^{\mathcal{B}}(a) = s^{\mathcal{A}}(a)$ and $s^{\mathcal{B}}(s^{\mathcal{B}}(a)) = s^{\mathcal{A}}(s^{\mathcal{A}}(a))$, and we already have either $s^{\mathcal{A}}(s^{\mathcal{A}}(a)) = a$ or $s^{\mathcal{A}}(s^{\mathcal{A}}(a)) = s^{\mathcal{A}}(a)$; for any $a \in A$, $s^{\mathcal{B}}(a) = a$, and so $s^{\mathcal{B}}(s^{\mathcal{B}}(a)) = a$; and for any $a \in B$, $s^{\mathcal{B}}(a) \in A$, meaning $s^{\mathcal{B}}(s^{\mathcal{B}}(a)) = s^{\mathcal{B}}(a)$; so $\mathcal{B}$ also satisfies $\psi_{\vee}$, and is therefore a $\mathcal{T}_f^{\vee}$-interpretation.

Now, let α be a term in ϕ, necessarily of the form $s^j(w_i)$: we know $w_i^{\mathcal{A}}$ and $s^{\mathcal{A}}(w_i^{\mathcal{A}})$ are in $V^{\mathcal{A}}$, and therefore so are $(s^j(w_i))^{\mathcal{A}}$ for all $j \in \mathbb{N}$, since $(s^{\mathcal{A}})^j(w_i^{\mathcal{A}})$ must equal either $w_i^{\mathcal{A}}$ or $s^{\mathcal{A}}(w_i^{\mathcal{A}})$; since $s^{\mathcal{B}}$ coincides with $s^{\mathcal{A}}$ on $V^{\mathcal{A}}$, and $w_i^{\mathcal{B}} = w_i^{\mathcal{A}}$, we have that $\alpha^{\mathcal{B}} = \alpha^{\mathcal{A}}$. Given that $wit(\phi)$ is quantifier and predicate-free, we have that it must receive the same value in either $\mathcal{A}$ or $\mathcal{B}$, and is therefore true in $\mathcal{B}$; furthermore, $x \mapsto x^{\mathcal{B}}$ is a bijection between $\{x_1, \ldots, x_{2^{k+1}}\}$ and $\sigma_1^{\mathcal{B}}$, and since $\{x_1, \ldots, x_{2^{k+1}}\} \subseteq vars_{\sigma_1}(wit(\phi))$, we have $vars_{\sigma_1}(wit(\phi))^{\mathcal{B}} = \sigma_1^{\mathcal{B}}$. Hence wit is indeed a witness. □

Lemma 71 *$\mathcal{T}_f^{\vee}$ is not strongly finitely witnessable.*

Proof We proceed as in Lemma 65, showing that

$$mincard(\phi) = \min\{|V/E| : E \in Eq(V) \text{ and } wit(\phi) \wedge \delta_V^E \text{ is } \mathcal{T}_f^{\vee} - \text{satisfiable}\},$$

what is absurd given the right side is computable: indeed, given ϕ, finding V and $Eq(V)$ can be easily done, as well as $|V/E|$; obtaining $wit(\phi) \wedge \delta_V^E$ can also be done algorithmically, if wit is computable; and, thanks to Theorem 67, testing whether $wit(\phi) \wedge \delta_V^E$ is $\mathcal{T}_f^{\vee}$-satisfiable is equivalent to testing whether the formula

$$\overline{wit(\phi) \wedge \delta_V^E} = wit(\phi) \wedge \delta_V^E \wedge \bigwedge_{w \in V} s^2(x) = x \vee s^2(x) = s(x)$$

is satisfiable, something that can be achieved algorithmically. □

Remark 5 We remark on the connection between the results regarding $\mathcal{T}_f$ and $\mathcal{T}_f^{\vee}$, and those of [22]. What we show here is that $\mathcal{T}_f$ ($\mathcal{T}_f^{\vee}$) is polite but not strongly polite. Figure 1 of [22] summarizes the relations between these two properties for the one-sorted case. It shows that polite theories that are axiomatized by a universal set of axioms, and whose quantifier-free satisfiability problem is decidable, are strongly polite. While $\mathcal{T}_f$ is decidable for quantifier-free formulas (this is a corollary of Lemma 62), its presentation here is definitely not as a universal theory. On the other hand, [22] also shows that decidable polite theories for which checking if a finite interpretation belongs to the theory is decidable are also strongly polite. However, it is undecidable, given an interpretation, to check whether it belongs to $\mathcal{T}_f$ (and $\mathcal{T}_f^{\vee}$): such an algorithm would lead to an algorithm to compute f as well. Thus, the theories $\mathcal{T}_f$ and $\mathcal{T}_f^{\vee}$ are polite, but do not meet the criteria for strong politeness from [22]. And indeed, they are not strongly polite.

Lemma 72 $\mathcal{T}_f^{\vee}$ *is not convex.*

Proof This proof is the same as the one for the non-convexity of $(\mathcal{T})_{\vee}$ in Theorem 19. □

A.3.4 $\mathcal{T}_{odd}^{\neq}$

$$\{\psi_{=1} \vee [\neg\psi_{=2k} \wedge \forall x.\, \neg(s(x) = x)] : k \in \mathbb{N}\} \qquad \textbf{(Axiomatization:)}$$

Lemma 73 $\mathcal{T}_{odd}^{\neq}$ *is not stably-infinite, and thus not smooth.*

Proof While $s(x) = x$ is satisfied by the $\mathcal{T}_{odd}^{\neq}$-interpretation $\mathcal{A}$ with $|\sigma_1^{\mathcal{A}}| = 1$, any infinite model of $\mathcal{T}_{odd}^{\neq}$ must satisfy $\forall x.\, \neg[s(x) = x]$ instead. □

Lemma 74 $\mathcal{T}_{odd}^{\neq}$ *is finitely witnessable.*

Proof Given a quantifier-free formula ϕ with $vars(\phi) = \{w_1, \ldots, w_n\}$ (notice $n \geq 1$, since the signature over which $\mathcal{T}_{odd}^{\neq}$ is defined has neither function nor predicate constants, being thus impossible to define a formula with no variables) and, for every $1 \leq i \leq n$, $M_i = \max\{j : s^j(w_i) \text{ is in } \phi\}$, we define its witness as

$$wit(\phi) = \phi \wedge (y = y) \wedge \bigwedge_{i=1}^{n} \bigwedge_{j=0}^{M_i+1} y_{i,j} = s^j(w_i),$$

where y and $y_{i,j}$ are all fresh variables. For $\overrightarrow{x} = vars(wit(\phi)) \setminus vars(\phi)$, we prove that ϕ and $\exists \overrightarrow{x}. wit(\phi)$ are $\mathcal{T}_{odd}^{\neq}$-equivalent: the right-to-left direction in this statement is trivial. For the converse, if $\mathcal{A}$ is a $\mathcal{T}_{odd}^{\neq}$-interpretation that satisfies ϕ, by changing the value given by $\mathcal{A}$ to the variables $y_{i,j}$ so that $y_{i,j}^{\mathcal{A}'} = (s^{\mathcal{A}})^j(w_i^{\mathcal{A}})$ we obtain a second $\mathcal{T}_{odd}^{\neq}$-interpretation $\mathcal{A}'$ that satisfies $wit(\phi)$; of course, this means that $\mathcal{A}$ itself satisfies $\exists \overrightarrow{x}. wit(\phi)$.

Now, assume that the $\mathcal{T}_{odd}^{\neq}$-interpretation $\mathcal{A}$ satisfies $wit(\phi)$. Let $V = vars(\phi) \cup \{y_{i,j} : 1 \leq i \leq n, 0 \leq j \leq M_i + 1\}$. We have three cases to consider.

1. If $|V^{\mathcal{A}}| = 1$, it is clear that $\mathcal{A}$ is the $\mathcal{T}_{odd}^{\neq}$-interpretation with only one element (since $M_i + 1 \geq 1$, and thus necessarily $(s^{\mathcal{A}})^1(w_i^{\mathcal{A}}) = w_i^{\mathcal{A}}$), and therefore $\mathcal{A}$ is already a $\mathcal{T}_{odd}^{\neq}$-interpretation that satisfies $wit(\phi)$ with $vars_{\sigma_1}(wit(\phi))^{\mathcal{A}} = \sigma_1^{\mathcal{A}}$, so there is nothing we need to do.
2. If $|V^{\mathcal{A}}|$ is an odd number greater than 1, we make a second $\mathcal{T}_{odd}^{\neq}$-interpretation $\mathcal{B}$ by proceeding as follows. Regarding the domain of $\mathcal{B}$, $\sigma_1^{\mathcal{B}} = V^{\mathcal{A}}$.
   If $a = (s^{\mathcal{A}})^j(w_i^{\mathcal{A}})$ for some $1 \leq i \leq n$ and $0 \leq j \leq M_i$, $s^{\mathcal{B}}(a) = s^{\mathcal{A}}(a)$ (and, this way, $s^{\mathcal{B}}(a) \neq a$, since $s^{\mathcal{A}}(a) \neq a$); and if $a = (s^{\mathcal{A}})^{M_i+1}(w_i^{\mathcal{A}})$, but a does not equal $(s^{\mathcal{A}})^j(w_k^{\mathcal{A}})$ for any $1 \leq k \leq n$ and $0 \leq j \leq M_k$, we simply make $s^{\mathcal{B}}(a)$ equal any element from $\sigma_1^{\mathcal{B}}$ different from a (and there is one, since $|V^{\mathcal{A}}| > 1$).
   Finally, $x^{\mathcal{B}} = x^{\mathcal{A}}$ for all variables $x \in V$, and arbitrarily otherwise (what includes y and all $y_{i,j}$). It is then easy to see that not only $\mathcal{B}$ is a $\mathcal{T}_{odd}^{\neq}$-interpretation that satisfies $wit(\phi)$, but also $\sigma_1^{\mathcal{B}} = vars(wit(\phi))^{\mathcal{B}}$.

3. For the last case, suppose that $|V^{\mathcal{A}}|$ is an even number. We then take an element $b \notin V^{\mathcal{A}}$ and define a new $\mathcal{T}_{odd}^{\neq}$-interpretation $\mathcal{B}$ as follows. For the domain, we use $\sigma_1^{\mathcal{B}} = V^{\mathcal{A}} \cup \{b\}$.
   If $a = (s^{\mathcal{A}})^j(w_i^{\mathcal{A}})$, for some $1 \leq i \leq n$ and $0 \leq j \leq M_i$, again we make $s^{\mathcal{B}}(a) = s^{\mathcal{A}}(a)$; and if $a = b$ or $a = (s^{\mathcal{A}})^{M_i+1}(w_i^{\mathcal{A}})$ but a does not equal $(s^{\mathcal{A}})^j(w_k^{\mathcal{A}})$, for any $1 \leq k \leq n$ and $0 \leq j \leq M_k$, $s^{\mathcal{B}}(a)$ may be an arbitrary element from $\sigma_1^{\mathcal{B}} \setminus \{a\}$.
   Finally, $x^{\mathcal{B}} = x^{\mathcal{A}}$ for all variables x in V, $y^{\mathcal{B}} = b$, and arbitrarily otherwise. Again one easily obtains that $\mathcal{B}$ is a $\mathcal{T}_{odd}^{\neq}$-interpretation that satisfies $wit(\phi)$ and has $\sigma_1^{\mathcal{B}} = vars(wit(\phi))^{\mathcal{B}}$.

□

Lemma 75 *$\mathcal{T}_{odd}^{\neq}$ is not strongly finitely witnessable.*

Proof Suppose that wit is a strong witness. We begin by noticing that there are $\mathcal{T}_{odd}^{\neq}$-interpretations $\mathcal{A}'$ with infinitely many elements, such as the one with domain $\mathbb{N}$ and $s^{\mathcal{A}'}(n) = n + 1$ for all $n \in \mathbb{N}$. Since $w = w$, which we shall denote by ϕ, is satisfied by all $\mathcal{T}_{odd}^{\neq}$-interpretations, including the infinite ones, there must exist an infinite $\mathcal{T}_{odd}^{\neq}$-interpretation $\mathcal{A}$ that satisfies $wit(w = w)$ (since ϕ and $\exists \overrightarrow{x}. wit(\phi)$ are $\mathcal{T}_{odd}^{\neq}$-equivalent, for $\overrightarrow{x} = vars(wit(\phi)) \setminus vars(\phi)$, and $\mathcal{A}'$ satisfies ϕ, there must exist a $\mathcal{T}_{odd}^{\neq}$-interpretation $\mathcal{A}$, differing from $\mathcal{A}'$ at most on $\overrightarrow{x}$, that satisfies $wit(\phi)$).

Consider now the set $W = vars(wit(\phi))$ and the equivalence relation F on W such that xFy iff $x^{\mathcal{A}} = y^{\mathcal{A}}$, with corresponding arrangement δ_W: of course $\mathcal{A}$ satisfies $wit(\phi) \wedge \delta_W$, and we have now two cases to consider.

1. If W/F has an even number of equivalence classes, we know there must exist a $\mathcal{T}_{odd}^{\neq}$-interpretation $\mathcal{B}$ that satisfies $wit(\phi) \wedge \delta_W$ with $\sigma_1^{\mathcal{B}} = W^{\mathcal{B}}$, what is absurd: if $\mathcal{B}$ satisfies

δ_W, $W^{\mathcal{B}}$ will have as many elements as W/F, and therefore have an even number of them, contradicting the fact that $\mathcal{B}$ is a $\mathcal{T}_{odd}^{\neq}$-interpretation.

2. So, assume that W/F has an odd number of equivalence classes, take some $z \notin W$, and define the equivalence relation E on $V = W \cup \{z\}$ such that xEy iff xFy or $x = y$, with corresponding arrangement δ_V.
To see that $wit(\phi) \wedge \delta_V$ is still $\mathcal{T}_{odd}^{\neq}$-satisfiable, remember that $\mathcal{A}$ not only is a $\mathcal{T}_{odd}^{\neq}$-interpretation that satisfies ϕ, but is also infinite: since W, and thus $W^{\mathcal{A}}$, must be finite, there exists an element $a \in \sigma_1^{\mathcal{A}} \setminus W^{\mathcal{A}}$; we then define $\mathcal{A}''$ to be the same interpretation as $\mathcal{A}$, except that $z^{\mathcal{A}''} = a$ (and, for all other variables x, $x^{\mathcal{A}''} = x^{\mathcal{A}}$). Given $\mathcal{A}$ and $\mathcal{A}''$ agree on the variables of the quantifier-free formula $wit(\phi) \wedge \delta_W$, and $\mathcal{A}$ satisfies $wit(\phi) \wedge \delta_W$, it follows that $\mathcal{A}''$ also satisfies that formula and, additionally, that $z^{\mathcal{A}''} \neq x^{\mathcal{A}''}$ for all $x \in W$; this, of course, means $\mathcal{A}''$ is a $\mathcal{T}_{odd}^{\neq}$-interpretation that satisfies $wit(\phi) \wedge \delta_V$.
So there must exist a $\mathcal{T}_{odd}^{\neq}$-interpretation $\mathcal{B}$ that satisfies $wit(\phi) \wedge \delta_V$ with $\sigma_1^{\mathcal{B}} = V^{\mathcal{B}}$. Of course, this is absurd: if W/F has an odd number of equivalence classes, V/E has an even number of equivalence classes, forcing $\mathcal{B}$ to have an even number of elements in its domain since it validates δ_V.

□

In the following proof, we need to use the fact that the theory $\mathcal{T}'$, axiomatized by the set of formulas

$$\{\neg\psi_{=1}\} \cup \{\forall x.\, \neg(s(x) = x)\} \cup \{\neg\psi_{2k} : k \in \mathbb{N}\},$$

is stably-infinite. This is actually easy to see: take a quantifier-free formula ϕ and a $\mathcal{T}'$-interpretation $\mathcal{A}$ that satisfies ϕ. Consider then a set $A = \{a_n, b_n : n \in \mathbb{N}\}$ disjoint from $\sigma_1^{\mathcal{A}}$, and define a $\mathcal{T}'$-interpretation $\mathcal{B}$ with: $\sigma_1^{\mathcal{B}} = \sigma_1^{\mathcal{A}} \cup A$; $s^{\mathcal{B}}(a) = s^{\mathcal{A}}(a)$ if $a \in \sigma_1^{\mathcal{A}}$, $s^{\mathcal{B}}(a_n) = b_n$ and $s^{\mathcal{B}}(b_n) = a_n$; and $x^{\mathcal{B}} = x^{\mathcal{A}}$ for all variables x. Then $\mathcal{B}$ is infinite, meaning it satisfies $\{\neg\psi_{=1}\} \cup \{\neg\psi_{2k} : k \in \mathbb{N}\}$, and in addition satisfies $\forall x.\neg(s(x) = x)$; furthermore, it satisfies ϕ, implying it is an infinite $\mathcal{T}'$-interpretation that satisfies this formula, what makes the theory stably-infinite.

Lemma 76 *$\mathcal{T}_{odd}^{\neq}$ is convex.*

Proof Suppose we have

$$\vdash_{\mathcal{T}_{odd}^{\neq}} \phi \rightarrow \bigvee_{k=1}^{n} x_k = y_k \quad \text{but} \quad \nvdash_{\mathcal{T}_{odd}^{\neq}} \phi \rightarrow x_k = y_k, \quad \text{for every } 1 \leq k \leq n,$$

where ϕ is a conjunction of literals. There must then exist $\mathcal{T}_{odd}^{\neq}$-interpretations $\mathcal{A}_k$ that satisfy ϕ but not $x_k = y_k$, for every $1 \leq k \leq n$; notice that, since $\mathcal{A}_k$ does not satisfy $x_k = y_k$, it cannot be the $\mathcal{T}_{odd}^{\neq}$-interpretation with only one element. Notice as well that, if we remove the model with domain of cardinality 1 from the class of models of $\mathcal{T}_{odd}^{\neq}$, we obtain the theory $\mathcal{T}'$ axiomatized by the set of formulas

$$\{\neg\psi_{=1}\} \cup \{\forall x.\, \neg(s(x) = x)\} \cup \{\neg\psi_{2k} : k \in \mathbb{N}\},$$

which, unlike $\mathcal{T}_{odd}^{\neq}$, is stably-infinite. Because, for each $1 \leq k \leq n$, the $\mathcal{A}_k$ are $\mathcal{T}_{odd}^{\neq}$-interpretations, and so $\mathcal{T}'$-interpretations, that satisfy ϕ but not $x_k = y_k$, using the fact that $\mathcal{T}'$ is stably-infinite (and $\phi \wedge \neg(x_k = y_k)$ is quantifier-free) we obtain $\mathcal{T}'$-interpretations, that are necessarily $\mathcal{T}^{\neq}_{odd}$-interpretations as well, $\mathcal{A}'_k$ that satisfy ϕ but not $x_k = y_k$ and that are infinite.

Appealing to Lemma 2 once again, there must exist $\mathcal{T}^{\neq}_{odd}$-interpretations $\mathcal{B}_k$ with countably infinite domains that satisfy ϕ but not $x_k = y_k$. Let $\{z_1, \dots, z_M\} = vars(\phi) \cup \{x_k, y_k : 1 \leq k \leq n\}$ and define M_i, for each $1 \leq i \leq M$, as either the maximum of j such that $s^j(z_i)$ appears in ϕ or, if no $s^j(z_i)$ is a a term in ϕ, as equal to 0. We then take a fresh set of variables $V = \{x_{i,j} : 1 \leq i \leq M, 0 \leq j \leq M_i\}$ and for each k define E_k as the smallest equivalence relation on V such that:

1. if $x_{i,j} E_k x_{p,q}$, $0 \leq j < M_i$ and $0 \leq q < M_p$, then $x_{i,j+1} E_k x_{p,q+1}$;
2. if $(s^{\mathcal{B}_k})^j(z_i^{\mathcal{B}_k}) = (s^{\mathcal{B}_k})^q(z_p^{\mathcal{B}_k})$, then $x_{i,j} E_k x_{p,q}$.

Notice that all of these equivalence relations are well-defined, since they are precisely the equivalences induced by $\mathcal{B}_k$ on the set V once we identify $(s^{\mathcal{B}_k})^j(z_i^{\mathcal{B}_k})$ with $x_{i,j}$ (observe that the first defining property of E_k comes from the fact that, if $a = b$, then $s^{\mathcal{B}_k}(a) = s^{\mathcal{B}_k}(b)$): since $s^{\mathcal{B}_k}(a) \neq a$ for all $a \in \sigma_1^{\mathcal{B}_k}$, we also easily derive that, if $0 \leq j < M_i$, $x_{i,j} \overline{E_k} x_{i,j+1}$, where $\overline{E_k}$ is the complement of E_k. We then finally define the equivalence E on V by setting $x_{i,j} E x_{p,q}$ iff $x_{i,j} E_k x_{p,q}$ for all $1 \leq k \leq n$.

We will denote by $[x_{i,j}]$ the equivalence class with representative $x_{i,j}$, and proceed now to define a $\mathcal{T}^{\neq}_{odd}$-interpretation $\mathcal{B}$ as follows.

1. $\sigma_1^{\mathcal{B}} = (V/E) \cup \mathbb{N}$ (which is infinite).
2. If there is an $x_{i,j}$ in $[x_{p,q}]$ such that $j < M_i$, we make $s^{\mathcal{B}}([x_{p,q}]) = [x_{i,j+1}]$, otherwise $s^{\mathcal{B}}([x_{p,q}]) = 0$; $s^{\mathcal{B}}(a) = a + 1$ for every $a \in \mathbb{N}$ (notice that we always have $s^{\mathcal{B}}(a) \neq a$, that being obviously true if $a \in \mathbb{N}$ or if $s^{\mathcal{B}}(a) = 0$; if $a \in V/E$, $a = [x_{i,j}]$ and $s^{\mathcal{B}}([x_{i,j}]) = [x_{i,j+1}]$, the fact that $s^{\mathcal{B}}(a) = a$, and therefore $[x_{i,j}] = [x_{i,j+1}]$, would imply that $x_{i,j} E_k x_{i,j+1}$, what is not possible).
   To prove that $s^{\mathcal{B}}$ is well-defined, suppose that $[x_{i,j}] = [x_{p,q}]$, $j < M_i$ and $q < M_p$: because of the first defining property of E_k, we have that $x_{i,j} E_k x_{p,q}$ implies $x_{i,j+1} E_k x_{p,q+1}$, and of course this will extend to E, meaning $s^{\mathcal{B}}([x_{i,j}]) = s^{\mathcal{B}}([x_{p,q}])$ as we needed to show.
3. For every variable $x_{i,j}$ in V, $x_{i,j}^{\mathcal{B}} = [x_{i,j}]$, and for every z_i we make $z_i^{\mathcal{B}} = [x_{i,0}]$ (and we may define $x^{\mathcal{B}}$ arbitrarily for variables x not in V).

Now, we state that $\mathcal{B}$ validates ϕ: in fact, let $s^j(z_i) = s^q(z_p)$ be a literal of ϕ not preceded by negation; in this case, $x_{i,j} E_k x_{p,q}$ for every $1 \leq k \leq n$ (by the third defining property of E_k), and so $[x_{i,j}] = [x_{p,q}]$, meaning $(s^{\mathcal{B}})^j(z_i^{\mathcal{B}}) = (s^{\mathcal{B}})^q(z_i^{\mathcal{B}})$. If, however, $\neg[s^j(z_i) = s^q(z_p)]$ is the literal in ϕ, since E_k is the smallest equivalence with its properties, $x_{i,j} \overline{E_k} x_{p,q}$, and so $(s^{\mathcal{B}})^j(z_i^{\mathcal{B}}) \neq (s^{\mathcal{B}})^q(z_i^{\mathcal{B}})$.

Finally, we see that $\mathcal{B}$ does not satisfy any $x_k = y_k$, leading to a contradiction: if $x_k = z_i$ and $y_k = z_p$, since $x_{i,0} \overline{E_k} x_{p,0}$, by construction of E_k, this means that $x_{i,0} \overline{E} x_{p,0}$, and so $\mathcal{B}$ does not satisfy $x_k = y_k$, for each $1 \leq k \leq n$; of course, this would imply that $\mathcal{B}$ does not satisfy $\bigvee_{k=1}^n x_k = y_k$, contradicting the fact that is satisfies ϕ, and thus we have $\vdash_{\mathcal{T}^{\neq}_{odd}} \phi \rightarrow \bigvee_{k=1}^n x_k = y_k$. □

A.3.5 $\mathcal{T}^{\neq}_{1,\infty}$

$$\{\psi_{=1} \vee [\psi_{\geq k} \wedge \forall x. \neg(s(x) = s)] : k \in \mathbb{N}\} \qquad \textbf{(Axiomatization:)}$$

Lemma 77 $\mathcal{T}^{\neq}_{1,\infty}$ *is not stably-infinite, and thus not smooth.*

Proof Notice that the quantifier-free formula $s(x) = x$ is satisfied by the $\mathcal{T}^{\neq}_{1,\infty}$-interpretation $\mathcal{A}$ with $|\sigma_1^{\mathcal{A}}| = 1$, but by no infinite interpretations of this theory. □

Lemma 78 $\mathcal{T}^{\neq}_{1,\infty}$ *is not finitely witnessable, and thus not strongly finitely witnessable.*

Proof If there existed a witness *wit*, there would also exist a finite $\mathcal{T}^{\neq}_{1,\infty}$-interpretation $\mathcal{A}$ satisfying $wit(\phi)$ with $\sigma_1^{\mathcal{A}} = vars(wit(\phi))^{\mathcal{A}}$, for ϕ equal to $\neg(x = y)$. This is absurd, since $\mathcal{A}$ would be a finite $\mathcal{T}^{\neq}_{1,\infty}$-interpretation that satisfies ϕ, and therefore has at least 2 elements. □

Lemma 79 $\mathcal{T}^{\neq}_{1,\infty}$ *is convex.*

Proof This proof, rather tedious, follows that of Lemma 76. □

A.3.6 $\mathcal{T}^{\neq}_{2,\infty}$

$$\{[\psi_{=2} \wedge \forall x.\, (s(x) = x)] \vee [\psi_{\geq k} \wedge \forall x.\, \neg(s(x) = x)] : k \in \mathbb{N}\} \qquad \textbf{(Axiomatization:)}$$

Lemma 80 $\mathcal{T}^{\neq}_{2,\infty}$ *is not stably-infinite, and thus not smooth.*

Proof Since the quantifier-free formula $s(x) = x$ is satisfied by the $\mathcal{T}^{\neq}_{2,\infty}$-interpretation with two elements, but by no infinite such interpretation, the theory cannot be stably-infinite. □

Lemma 81 $\mathcal{T}^{\neq}_{2,\infty}$ *is not finitely witnessable, and thus not strongly finitely witnessable.*

Proof If $\mathcal{T}^{\neq}_{2,\infty}$ were finitely witnessable with witness *wit*, we would have that there is a finite $\mathcal{T}^{\neq}_{2,\infty}$-interpretation $\mathcal{A}$ satisfying $wit(\phi)$ with $\sigma_1^{\mathcal{A}} = vars(wit(\phi))^{\mathcal{A}}$, for

$$\phi = \neg(x = y) \wedge \neg(x = z) \wedge \neg(y = z).$$

This is absurd, since $\mathcal{A}$ would be a finite $\mathcal{T}^{\neq}_{2,\infty}$-interpretation that satisfies ϕ, and therefore has at least 3 elements. □

Lemma 82 $\mathcal{T}^{\neq}_{2,\infty}$ *is not convex.*

Proof Given Theorem 10, and the facts that $\mathcal{T}^{\neq}_{2,\infty}$ has no models with domains of cardinality 1 and is not stably-infinite (see Lemma 80), the theory cannot be convex. □

B Proofs for Theory Operators

We now prove the many properties that are needed from our theory operators.

B.1 Proof of Theorem 16

To prove our theorem, we shall need Lemmas 83 and 84.

For simplicity of notation, the symbols $\mathcal{A}$ and $\mathcal{B}$ will be reserved, in the proofs of Lemmas 83 and 84, and Theorem 16, for Σ_1-interpretations, while $\mathcal{C}$ and $\mathcal{D}$ will denote Σ_2-interpretations. Still in the following results, given a Σ_2-interpretation $\mathcal{C}$, we denote by $\mathcal{C}_1$ the Σ_1-interpretation with $\sigma_1^{\mathcal{C}_1} = \sigma_1^{\mathcal{C}}$, and $x^{\mathcal{C}_1} = x^{\mathcal{C}}$ for every variable x of sort σ_1; analogously, given a $\Sigma_{2,s}$-interpretation $\mathcal{C}$, we denote by $\mathcal{C}_1$ the Σ_s-interpretation with $\sigma_1^{\mathcal{C}_1} = \sigma_1^{\mathcal{C}}$, $x^{\mathcal{C}_1} = x^{\mathcal{A}}$ for every variable x of sort σ_1, and $s^{\mathcal{C}_1}$ the same function as $s^{\mathcal{A}}$. Given we will not use both Σ_1 and Σ_s at the same time in a proof, the risk of confusing the notations is low.

Lemma 83 *Take a Σ_2-interpretation (respectively $\Sigma_{2,s}$-interpretation) $\mathcal{C}$. It is then true that, for any Σ_1-formula (respectively Σ_s-formula) φ, $\mathcal{C}_1$ satisfies φ if, and only if, $\mathcal{C}$ satisfies φ.*

Proof We focus on the case of empty signatures, and prove that $\mathcal{C}_1$ satisfies φ iff $\mathcal{C}$ satisfies φ by structural induction on φ.

1. Suppose φ is $x = y$, for x and y variables of sort σ_1; then, since $x^{\mathcal{C}_1} = x^{\mathcal{C}}$ and $y^{\mathcal{C}_1} = y^{\mathcal{C}}$, $\mathcal{C}_1$ satisfies φ iff $\mathcal{C}$ does so.
2. Suppose $\varphi = \neg\psi$ or $\varphi = \psi \vee \xi$, the proof for the connectives $\wedge$ and $\rightarrow$ being analogous. In the first case, $\mathcal{C}_1$ satisfies φ iff it does not satisfy ψ, what happens by induction hypothesis iff $\mathcal{C}$ does not satisfy ψ, equivalent to $\mathcal{C}$ satisfying φ. In the second case, $\mathcal{C}_1$ satisfies φ iff it satisfies either ψ or ξ, what happens by induction hypothesis iff $\mathcal{C}$ satisfies either ψ or ξ; of course, this is equivalent to $\mathcal{C}$ satisfying φ.
3. Finally, suppose $\varphi = \exists x.\, \psi$, the case for the quantifier $\forall$ being very similar. $\mathcal{C}_1$ satisfies φ iff there exists a second Σ_1-interpretation $\mathcal{A}$, differing from $\mathcal{C}_1$ at most on x, such that $\mathcal{A}$ satisfies ψ. Taking the Σ_2-interpretation $\mathcal{D}$ that differs from $\mathcal{C}$ at most on x, where $x^{\mathcal{D}} = x^{\mathcal{A}}$, one has $\mathcal{D}_1 = \mathcal{A}$, and so it is clear by induction hypothesis that $\mathcal{A}$ satisfies ψ iff $\mathcal{D}$ satisfies ψ. This is equivalent to the fact that $\mathcal{C}$ satisfies φ.

□

The following is an interesting technical lemma that allows us to "cut and paste" interpretations. We start by defining a transformation from some two-sorted signatures to their one-sorted counterparts.

Definition 11 Given a quantifier-free Σ_2 or $\Sigma_{2,s}$-formula ϕ, and an equivalence E on $vars(\phi)$, we define the (still quantifier-free) formula ϕ_E by replacing an equality $u = v$, for u and v of sort σ_2, by a tautology (whose variables are already in ϕ, we assume for simplicity, or if there are none we use the fresh variable z) in Σ_1 if uEv, and a contradiction (again in Σ_1, with variables in ϕ or equal to z) otherwise.

Lemma 84 *If the Σ_2-interpretation ($\Sigma_{2,s}$-interpretation) $\mathcal{C}$ satisfies the quantifier-free Σ_2-formula ($\Sigma_{2,s}$-formula) ϕ, and E is the equivalence relation on $vars(\phi)$ such that xEy iff $x^{\mathcal{C}} = y^{\mathcal{C}}$, a Σ_1-interpretation (Σ_s-interpretation) $\mathcal{A}$ satisfies the formula ϕ_E from Definition 11 iff the Σ_2-interpretation ($\Sigma_{2,s}$-interpretation) $\mathcal{D}$ satisfies ϕ, where: $\mathcal{D}_1 = \mathcal{A}$, $\sigma_2^{\mathcal{D}} = \sigma_2^{\mathcal{C}}$, and $u^{\mathcal{D}} = u^{\mathcal{C}}$ for all variables u of sort σ_2.*

Proof We prove the result by structural induction on ϕ, and for only the case of Σ_1 and Σ_2, the case of Σ_s and $\Sigma_{2,s}$ being completely analogous.

1. Suppose ϕ is atomic, meaning it equals either $x = y$ or $u = v$, for x and y of sort σ_1, and u and v of sort σ_2; also assume that $\mathcal{A}$ satisfies ϕ_E. In the former case, ϕ_E equals $x = y$, and from the fact $\mathcal{A}$ satisfies ϕ_E we get $x^{\mathcal{A}} = y^{\mathcal{A}}$, meaning $x^{\mathcal{D}} = y^{\mathcal{D}}$ and therefore that $\mathcal{D}$ satisfies ϕ. If it is the latter, since $\mathcal{C}$ satisfies ϕ we get $u^{\mathcal{C}} = v^{\mathcal{C}}$, and thus $u^{\mathcal{D}} = v^{\mathcal{D}}$, meaning $\mathcal{D}$ again satisfies ϕ. Reciprocally, assume $\mathcal{D}$ satisfies ϕ. If ϕ is $x = y$, $x^{\mathcal{D}} = y^{\mathcal{D}}$, meaning that $x^{\mathcal{A}} = y^{\mathcal{A}}$ and thus $\mathcal{A}$ satisfies ϕ_E. If ϕ is $u = v$, $u^{\mathcal{C}} = v^{\mathcal{C}}$, thus uEv, uFv and therefore ϕ_E is a tautology, meaning $\mathcal{A}$ satisfies that formula.
2. Suppose $\phi = \neg\phi^1$, that the result is true for ϕ^1, and that $\mathcal{A}$ satisfies ϕ_E. Since $\phi_E = \neg\phi^1_E$, we have that $\mathcal{A}$ does not satisfy ϕ^1_E, and thus $\mathcal{D}$ does not satisfy ϕ^1, meaning it satisfies $\neg\phi^1 = \phi$. Reciprocally, if $\mathcal{D}$ satisfies ϕ, it does not satisfy ϕ^1, and thus $\mathcal{A}$ does not satisfy ϕ^1_E, meaning it satisfies ϕ as we wanted to prove.
3. Suppose $\phi = \phi^1 \vee \phi^2$, that the result is true for ϕ^1 and ϕ^2, and that $\mathcal{A}$ satisfies ϕ_E. Since $\phi_E = \phi^1_E \vee \phi^2_E$, $\mathcal{A}$ satisfies either ϕ^1_E or ϕ^2_E, thus $\mathcal{D}$ satisfies either ϕ^1 or ϕ^2, and therefore $\phi^1 \vee \phi^2 = \phi$. Reciprocally, if $\mathcal{D}$ satisfies ϕ, it satisfies either ϕ^1 or ϕ^2, and thus $\mathcal{A}$ satisfies either ϕ^1_E or ϕ^2_E, meaning it satisfies $\phi^1_E \vee \phi^2_E = \phi_E$

□

Theorem 16 *A Σ_1 or Σ_s-theory $\mathcal{T}$ is stably infinite, smooth, finitely witnessable, strongly finitely witnessable, or convex w.r.t. $\{\sigma_1\}$ if and only if $(\mathcal{T})^2$ is, respectively, stably infinite, smooth, finitely witnessable, strongly finitely witnessable, or convex w.r.t. $\{\sigma_1, \sigma_2\}$.*

Proof As expected, we must use Lemma 83 again and again.

1. If $\mathcal{T}$ is stably-infinite, take a quantifier-free Σ_2-formula ϕ and a $(\mathcal{T})^2$-interpretation $\mathcal{C}$ that satisfies ϕ; we then have that $\mathcal{C}_1$ is a $\mathcal{T}$-interpretation (since, if $\mathcal{C}$ satisfies ψ, for a ψ in $Ax(\mathcal{T})$, $\mathcal{C}_1$ certainly satisfies ψ as well) that satisfies $\phi_{\mathcal{C}}$. Since $\mathcal{T}$ is assumed to be stably-infinite, there exists an infinite $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies $\phi_{\mathcal{C}}$. By then picking a $(\mathcal{T})^2$-interpretation $\mathcal{D}$ such that: $\mathcal{D}_1 = \mathcal{A}$; $|\sigma_2^{\mathcal{D}}| \geq \aleph_0$; and $u^{\mathcal{D}} = v^{\mathcal{D}}$ iff $u^{\mathcal{C}} = v^{\mathcal{C}}$ for variables u and v of sort σ_2, we get that $\mathcal{D}$ is infinite in both domains and satisfies ϕ.
Reciprocally, suppose now that $(\mathcal{T})^2$ is stably-infinite, and then take a quantifier-free Σ_1-formula ϕ and a $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies ϕ; it follows that any $(\mathcal{T})^2$-interpretation $\mathcal{C}$ with $\mathcal{C}_1 = \mathcal{A}$ must satisfy ϕ, and from the fact that $(\mathcal{T})^2$ is stably-infinite, there exists a $(\mathcal{T})^2$-interpretation $\mathcal{D}$, infinite on both domains, that satisfies ϕ. Then the $\mathcal{T}$-interpretation $\mathcal{D}_1$ has an infinite domain and satisfies ϕ, proving $\mathcal{T}$ is stably-infinite.
2. Start assuming $\mathcal{T}$ is smooth; then take a quantifier-free Σ_2-formula ϕ, a $(\mathcal{T})^2$-interpretation $\mathcal{C}$ that satisfies ϕ, and cardinals $\kappa(\sigma) \geq |\sigma_1^{\mathcal{C}}|$ and $\kappa(\sigma_2) \geq |\sigma_2^{\mathcal{C}}|$. We know that $\mathcal{C}_1$ is a $\mathcal{T}$-interpretation that satisfies $\phi_{\mathcal{C}}$, and then there exists a $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies $\phi_{\mathcal{C}}$ with $|\sigma_1^{\mathcal{A}}| = \kappa(\sigma)$. Taking a $(\mathcal{T})^2$-interpretation $\mathcal{D}$ with: $\mathcal{D}_1 = \mathcal{A}$; $|\sigma_2^{\mathcal{D}}| = \kappa(\sigma_2)$; and $u^{\mathcal{D}} = v^{\mathcal{D}}$, for all variables u and v of sort σ_2, iff $u^{\mathcal{C}} = v^{\mathcal{C}}$, we get that $\mathcal{D}$ satisfies ϕ, and $|\sigma_1^{\mathcal{D}}| = \kappa(\sigma)$ and $|\sigma_2^{\mathcal{D}}| = \kappa(\sigma_2)$.
Reciprocally, if $(\mathcal{T})^2$ is smooth, take a quantifier-free Σ_1-formula ϕ, a $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies ϕ, and a cardinal $\kappa \geq |\sigma_1^{\mathcal{A}}|$. Any $(\mathcal{T})^2$- interpretation $\mathcal{C}$ with $\mathcal{C}_1 = \mathcal{A}$ must satisfy ϕ, and then there must exist a $(\mathcal{T})^2$-interpretation $\mathcal{D}$, since this theory is smooth, with $|\sigma_1^{\mathcal{D}}| = \kappa$, $|\sigma_2^{\mathcal{D}}| = |\sigma_2^{\mathcal{C}}|$ and that satisfies ϕ. Of course $\mathcal{D}_1$ is then a $\mathcal{T}$-interpretation with $|\sigma_1^{\mathcal{D}_1}| = \kappa$ and that satisfies ϕ.
3. (a) Suppose $\mathcal{T}$ is finitely witnessable, with witness *wit*. We define a function $wit_2 : QF(\Sigma_2) \rightarrow QF(\Sigma_2)$ by

$$wit_2(\phi) = \phi \wedge (w = w) \wedge \bigvee_{E \in Eq(U)} [wit(\phi_E) \wedge \delta_U^E],$$

where: w is a fresh variable of sort σ_2; $U = vars_{\sigma_2}(\phi)$; $Eq(U)$ is the (computable) set of equivalence relations on U; δ_U^E is the arrangement on U induced by E. It is easy to see that wit_2 maps quantifier-free formulas into themselves, and is computable.

Suppose now that $\mathcal{C}$ is a $(\mathcal{T})^2$-interpretation that satisfies ϕ: let E be the equivalence induced by $\mathcal{C}$ on U (so $\mathcal{C}$ satisfies δ_U^E), so that $\mathcal{C}_1$ satisfies ϕ_E (from Definition 11), and thus $\exists \overrightarrow{y}\,.\, wit(\phi_E)$ for $\overrightarrow{y} = vars(wit(\phi_E)) \setminus vars(\phi_E)$, meaning $\mathcal{C}$ satisfies $\exists \overrightarrow{y}\,.\, wit(\phi_E)$. Since $\overrightarrow{y}$ is contained in $\overrightarrow{x} = vars(wit_2(\phi)) \setminus vars(\phi)$, $\mathcal{C}$ satisfies $\exists \overrightarrow{x}\,.\, wit(\phi_E)$, and thus $\exists \overrightarrow{x}\,.\, wit_2(\phi)$ (notice neither ϕ nor δ_U^E contain the variables on $\overrightarrow{x}$). The reciprocal is obvious.

Suppose now that $\mathcal{C}$ satisfies $wit_2(\phi)$, and thus $wit(\phi_E)$ for some $E \in Eq(U)$: this means $\mathcal{C}_1$ satisfies $wit(\phi_E)$ (due to Lemma 83), and so there is a $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies $wit(\phi_E)$ with $\sigma_1^{\mathcal{A}} = vars_{\sigma_1}(wit(\phi_E))^{\mathcal{A}}$. Let then $\mathcal{D}$ be a $(\mathcal{T})^2$-interpretation with $\mathcal{D}_1 = \mathcal{A}$, and $\sigma_2^{\mathcal{D}} = U/E$ (unless $U = \emptyset$, when we make $\sigma_2^{\mathcal{D}}$ a singleton), where $u^{\mathcal{D}}$ for $u \in U$ equals the equivalence class of representative u according to E, and $w^{\mathcal{D}}$ is set arbitrarily (where we remind the reader that w is fresh and thus not in U). $\mathcal{D}$ satisfies $wit(\phi_E)$ and, since $\mathcal{D}_1$ satisfies ϕ_E and $\mathcal{D}$ induces the equivalence E on U, also ϕ and δ_U^E; furthermore, $\sigma_1^{\mathcal{D}}$ equals $vars_{\sigma_1}(wit_2(\phi))^{\mathcal{D}}$, as the former set equals $\sigma_1^{\mathcal{D}_1} = \sigma_1^{\mathcal{A}}$ and the latter contains $vars_{\sigma_1}(wit(\phi_E))^{\mathcal{A}}$, and obviously $\sigma_2^{\mathcal{D}} = vars_{\sigma_2}(wit_2(\phi))^{\mathcal{D}}$.

(b) Suppose $(\mathcal{T})^2$ is finitely witnessable, with witness wit, and given a quantifier-free Σ_1-formula ϕ, we define a function wit_1 by making

$$wit_1(\phi) = \phi \wedge \bigvee_{E \in Eq(U)} wit(\phi)_E,$$

where: $U = vars_{\sigma_2}(wit(\phi))$; $Eq(U)$ is the set of equivalence relations on U (which can be found algorithmically); and $wit(\phi)_E$ is obtained from $wit(\phi)$ by replacing an equality $u = v$, for u and v of sort σ_2, by a tautology (whose variables are already in ϕ, for simplicity) in Σ_1 if uEv, and a contradiction (again in Σ_1, with variables in ϕ) otherwise. Of course wit_1 maps quantifier-free formulas into themselves, and is computable.

First, suppose a $\mathcal{T}$-interpretation $\mathcal{A}$ satisfies ϕ, and take a $(\mathcal{T})^2$-interpretation $\mathcal{C}$ with $\mathcal{C}_1 = \mathcal{A}$: $\mathcal{C}$ then satisfies ϕ (by Lemma 83), and thus $\exists \overrightarrow{y}\,.\, wit(\phi)$ for $\overrightarrow{y} = vars(wit(\phi)) \setminus vars(\phi)$; this way, some interpretation $\mathcal{C}'$, differing from $\mathcal{C}$ at most on the value assigned to $\overrightarrow{y}$, satisfies $wit(\phi)$ (and thus $\exists \overrightarrow{y}\,.\, wit(\phi)$ and ϕ). If E is the equivalence induced by $\mathcal{C}'$ on U, it is clear that $\mathcal{C}'_1$ satisfies $wit(\phi)_E$ (again by Lemma 83), as well as ϕ; since $\mathcal{C}'_1$ differs from $\mathcal{A}$ at most on the value assigned to the variables in $\overrightarrow{y}$ of sort σ_1, what equals $\overrightarrow{x} = vars(wit_1(\phi)) \setminus vars(\phi)$, $\mathcal{A}$ satisfies $\exists \overrightarrow{x}\,.\, wit(\phi)_E$ and ϕ, and thus $\exists \overrightarrow{x}\,.\, wit_1(\phi)$. The reciprocal is obvious, as ϕ has none of the variables in $\overrightarrow{x}$.

Now, suppose that the $\mathcal{T}$-interpretation $\mathcal{A}$ satisfies $wit_1(\phi)$, and thus $wit(\phi)_E$ for some $E \in Eq(U)$: take then a $(\mathcal{T})^2$-interpretation $\mathcal{C}$ with $\mathcal{C}_1 = \mathcal{A}$ and that induces the equivalence E on U, so $\mathcal{C}$ satisfies $wit(\phi)$. There is then a $(\mathcal{T})^2$-interpretation $\mathcal{D}$ that satisfies $wit(\phi)$ (and thus $\exists \overrightarrow{x}\,.\, wit(\phi)$ and ϕ) with $\sigma_1^{\mathcal{D}} = vars_{\sigma_1}(wit(\phi))^{\mathcal{D}}$ and $\sigma_2^{\mathcal{D}} = vars_{\sigma_2}(wit(\phi))^{\mathcal{D}}$: if F is the equivalence induced by $\mathcal{D}$ on U, we have that $\mathcal{D}_1$ satisfies ϕ and $wit(\phi)_F$, and therefore $wit_1(\phi)$, and since $vars_{\sigma_1}(wit_1(\phi)) \supseteq vars_{\sigma_1}(wit(\phi))$ we get $\sigma_1^{\mathcal{D}_1} = vars_{\sigma_1}(wit_1(\phi))^{\mathcal{D}_1}$, proving wit_1 is a witness.

4. (a) Suppose $\mathcal{T}$ is now strongly finitely witnessable, with strong witness wit, and given a quantifier-free Σ_2-formula ϕ, we consider the same function wit_2 from item $3(a)$ above. We do not need to prove that wit_2 is computable, and that ϕ and $\exists \overrightarrow{x} . wit_2(\phi)$ are $(\mathcal{T})^2$-equivalent, as this is done on item $3(a)$.
So assume V is a set of variables, δ_V an arrangement on V, and $\mathcal{C}$ a $(\mathcal{T})^2$-interpretation that satisfies $wit_2(\phi)\wedge\delta_V$: let V_1 be the variables of sort σ_1 in V, V_2 the ones with sort σ_2, and write $\delta_V = \delta_{V_1} \wedge \delta_{V_2}$ in the obvious way. $\mathcal{C}$ satisfies $wit(\phi_E) \wedge \delta_U^E$ for some $E \in Eq(U)$, and thus $\mathcal{C}_1$ satisfies $wit(\phi_E)$ and δ_{V_1}, so there is a $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies $wit(\phi_E) \wedge \delta_{V_1}$ with $\sigma_1^{\mathcal{A}} = vars_{\sigma_1}(wit(\phi_E) \wedge \delta_{V_1})^{\mathcal{A}}$. We define a $(\mathcal{T})^2$-interpretation $\mathcal{D}$ by making: $\mathcal{D}_1 = \mathcal{A}$; $\sigma_2^{\mathcal{D}}$ equal to $U \cup V_2$ modulo the equivalence relative to $\delta_{V_2} \wedge \delta_U^E$ (this is possible as the formula $\delta_{V_2} \wedge \delta_U^E$ is not contradictory, being satisfied by $\mathcal{C}$), unless $U \cup V_2$ is empty when we make $\sigma_2^{\mathcal{D}}$ a singleton; $u^{\mathcal{D}}$ equal to the equivalence class with representative $u \in U \cup V_2$, and $w^{\mathcal{D}}$ arbitrary. Since $\mathcal{A}$ satisfies $wit(\phi_E) \wedge \delta_{V_1}$, so does $\mathcal{D}$, and $\mathcal{D}$ satisfies $\delta_{V_2} \wedge \delta_U^E$ by design, meaning it satisfies $wit_2(\phi)$. That $\sigma_1^{\mathcal{D}} = vars_{\sigma_1}(wit_2(\phi) \wedge \delta_V)^{\mathcal{D}}$ and $\sigma_2^{\mathcal{D}} = vars_{\sigma_2}(wit_2(\phi) \wedge \delta_V)^{\mathcal{D}}$ follows from the definition of $\mathcal{D}$ and choice of $\mathcal{A}$.
(b) Suppose $(\mathcal{T})^2$ is now strongly finitely witnessable, with strong witness wit, and given a quantifier-free Σ_1-formula ϕ, we consider the same function wit_1 from item $3(b)$ above. We already know that wit_1 is computable and that, for $\overrightarrow{x} = vars(wit_1(\phi)) \setminus vars(\phi)$, ϕ and $\exists \overrightarrow{x} . wit_1(\phi)$ are $\mathcal{T}$-equivalent.
So, take a set V of variables of sort σ_1, let δ_V be an arrangement on V, and $\mathcal{A}$ a $\mathcal{T}$-interpretation that satisfies $wit_1(\phi) \wedge \delta_V$: there is an equivalence E on U such that $\mathcal{A}$ satisfies $wit(\phi)_E \wedge \delta_V$, so take a $(\mathcal{T})^2$-interpretation $\mathcal{C}$ with $\mathcal{C}_1 = \mathcal{A}$ and that induces the equivalence E on U, so that $\mathcal{C}$ satisfies $wit(\phi) \wedge \delta_V$. There is then a $(\mathcal{T})^2$-interpretation $\mathcal{D}$ that satisfies $wit(\phi) \wedge \delta_V$ (and thus $\exists \overrightarrow{x} . wit(\phi)$ and ϕ) with $\sigma_1^{\mathcal{D}} = vars_{\sigma_1}(wit(\phi) \wedge \delta_V)^{\mathcal{D}}$ and $\sigma_2^{\mathcal{D}} = vars_{\sigma_2}(wit(\phi) \wedge \delta_V)^{\mathcal{D}}$: $\mathcal{D}_1$ is then a $\mathcal{T}$-interpretation that satisfies ϕ, $wit(\phi)_F \wedge \delta_V$ (and thus $wit_1(\phi) \wedge \delta_V$), for F the equivalence induced by $\mathcal{D}$ on U, with $\sigma_1^{\mathcal{D}_1} = vars_{\sigma_1}(wit_1(\phi) \wedge \delta_V)^{\mathcal{D}_1}$, proving wit_1 is a strong witness.
5. Suppose $\mathcal{T}$ is convex, let ϕ be a cube in Σ_2 and assume that

$$\vdash_{(\mathcal{T})^2} \phi \rightarrow \bigvee_{i=1}^{n} x_i = y_i \vee \bigvee_{j=1}^{m} u_j = v_j,$$

where the x_i and y_i are of sort σ_1, and the u_j and v_j are of sort σ_2; because ϕ is a conjunction of literals, and each literal can only have variables of one sort, we may write $\phi = \phi_1 \wedge \phi_2$, where ϕ_1 has only variables of sort σ_1, and ϕ_2 has only variables of sort σ_2.
It follows that $\vdash_{(\mathcal{T})^2} \phi_1 \rightarrow \bigvee_{i=1}^{n} x_i = y_i$ or $\vdash_{(\mathcal{T})^2} \phi_2 \rightarrow \bigvee_{j=1}^{m} u_j = v_j$: indeed, suppose this were not true, and so there exist $(\mathcal{T})^2$-interpretations $\mathcal{C}_1$ and $\mathcal{C}_2$ where, respectively, $\phi_1 \rightarrow \bigvee_{i=1}^{n} x_i = y_i$ and $\phi_2 \rightarrow \bigvee_{j=1}^{m} u_j = v_j$ are not satisfied; hence $\mathcal{C}_1$ satisfies ϕ_1 but not $\bigvee_{i=1}^{n} x_i = y_i$, and $\mathcal{C}_2$ satisfies ϕ_2, but not $\bigvee_{j=1}^{m} u_j = v_j$. So define the $(\mathcal{T})^2$-interpretation $\mathcal{D}$ where: $\sigma_1^{\mathcal{D}} = \sigma_1^{\mathcal{C}_1}$, $\sigma_2^{\mathcal{D}} = \sigma_2^{\mathcal{C}_2}$, $x^{\mathcal{D}} = x^{\mathcal{C}_1}$ for every variable x of sort σ_1, and $u^{\mathcal{D}} = u^{\mathcal{C}_2}$ for every variable u of sort σ_2. Because $\mathcal{D}$ agrees with $\mathcal{C}_1$ on the variables of sort σ_1, it satisfies ϕ but not $\bigvee_{i=1}^{n} x_i = y_i$; and because $\mathcal{D}$ agrees with $\mathcal{C}_2$ on the variables of sort σ_2, it satisfies ϕ_2 but not $\bigvee_{j=1}^{m} u_j = v_j$. Of course, $\mathcal{D}$ then satisfies ϕ but not $\bigvee_{i=1}^{n} x_i = y_i \vee \bigvee_{j=1}^{m} u_j = v_j$, leading to a contradiction.
If we have $\vdash_{(\mathcal{T})^2} \phi_2 \rightarrow \bigvee_{j=1}^{m} u_j = v_j$, because $(\mathcal{T})^2$ has an axiomatization of formulas

with no variables of sort σ_2, one gets $\vdash_{\mathcal{T}'} \phi_2 \rightarrow \bigvee_{j=1}^{m} u_j = v_j$, for $\mathcal{T}'$ the theory over the signature with only one sort σ_2 and no symbols, axiomatized by the empty set; since $\mathcal{T}'$ is convex, according to Theorem 6, this means $\vdash_{\mathcal{T}'} \phi_2 \rightarrow u_j = v_j$ for some $1 \leq j \leq m$, and thus $\vdash_{(\mathcal{T})^2} \phi \rightarrow u_j = v_j$, in which case we would be done.
Suppose then that $\vdash_{(\mathcal{T})^2} \phi_1 \rightarrow \bigvee_{i=1}^{n} x_i = y_i$. Since $\phi_1 \rightarrow \bigvee_{i=1}^{n} x_i = y_i$ has no variables of sort σ_2, we obtain $\vdash_{\mathcal{T}} \phi_1 \rightarrow \bigvee_{i=1}^{n} x_i = y_i$ (from Lemma 84), and since this theory is convex, $\vdash_{\mathcal{T}} \phi_1 \rightarrow x_i = y_i$, for some $1 \leq i \leq n$, and thus $\vdash_{(\mathcal{T})^2} \phi \rightarrow x_i = y_i$.
Reciprocally, assume this time $(\mathcal{T})^2$ is convex, and let ϕ be a conjunction of literals such that $\vdash_{\mathcal{T}} \phi \rightarrow \bigvee_{i=1}^{n} x_i = y_i$, where x_i and y_i, for $i \in [1, n]$, are variables of sort σ_1. It follows that $\vdash_{(\mathcal{T})^2} \phi \rightarrow \bigvee_{i=1}^{n} x_i = y_i$, and so $\vdash_{(\mathcal{T})^2} \phi \rightarrow x_i = y_i$ for some $1 \leq i \leq n$, meaning $\vdash_{\mathcal{T}} \phi \rightarrow x_i = y_i$.

□

B.2 Proof of Theorem 17

To prove this section's main result, Theorem 17, we will use Lemma 85 below. In the proofs of Lemma 85 and Theorem 17, given the enormous number of involved interpretations in the demonstrations, we agree that $\mathcal{A}$ and $\mathcal{B}$ shall be Σ_n-interpretations (often these will be in addition $\mathcal{T}$-interpretations), while $\mathcal{C}$ and $\mathcal{D}$ will denote $\Sigma_{n,s}$-interpretations (that will be, many times, $(\mathcal{T})_s$-interpretations as well).

Definition 12 Using the signatures from Definition 4, given a Σ_n-interpretation $\mathcal{A}$, we define a $\Sigma_{n,s}$-interpretation $s(\mathcal{A})$ by making:

1. $\sigma^{s(\mathcal{A})} = \sigma^{\mathcal{A}}$ for each $\sigma \in S$;
2. $s^{s(\mathcal{A})}(a) = a$ for all $a \in \sigma_1^{\mathcal{A}}$;
3. and $x^{s(\mathcal{A})} = x^{\mathcal{A}}$ for every variable x.

Reciprocally, given any $\Sigma_{n,s}$-interpretation $\mathcal{C}$, we may consider the Σ_n-interpretation $(\mathcal{C})$ with:

1. $\sigma_1{}^{(\mathcal{C})} = \sigma_1^{\mathcal{C}}$ for each $\sigma \in S$;
2. and $x^{(\mathcal{C})} = x^{\mathcal{C}}$, for every variable x.

Finally, given a $\Sigma_{n,s}$-formula φ, we repeatedly replace each occurrence of $s(x)$ in φ by x until we obtain a Σ_n-formula φ .

Lemma 85 *A $\Sigma_{n,s}$-interpretation $\mathcal{C}$ that satisfies $\forall x. [s(x) = x]$ (where x is of sort σ_1) also satisfies φ iff $(\mathcal{C})$ satisfies φ (check Definition 12 for the relevant definitions); of course, given that for any Σ_n-interpretation $\mathcal{A}$, $(s(\mathcal{A})) = \mathcal{A}$, $\mathcal{A}$ satisfies a Σ_n-formula φ iff $s(\mathcal{A})$ satisfies φ (since $\varphi = \varphi$).*

The proof of this lemma is a simple exercise in structural induction, quite similar to the proof of Lemma 83. One small difference is in the fact that, in this case, the step dealing with atomic formulas must consider both formulas of the form $s^i(x) = s^j(y)$, for x and y of sort σ_1, or of the form $u = v$ for u and v of a sort in $S \setminus \{\sigma_1\}$.

Theorem 17 *For every theory $\mathcal{T}$ over an empty signature Σ_n with sorts $S = \{\sigma_1, \ldots, \sigma_n\}$: $\mathcal{T}$ is stably infinite, smooth, finitely witnessable, strongly finitely witnessable, or convex w.r.t. S if and only if $(\mathcal{T})_s$ is, respectively, stably infinite, smooth, finitely witnessable, strongly finitely witnessable, or convex w.r.t. S.*

Proof We heavily and repeatedly use Lemma 85 throughout this proof. Start by noticing that $\mathcal{C}$ is a $(\mathcal{T})_s$-interpretation iff $(\mathcal{C})$ is a $\mathcal{T}$-interpretation, given that for any formula φ in $Ax(\mathcal{T})$, $\varphi = \varphi$ (see Lemma 85 for the definition of φ).

1. (a) Suppose $\mathcal{T}$ is stably-infinite: given a quantifier-free $\Sigma_{n,s}$-formula ϕ and a $(\mathcal{T})_s$-interpretation $\mathcal{C}$ that satisfies ϕ, we have that $(\mathcal{C})$ is a $\mathcal{T}$-interpretation that satisfies ϕ ; since ϕ is also quantifier-free, there must exist a $\mathcal{T}$-interpretation $\mathcal{A}$, with all infinite domains, that satisfies ϕ , and then $s(\mathcal{A})$ is a $(\mathcal{T})_s$-interpretation, infinite in all of its domains, that satisfies ϕ.
   (b) Reciprocally, suppose $(\mathcal{T})_s$ is stably-infinite, let ϕ be a quantifier-free Σ_n-formula and $\mathcal{A}$ a $\mathcal{T}$-interpretation that satisfies ϕ; since $\phi = \phi$, $s(\mathcal{A})$ satisfies ϕ, and there must exist a $(\mathcal{T})_s$-interpretation $\mathcal{C}$ that satisfies ϕ and has all infinite domains. Of course, $(\mathcal{C})$ then satisfies ϕ.
2. (a) Suppose $\mathcal{T}$ is smooth: given a quantifier-free $\Sigma_{n,s}$-formula ϕ, a $(\mathcal{T})_s$-interpretation $\mathcal{C}$ that satisfies ϕ, and a function κ from S to the class of cardinals such that $\kappa(\sigma) \geq |\sigma^{\mathcal{C}}|$ for each $\sigma \in S$, we have that $(\mathcal{C})$ is a $\mathcal{T}$-interpretation that satisfies ϕ . Given that $\mathcal{T}$ is supposed to be smooth, ϕ is quantifier-free and $|\sigma^{(\mathcal{C})}| = |\sigma^{\mathcal{C}}| \leq \kappa(\sigma)$ for each $\sigma \in S$, there exists a $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies ϕ with $|\sigma^{\mathcal{A}}| = \kappa(\sigma)$ (again, for every $\sigma \in S$); of course, $s(\mathcal{A})$ is then a $(\mathcal{T})_s$-interpretation with $|\sigma^{s(\mathcal{A})}| = |\sigma^{\mathcal{A}}| = \kappa(\sigma)$, for any $\sigma \in S$, that satisfies ϕ.
   (b) Reciprocally, suppose $(\mathcal{T})_s$ is smooth: then, for any quantifier-free Σ_n-formula ϕ, $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies ϕ, and function κ from S to the class of cardinals such that $\kappa(\sigma) \geq |\sigma^{\mathcal{A}}|$ for all $\sigma \in S$, we have that $s(\mathcal{A})$ satisfies ϕ, since $\phi = \phi$; given that $|\sigma^{s(\mathcal{A})}| = |\sigma^{\mathcal{A}}| \leq \kappa(\sigma)$, one obtains there must exist a $(\mathcal{T})_s$-interpretation $\mathcal{C}$ that satisfies ϕ with $|\sigma^{\mathcal{C}}| = \kappa(\sigma)$ for every $\sigma \in S$. And then, $(\mathcal{C})$ is a $\mathcal{T}$-interpretation that satisfies ϕ with $|\sigma^{(\mathcal{C})}| = \kappa(\sigma), \forall\sigma \in S$.
3. (a) Suppose now $\mathcal{T}$ is finitely witnessable, with witness *wit*: we shall prove that $(\mathcal{T})_s$ is also finitely witnessable, with witness

$$wit_s(\phi) = \phi \wedge wit(\phi),$$

which clearly maps quantifier-free formulas to quantifier-free formulas, and is computable. To start with, given a quantifier-free $\Sigma_{n,s}$-formula ϕ, ϕ is also quantifier-free, and thus ϕ and $\exists\,\overrightarrow{x}\,.\,wit(\phi)$ are $\mathcal{T}$-equivalent, for $\overrightarrow{x} = vars(wit(\phi)) \setminus vars(\phi) = vars(wit_s(\phi)) \setminus vars(\phi)$. If the $(\mathcal{T})_s$-interpretation $\mathcal{C}$ satisfies ϕ, $(\mathcal{C})$ satisfies ϕ and thus $\exists\,\overrightarrow{x}\,.\,wit(\phi)$; since $(\exists\,\overrightarrow{x}\,.\,wit(\phi)) = \exists\,\overrightarrow{x}\,.\,wit(\phi)$, we have that $\mathcal{C}$ satisfies $\exists\,\overrightarrow{x}\,.\,wit(\phi)$ and thus $\exists\,\overrightarrow{x}\,.\,wit_s(\phi) = \exists\,\overrightarrow{x}\,.\,(\phi \wedge wit(\phi))$, given that ϕ has none of the variables in $\overrightarrow{x}$. Of course, if the $(\mathcal{T})_s$-interpretation $\mathcal{C}$ satisfies $\exists\,\overrightarrow{x}\,.\,wit_s(\phi)$, it must satisfy ϕ, and so the two formulas are $(\mathcal{T})_s$-equivalent.
Suppose now that a $(\mathcal{T})_s$-interpretation $\mathcal{C}$ satisfies $wit_s(\phi)$, and we have that $(\mathcal{C})$ is a $\mathcal{T}$-interpretation that satisfies $(wit_s(\phi)) = \phi \wedge wit(\phi)$; from the facts that $\mathcal{T}$ is finitely witnessable, with witness *wit*, and $(\mathcal{C})$ satisfies $wit(\phi)$, it follows that there exists a $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies $wit(\phi)$, and thus ϕ , with $\sigma^{\mathcal{A}} = vars_\sigma(wit(\phi))^{\mathcal{A}}$ for each $\sigma \in S$. Then, since $(wit(\phi)) = wit(\phi)$, $s(\mathcal{A})$ is a $(\mathcal{T})_s$-interpretation that satisfies $wit(\phi)$; since $\mathcal{A}$ also satisfies ϕ , $s(\mathcal{A})$ satisfies ϕ and thus $wit_s(\phi) = \phi \wedge wit(\phi)$ as well; and, given that $\sigma^{s(\mathcal{A})} = \sigma^{\mathcal{A}}$ and $vars_\sigma(wit(\phi)) = vars_\sigma(\phi \wedge wit(\phi))$ (both for any $\sigma \in S$), we get $\sigma^{s(\mathcal{A})} = vars(wit_s(\phi))^{s(\mathcal{A})}$ for all $\sigma \in S$, proving $(\mathcal{T})_s$ is indeed finitely witnessable.

(b) Reciprocally, suppose $(\mathcal{T})_s$ is finitely witnessable with witness *wit*, and we want to prove that $\mathcal{T}$ is finitely witnessable with witness $wit_0(\phi) = (wit(\phi))$, being this obviously a computable function from quantifier-free formulas into themselves. We start by taking a quantifier-free Σ_n-formula ϕ, and since ϕ is a quantifier-free $\Sigma_{n,s}$-formula as well, ϕ and $\exists\overrightarrow{x}.\, wit(\phi)$ are $(\mathcal{T})_s$-equivalent, where $\overrightarrow{x} = vars(wit(\phi)) \setminus vars(\phi) = vars(wit(\phi)\ ) \setminus vars(\phi)$. So, suppose the $\mathcal{T}$-interpretation $\mathcal{A}$ satisfies $\phi = \phi$: then $s(\mathcal{A})$ satisfies ϕ and thus $\exists\overrightarrow{x}.\, wit(\phi)$, meaning $\mathcal{A}$ satisfies

$$(\exists\overrightarrow{x}.\, wit(\phi)) = \exists\overrightarrow{x}.\, wit(\phi) = \exists\overrightarrow{x}.\, wit_0(\phi).$$

If the $\mathcal{T}$-interpretation $\mathcal{A}$ satisfies $\exists\overrightarrow{x}.\, wit_0(\phi)$, $s(\mathcal{A})$ satisfies $\exists\overrightarrow{x}.\, wit(\phi)$ and hence ϕ, meaning $\mathcal{A}$ satisfies ϕ and proving that ϕ and $\exists\overrightarrow{x}.\, wit_0(\phi)$ are $\mathcal{T}$-equivalent. Now suppose that $\mathcal{A}$ is a $\mathcal{T}$-interpretation that satisfies $wit_0(\phi)$, and thus $s(\mathcal{A})$ is a $(\mathcal{T})_s$-interpretation that satisfies $wit(\phi)$, since $wit_0(\phi) = (wit(\phi))$. Given that $(\mathcal{T})_s$ is finitely witnessable with witness *wit*, there exists a $(\mathcal{T})_s$-interpretation $\mathcal{C}$ that satisfies $wit(\phi)$ with $\sigma^{\mathcal{C}} = vars_\sigma(wit(\phi))^{\mathcal{C}}$ for any $\sigma \in S$. It follows that $(\mathcal{C})$ is a $\mathcal{T}$-interpretation that satisfies $(wit(\phi)) = wit_0(\phi)$ with $\sigma^{\ (\mathcal{C})} = vars(wit_0(\phi))^{\ (\mathcal{C})}$, $\forall \sigma \in S$ (since $\sigma^{\ (\mathcal{C})} = \sigma^{\mathcal{C}}$ and $vars_\sigma(wit(\phi)) = vars_\sigma(wit_0(\phi))$, both for all $\sigma \in S$).

4. (a) Now, let us look at strong finite witnessability. If $\mathcal{T}$ is strongly finitely witnessable with witness *wit*, we state $(\mathcal{T})_s$ is also strongly finitely witnessable with witness $wit_s(\phi) = \phi \wedge wit(\phi\ )$, and we already know that ϕ and $\exists\overrightarrow{x}.\, wit_s(\phi)$ are $(\mathcal{T})_s$-equivalent from our discussion above about finite witnessability. So let V be a set of variables, δ_V an arrangement on V, and $\mathcal{C}$ a $(\mathcal{T})_s$-interpretation that satisfies $wit_s(\phi) \wedge \delta_V$. Since $(wit_s(\phi) \wedge \delta_V) = \phi \ \wedge wit(\phi\ ) \wedge \delta_V$, $(\mathcal{C})$ satisfies $wit(\phi\ ) \wedge \delta_V$; it follows, from the fact that $\mathcal{T}$ has as strong witness *wit*, that there is a $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies $wit(\phi\ ) \wedge \delta_V$ (and hence ϕ as well) with $\sigma^{\mathcal{A}} = vars_\sigma(wit(\phi\ ) \wedge \delta_V)^{\mathcal{A}}$, for each and any $\sigma \in S$. $s(\mathcal{A})$ is, therefore, a $(\mathcal{T})_s$-interpretation that satisfies $\phi \wedge wit(\phi\ ) \wedge \delta_V$, what amounts to $wit_s(\phi) \wedge \delta_V$, with $\sigma^{s(\mathcal{A})} = vars_\sigma(wit_s(\phi) \wedge \delta_V)^{s(\mathcal{A})}$ once one notices that $\sigma^{s(\mathcal{A})} = \sigma^{\mathcal{A}}$ and $vars_\sigma(wit_s(\phi) \wedge \delta_V) = vars_\sigma(wit(\phi\ ) \wedge \delta_V)$, both of these for all sorts σ in S. So $(\mathcal{T})_s$ is indeed strongly finitely witnessable.

(b) Reciprocally, assume $(\mathcal{T})_s$ is strongly finitely witnessable with strong witness *wit*, and we will prove that $\mathcal{T}$ is also strongly finitely witnessable with strong witness

$$wit_0(\phi) = (wit(\phi))\ .$$

Of course, from our discussion about finite witnessability we already know that ϕ and $\exists\overrightarrow{x}.\, wit_0(\phi)$ are $\mathcal{T}$-equivalent, where $\overrightarrow{x} = vars(wit_0(\phi)) \setminus vars(\phi)$. So take a set of variables V, an arrangement δ_V on V, and a $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies $wit_0(\phi) \wedge \delta_V$. Since $(wit(\phi) \wedge \delta_V)$ equals $wit_0(\phi) \wedge \delta_V$, we have that $s(\mathcal{A})$ satisfies $wit(\phi) \wedge \delta_V$, and so there exists a $(\mathcal{T})_s$-interpretation $\mathcal{C}$ that satisfies $wit(\phi) \wedge \delta_V$ with $\sigma^{\mathcal{C}} = vars_\sigma(wit(\phi) \wedge \delta_V)^{\mathcal{C}}$ for every σ in S. Then $(\mathcal{C})$ satisfies $wit_0(\phi) \wedge \delta_V$ and, since $\sigma^{\ (\mathcal{C})} = \sigma^{\mathcal{C}}$ and $vars_\sigma(wit(\phi) \wedge \delta_V) = vars_\sigma(wit_0(\phi) \wedge \delta_V)$ for any σ in S, $\sigma^{\ (\mathcal{C})} = vars_\sigma(wit_0(\phi) \wedge \delta_V)^{\mathcal{C}}$ again for any σ in S, what finishes the proof.

5. (a) Finally, suppose $\mathcal{T}$ is convex. Let ϕ be a conjunction of literals in $\Sigma_{n,s}$ (notice ϕ is then a conjunction of literals in Σ_n), and assume that $\vdash_{(\mathcal{T})_s} \phi \rightarrow \bigvee_{i=1}^{n} x_i = y_i$ (implying $\vdash_{\mathcal{T}} \phi \ \rightarrow \bigvee_{i=1}^{n} x_i = y_i$): it follows that $\vdash_{\mathcal{T}} \phi \ \rightarrow x_i = y_i$ for some $1 \leq i \leq n$, and since $(x_i = y_i) \ = (x_i = y_i)$, that $\vdash_{(\mathcal{T})_s} \phi \rightarrow x_i = y_i$, proving $(\mathcal{T})_s$ is convex.

(b) Reciprocally, assume $(\mathcal{T})_s$ is convex, and let ϕ be a cube in Σ_n (and so $\phi \ = \phi$) such that $\vdash_{\mathcal{T}} \phi \rightarrow \bigvee_{i=1}^{n} x_i = y_i$ (and thus $\vdash_{(\mathcal{T})_s} \phi \rightarrow \bigvee_{i=1}^{n} x_i = y_i$); we must then have $\vdash_{(\mathcal{T})_s} \phi \rightarrow x_i = y_i$ for some $1 \leq i \leq n$, and again by using that $(x_i = y_i) \; = (x_i = y_i)$ we get $\vdash_{\mathcal{T}} \phi \rightarrow x_i = y_i$, proving $\mathcal{T}$ is convex.

□

B.3 Proof of Theorem 19

In the proofs of Lemma 18 and Theorem 19, we will use the same convention found in the proofs of Lemma 85 and Theorem 17: $\mathcal{A}$ and $\mathcal{B}$ will be Σ_n-interpretations (often $\mathcal{T}$-interpretations in addition), while $\mathcal{C}$ and $\mathcal{D}$ will be $\Sigma_{n,s}$-interpretations (often $(\mathcal{T})_\vee$-interpretations as well). Also important, on what is to follow, given a $\Sigma_{n,s}$ formula ϕ and a $\Sigma_{n,s}$-interpretation $\mathcal{C}$ that satisfies ϕ, is to consider the formula $\phi^\dagger_{\mathcal{C}}$. To define this formula, let $vars_{\sigma_1}(\phi) = \{z_1, \ldots, z_n\}$, M_i be the maximum of j such that $s^j(z_i)$ appears in ϕ, and $\{y_{i,j} : 1 \leq i \leq n, 0 \leq j \leq M_i + 2\}$ be fresh variables of sort σ_1.

1. We define $\phi^\dagger$ by replacing each atomic subformula $s^j(z_i) = s^q(z_p)$ of ϕ by $y_{i,j} = y_{p,q}$;
2. δ_V is the arrangement induced on the set of variables $V = \{y_{i,j} : 1 \leq i \leq n, 0 \leq j \leq M_i + 2\}$ by making x related to y iff $x^{\mathcal{C}} = y^{\mathcal{C}}$, for any $x, y \in V$.

We then make $\phi^\dagger_{\mathcal{C}} = \phi^\dagger \wedge \delta_V$.

Given a Σ_n-interpretation, $\hat{s}(\mathcal{A})$ is the same $\Sigma_{n,s}$-interpretation as $s(\mathcal{A})$: we choose to use different symbols given that the two operators are used in different contexts. Similarly, given a $\Sigma_{n,s}$-interpretation $\mathcal{C}$, we define $\Theta(\mathcal{C})$ to be the same as $\mathcal{C}$.

Lemma 18 *Every theory $\mathcal{T}$ defined over an empty signature Σ that is finitely witnessable (respectively strongly finitely witnessable) w.r.t. $S \subseteq \mathcal{S}_\Sigma$, has a witness (strong witness) that is variable-dependent.*

Proof Take a quantifier-free formula ϕ. Let $V = vars(\phi)$, take the set $Eq(V)$ of equivalence relations on V, and for a $E \in Eq(V)$, let δ^E_V be the arrangement induced by E on V; we then define

$$\chi(V) = \bigvee_{E \in Eq(V)} wit(\delta^E_V) \quad \text{and} \quad wit_0(\phi) = \phi \wedge \chi(V),$$

and we state that, if wit is a witness, respectively a strong witness, wit_0 is also a witness, respectively a strong witness. Of course, since wit is computable, wit_0 is a computable function from quantifier-free formulas to quantifier-free formulas.

1. We start by showing that, if $\overrightarrow{x} = vars(wit_0(\phi)) \setminus V$, ϕ and $\exists \overrightarrow{x}. wit_0(\phi)$ are $\mathcal{T}$-equivalent; of course, if $\exists \overrightarrow{x}. wit_0(\phi) = \phi \wedge \exists \overrightarrow{x}. \chi(V)$ (what we can do since ϕ contain none of the variables in $\overrightarrow{x}$) is satisfied by a $\mathcal{T}$-interpretation $\mathcal{A}$, so is ϕ, so let us focus on the other direction instead.
   So assume the $\mathcal{T}$-interpretation $\mathcal{A}$ satisfies ϕ, and let E_0 be the equivalence on V such that $x E_0 y$ iff $x^{\mathcal{A}} = y^{\mathcal{A}}$, meaning $\mathcal{A}$ satisfies $\delta^{E_0}_V$. Then $\mathcal{A}$ must also satisfy $\exists \overrightarrow{y}. wit(\delta^{E_0}_V)$, for $\overrightarrow{y} = vars(wit(\delta^{E_0}_V)) \setminus V$, and thus there exists a $\mathcal{T}$-interpretation $\mathcal{A}'$, differing from $\mathcal{A}$ at most on $\overrightarrow{y}$, that satisfies $wit(\delta^{E_0}_V)$ (and thus $\exists \overrightarrow{y}. wit(\delta^{E_0}_V)$ and $\delta^{E_0}_V$ as well).
   Because we are in an empty signature, all atomic subformulas of ϕ are equalities of variables $x = y$: and this formula is satisfied in $\mathcal{A}'$ iff $x E_0 y$, what happens in turn iff $x^{\mathcal{A}} = y^{\mathcal{A}}$; this means the atomic subformulas of ϕ receive the same truth-values in $\mathcal{A}$ and $\mathcal{A}'$, and since ϕ is quantifier-free we get that $\mathcal{A}'$ satisfies ϕ. Of course, $\mathcal{A}'$ also

satisfies $wit(\delta_V^{E_0})$, and thus satisfies $\phi \wedge \chi(V)$, meaning $\mathcal{A}$ satisfies $\exists \overrightarrow{y}\,.\,\phi \wedge \chi(V)$; since the variables in $\overrightarrow{y}$ are a subset of the variables in $\overrightarrow{x}$, we get $\mathcal{A}$ satisfies $\exists \overrightarrow{x}\,.\,\phi \wedge \chi(V)$. Thus ϕ and $\exists \overrightarrow{x}\,.\,wit_0(\phi)$ are indeed $\mathcal{T}$-equivalent.

2. (a) Suppose that *wit* is a witness, and let $\mathcal{A}$ be a $\mathcal{T}$-interpretation that satisfies $wit_0(\phi)$. Let E_0 be the equivalence on V such that xE_0y iff $x^{\mathcal{A}} = y^{\mathcal{A}}$, and we know that $\mathcal{A}$ also satisfies $\delta_V^{E_0}$, and thus $\exists \overrightarrow{y}\,.\,wit(\delta_V^{E_0})$, for $\overrightarrow{y} = vars(wit(\delta_V^{E_0})) \setminus vars(\delta_V^{E_0})$ (contained in $\overrightarrow{x} = vars(wit_0(\phi)) \setminus V$). Changing the value of $\mathcal{A}$ at most on the variables $\overrightarrow{y}$, we obtain a second $\mathcal{T}$-interpretation $\mathcal{A}'$ that satisfies $wit(\delta_V^{E_0})$; but of course $\mathcal{A}'$ also satisfies $\exists \overrightarrow{y}\,.\,wit(\delta_V^{E_0})$ and thus $\delta_V^{E_0}$.

   Now, since *wit* is a witness, there must exist a $\mathcal{T}$-interpretation $\mathcal{B}$ that satisfies $wit(\delta_V^{E_0})$ with

$$\sigma^{\mathcal{B}} = vars_\sigma(wit(\delta_V^{E_0}))^{\mathcal{B}}$$

   for each $\sigma \in S$. But, because $\mathcal{B}$ satisfies $wit(\delta_V^{E_0})$, it satisfies $\exists \overrightarrow{y}\,.\,wit(\delta_V^{E_0})$ and thus $\delta_V^{E_0}$; again, since we are on the empty signature, this means the atomic subformulas of ϕ receive the same truth-value in either $\mathcal{A}$ or $\mathcal{B}$, and so $\mathcal{B}$ satisfies ϕ (and, since $\mathcal{B}$ also satisfies $wit(\delta_V^{E_0})$ and thus $\bigvee_{E\in Eq(V)} wit(\delta_V^{E})$, $\mathcal{B}$ then satisfies $\phi \wedge \chi(V)$). Furthermore, $vars_\sigma(wit(\delta_V^{E_0})) \subseteq vars_\sigma(wit_0(\phi))$, meaning $\sigma^{\mathcal{B}} = vars_\sigma(wit_0(\phi))^{\mathcal{B}}$ and, therefore, that wit_0 is also a witness.

   (b) Suppose now *wit* is a strong witness, U is a set of variables, F is an equivalence on U, δ_U^F is the corresponding arrangement, and $\mathcal{A}$ is a $\mathcal{T}$-interpretation that satisfies $wit_0(\phi) \wedge \delta_U^F$. Let E_0 be the equivalence relation on V such that xE_0y iff $x^{\mathcal{A}} = y^{\mathcal{A}}$: this means that $\mathcal{A}$ satisfies $\delta_V^{E_0}$. Since $\mathcal{A}$ also satisfies $\chi(V)$, there is an equivalence $E \in Eq(V)$ such that $\mathcal{A}$ satisfies $wit(\delta_V^E)$, but we state that $E = E_0$: indeed, since $\mathcal{A}$ satisfies $wit(\delta_V^E)$, it also satisfies $\exists \overrightarrow{y}\,.\,wit(\delta_V^E)$ (for $\overrightarrow{y} = vars(wit(\delta_V^E)) \setminus vars(\delta_V^E)$) and therefore δ_V^E (since δ_V^E and $\exists \overrightarrow{y}\,.\,wit(\delta_V^E)$ are $\mathcal{T}$-equivalent); given $\mathcal{A}$ satisfies $\delta_V^{E_0}$ and δ_V^E, we have that $E = E_0$.

   Define then the set of variables $W = U \cup V$ with the equivalence G such that xGy iff $x^{\mathcal{A}} = y^{\mathcal{A}}$, and notice that $\mathcal{A}$ satisfies δ_W^G, a formula that implies both $\delta_V^{E_0}$ and δ_U^F. Since $\mathcal{A}$ satisfies $wit(\delta_V^{E_0}) \wedge \delta_W^G$, and *wit* is a strong witness, there must exist a $\mathcal{T}$-interpretation $\mathcal{B}$ that satisfies $wit(\delta_V^{E_0}) \wedge \delta_W^G$ with

$$\sigma^{\mathcal{B}} = vars_\sigma(wit(\delta_V^{E_0}) \wedge \delta_W^G)^{\mathcal{B}}$$

   for each $\sigma \in S$. Because $\mathcal{B}$ satisfies δ_W^G, it must satisfy $\delta_V^{E_0}$, meaning $\mathcal{B}$ and $\mathcal{A}$ satisfy precisely the same atomic subformulas of ϕ, and thus $\mathcal{B}$ satisfies ϕ (since that is a quantifier-free formula), and consequently $wit_0(\phi)$ (since $\mathcal{B}$ also satisfies $wit(\delta_V^{E_0})$ and thus $\chi(V)$); furthermore, because $\mathcal{B}$ satisfies δ_W^G, it also satisfies δ_U^F. Finally, notice the variables of sort σ of $wit(\delta_V^{E_0}) \wedge \delta_W^G$ are a subset of the set of variables of sort σ of $wit_0(\phi) \wedge \delta_U^F$: first of all, because $wit(\delta_V^{E_0})$ is a subformula of $\chi(V)$ and thus $wit_0(\phi)$; and second, because the variables of δ_W^G ($W = U \cup V$) are also variables of δ_U^F and ϕ (and thus $wit_0(\phi)$). So we have that $\sigma^{\mathcal{B}} = vars_\sigma(wit_0(\phi) \wedge \delta_U^F)^{\mathcal{B}}$, proving *wit* is indeed a strong witness.

□

Theorem 19 *Let $\mathcal{T}$ be a theory over an empty signature Σ_n with sorts $S = \{\sigma_1, \ldots, \sigma_n\}$. Then: $(\mathcal{T})_\vee$ is stably infinite, smooth, finitely witnessable, or strongly finitely witnessable w.r.t. S if and only if $\mathcal{T}$ is, respectively, stably infinite, smooth, finitely witnessable, or strongly finitely witnessable w.r.t. S. In addition, if $\mathcal{T}$ has a model $\mathcal{A}$ with $|\sigma_1^{\mathcal{A}}| \geq 2$, $(\mathcal{T})_\vee$ is not convex with respect to S.*

Proof Let $\mathcal{A}$ be a Σ_n-structure, and $\mathcal{C}$ a $\Sigma_{n,s}$-structure: of course, from Lemma 85, $\mathcal{A}$ is a model of $\mathcal{T}$ iff $\hat{s}(\mathcal{A})$ is a model of $(\mathcal{T})_\vee$; and $\mathcal{C}$ is a model of $(\mathcal{T})_\vee$ iff it satisfies $\psi_\vee$ and $\Theta(\mathcal{C})$ is a model of $\mathcal{T}$. This last observation is true since, although $\hat{s}(\Theta(\mathcal{C}))$ may differ from $\mathcal{C}$ in the function assigned to s, any formula on the axiomatization of $\mathcal{T}$ is free of symbols, thus receiving the same value in both $\mathcal{C}$ and $\hat{s}(\Theta(\mathcal{C}))$ (and therefore $\Theta(\mathcal{C})$).

1. (a) Suppose that $\mathcal{T}$ is stably-infinite, let ϕ be a quantifier-free $\Sigma_{n,s}$-formula and $\mathcal{C}$ a $(\mathcal{T})_\vee$-interpretation that satisfies ϕ. Then the $(\mathcal{T})_\vee$-interpretation $\mathcal{C}'$, obtained from $\mathcal{C}$ by making $y_{i,j}^{\mathcal{C}'} = (s^{\mathcal{C}})^j(z_i^{\mathcal{C}})$, satisfies the quantifier and symbol-free formula $\phi_{\mathcal{C}}^{\dagger}$, meaning $\Theta(\mathcal{C}')$ satisfies $\phi_{\mathcal{C}}^{\dagger}$. Since $\mathcal{T}$ is assumed to be stably-infinite, there is an infinite (on all domains) $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies $\phi_{\mathcal{C}}^{\dagger}$. By defining the $(\mathcal{T})_\vee$-interpretation $\mathcal{D}$ such that: $\sigma^{\mathcal{D}} = \sigma^{\mathcal{A}}$ for all $\sigma \in S$; for all $y_{i,j}$ with $j \leq M_i + 1$, $s^{\mathcal{D}}(y_{i,j}^{\mathcal{A}}) = y_{i,j+1}^{\mathcal{A}}$, and for all other elements a of $\sigma_1^{\mathcal{A}}$, $s^{\mathcal{D}}(a) = a$;[5] and, for all z_i, $z_i^{\mathcal{D}} = y_{i,0}^{\mathcal{A}}$, and for all other variables x, $x^{\mathcal{D}} = x^{\mathcal{A}}$, we see that $\mathcal{D}$ is a $(\mathcal{T})_\vee$-interpretation, infinite on all domains, that satisfies ϕ.
   Indeed, looking at $\psi_\vee$, for each element a of $\sigma_1^{\mathcal{D}}$, either $s^{\mathcal{D}}(a) = a$, or $a = y_{i,j}^{\mathcal{A}}$ for $j \leq M_i + 1$: if $j < M_i + 1$, $s^{\mathcal{D}}(s^{\mathcal{D}}(a)) = y_{i,j+2}^{\mathcal{A}}$, which we know to equal $y_{i,j}^{\mathcal{A}}$; if $j = M_i + 1$, $s^{\mathcal{D}}(a) = y_{i,j+1}^{\mathcal{A}}$ equals either $y_{i,j-1}^{\mathcal{A}}$ or $y_{i,j}^{\mathcal{A}}$, and in the former case $s^{\mathcal{D}}(s^{\mathcal{D}}(a)) = y_{i,j}^{\mathcal{A}}$. And $s^{\mathcal{D}}$ is a well-defined function since, if $y_{i,j}^{\mathcal{A}} = y_{p,q}^{\mathcal{A}}$ for $j < M_i + 1$ and $q < M_p + 1$, $\mathcal{A}$ satisfies $y_{i,j} = y_{p,q}$, and thus that is a literal of δ_V; this means in $\mathcal{C}'$ that $(s^{\mathcal{C}'})^j(z_i) = (s^{\mathcal{C}'})^q(z_p)$, and so $(s^{\mathcal{C}'})^{j+1}(z_i) = (s^{\mathcal{C}'})^{q+1}(z_p)$, from what $y_{i,j+1} = y_{p,q+1}$ is also in δ_V and thus both are the same in $\mathcal{A}$. Now, for ϕ: if $s^j(z_i) = s^q(z_p)$ is an atomic subformula of ϕ, it is satisfied by $\mathcal{C}$ iff $y_{i,j} = y_{p,q}$ shows up in δ_V, what happens if and only if $(s^{\mathcal{D}})^j(z_i^{\mathcal{D}}) = (s^{\mathcal{D}})^q(z_p)$; of course, since the atomic subformulas of ϕ receive the same value in $\mathcal{C}$ and $\mathcal{D}$, and ϕ is quantifier-free, $\mathcal{D}$ satisfies ϕ.

   (b) Reciprocally, assume $(\mathcal{T})_\vee$ is stably-infinite, take a quantifier-free Σ_n-formula ϕ and a $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies ϕ, meaning $\hat{s}(\mathcal{A})$ satisfies ϕ. There must then exist a $(\mathcal{T})_\vee$-interpretation $\mathcal{C}$, infinite in all domains, that satisfies ϕ, and since ϕ is free of symbols, $\Theta(\mathcal{C})$ satisfies ϕ and has $\sigma^{\Theta(\mathcal{C})}$ infinite for all $\sigma \in S$.
2. (a) Suppose now that $\mathcal{T}$ is smooth, let ϕ be a quantifier-free $\Sigma_{n,s}$-formula, $\mathcal{C}$ a $(\mathcal{T})_\vee$-interpretation that satisfies ϕ, and κ a function from S to the class of cardinals such that $\kappa(\sigma) \geq |\sigma^{\mathcal{C}}|$ for each $\sigma \in S$. We take the $(\mathcal{T})_\vee$-interpretation $\mathcal{C}'$ (obtained from $\mathcal{A}$ by changing $y_{i,j}^{\mathcal{C}'}$ so that it equals $(s^{\mathcal{C}})^j(z_i^{\mathcal{C}})$), which satisfies the quantifier and symbol-free formula $\phi_{\mathcal{C}}^{\dagger}$, meaning $\Theta(\mathcal{C}')$ satisfies $\phi_{\mathcal{C}}^{\dagger}$. Using $\mathcal{T}$ is smooth, there must exist a $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies $\phi_{\mathcal{C}}^{\dagger}$ and $|\sigma^{\mathcal{A}}| = \kappa(\sigma)$ for each $\sigma \in S$. We finally define a $(\mathcal{T})_\vee$-interpretation $\mathcal{D}$ such that: $\sigma^{\mathcal{D}} = \sigma^{\mathcal{A}}$ for all $\sigma \in S$; for all $a = y_{i,j}^{\mathcal{A}}$ with $j < M_i + 1$, $s^{\mathcal{D}}(a) = y_{i,j+1}^{\mathcal{A}}$, and for all other $a \in \sigma_1^{\mathcal{A}}$, $s^{\mathcal{D}}(a) = a$;

[5] Notice that the first step already defines $s^{\mathcal{D}}(y_{i,j}^{\mathcal{A}})$ for $j = M_i + 2$ since we must have either $y_{i,j+2} = y_{i,j}$ or $y_{i,j+2} = y_{i,j+1}$ in $\mathcal{C}'$ for $0 \leq j \leq M_i$, what remains true in δ_V and thus $\mathcal{A}$.

and $z_i^{\mathcal{D}} = y_{i,0}^{\mathcal{A}}$, and for all other variables x, $x^{\mathcal{D}} = x^{\mathcal{A}}$. Given that $\mathcal{D}$ satisfies an atomic subformula of ϕ iff $\mathcal{C}$ does so, it satisfies ϕ; it is also a $(\mathcal{T})_{\vee}$-interpretation such that $|\sigma^{\mathcal{D}}| = \kappa(\sigma)$ for each $\sigma \in S$, the argument being the same as the one for the stably infinite case.

(b) Reciprocally, suppose $(\mathcal{T})_{\vee}$ is smooth, and then take a quantifier-free Σ_n-formula ϕ, a $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies ϕ and a function κ from S to the class of cardinals such that $\kappa(\sigma) \geq |\sigma^{\mathcal{A}}|$ for all $\sigma \in S$; it follows that $\hat{s}(\mathcal{A})$ satisfies ϕ, and since $(\mathcal{T})_{\vee}$ is smooth there must exist a $(\mathcal{T})_{\vee}$-interpretation $\mathcal{C}$ that satisfies ϕ with $|\sigma^{\mathcal{C}}| = \kappa(\sigma)$ for each $\sigma \in S$. Since ϕ has no function symbols, $\Theta(\mathcal{C})$ is then a $\mathcal{T}$-interpretation that satisfies ϕ with $|\sigma^{\Theta(\mathcal{C})}| = \kappa(\sigma)$ for any $\sigma \in S$, finishing the proof that $\mathcal{T}$ is also smooth.

3. (a) Assume $\mathcal{T}$ is finitely witnessable, with a witness $wit(\phi) = \phi \wedge \psi(vars(\phi))$, where $\psi(vars(\phi))$ is a formula that depends only on the variables of ϕ; there is always such a witness as proved in Lemma 18. We state that $(\mathcal{T})_{\vee}$ is also finitely witnessable, with witness

$$wit_s(\phi) = \phi \wedge \psi(vars(\phi) \cup \overrightarrow{y}) \wedge \bigwedge_{i=1}^{n} \bigwedge_{j=0}^{M_i+2} [y_{i,j} = s^j(z_i)] \wedge \Psi_{\vee}(\overrightarrow{y}) \wedge Fun(\overrightarrow{y}),$$

where now ϕ may be a $\Sigma_{n,s}$-formula, $vars_{\sigma_1}(\phi) = \{z_1, \ldots, z_n\}$, M_i is the maximum of j such that $s^j(z_i)$ occurs in ϕ, and $\overrightarrow{y} = \{y_{i,j} : 1 \leq i \leq n, 0 \leq j \leq M_i + 2\}$ are fresh variables of sort σ_1. Take a $\Sigma_{n,s}$-formula ϕ and a $(\mathcal{T})_{\vee}$-interpretation $\mathcal{C}$ that satisfies ϕ. If we take the interpretation $\mathcal{A}$ that differs from $\Theta(\mathcal{C})$ on $\overrightarrow{y}$, where $y_{i,j}^{\mathcal{A}} = (s^{\mathcal{C}})^j(z_i^{\mathcal{C}})$, we have that it satisfies $\Psi_{\vee}(\overrightarrow{y})$ (since $\mathcal{C}$ satisfies $\psi_{\vee}$), $Fun(\overrightarrow{y})$ (since $s^{\mathcal{C}}$ is a function in $\mathcal{C}$), and $\phi_{\mathcal{C}}^{\dagger}$, and thus $\exists \overrightarrow{x}. wit(\phi_{\mathcal{C}}^{\dagger}) = \phi_{\mathcal{C}}^{\dagger} \wedge \exists \overrightarrow{x}. \psi(vars(\phi_{\mathcal{C}}^{\dagger}))$ for

$$\overrightarrow{x} = vars(wit(\phi_{\mathcal{C}}^{\dagger})) \setminus vars(\phi_{\mathcal{C}}^{\dagger}) = vars(wit_s(\phi)) \setminus \left[vars(\phi) \cup \overrightarrow{y}\right],$$

where we remind the reader that $vars(\phi_{\mathcal{C}}^{\dagger}) = vars(\phi) \cup \overrightarrow{y}$; there is then an interpretation $\mathcal{A}'$, differing from $\mathcal{A}$ at most on $\overrightarrow{x}$, that satisfies $\phi_{\mathcal{C}}^{\dagger} \wedge \psi(vars(\phi_{\mathcal{C}}^{\dagger}))$. We now take the $(\mathcal{T})_{\vee}$-interpretation $\mathcal{D}$ that differs from $\mathcal{C}$ at most on $\overrightarrow{x} \cup \overrightarrow{y}$, with $\Theta(\mathcal{D}) = \mathcal{A}'$, and we have that $\mathcal{D}$ satisfies: ϕ, since $\mathcal{C}$ satisfies ϕ, and $\mathcal{D}$ only differs from $\mathcal{C}$ at most on $\overrightarrow{x} \cup \overrightarrow{y}$, none of these variables present in ϕ; $\psi(vars(\phi) \cup \overrightarrow{y})$, since this formula is satisfied by $\mathcal{A}'$ and has no function symbols; $\Psi_{\vee}(\overrightarrow{y})$ and $Fun(\overrightarrow{y})$, since both are satisfied by $\mathcal{A}$, and $\mathcal{D}$ only differs on the values given for variables from $\mathcal{A}$ on $\overrightarrow{x}$; and $\bigwedge_{j=0}^{M_i+2}[y_{i,j} = s^j(z_i)]$, since

$$y_{i,j}^{\mathcal{D}} = y_{i,j}^{\mathcal{A}'} = y_{i,j}^{\mathcal{A}} = (s^{\mathcal{C}})^j(z_i^{\mathcal{C}}) = (s^{\mathcal{D}})^j(z_i^{\mathcal{D}}).$$

This means, of course, that $\mathcal{C}$ satisfies $\exists \overrightarrow{x}. \exists \overrightarrow{y}.\, wit_s(\phi)$, as we needed to show. Reciprocally, if the $(\mathcal{T})_{\vee}$-interpretation $\mathcal{C}$ satisfies $\exists \overrightarrow{x}. \exists \overrightarrow{y}.\, wit_s(\phi)$, it is obvious that it satisfies ϕ (since the variables in $\overrightarrow{x} \cup \overrightarrow{y}$ do not occur in ϕ), meaning ϕ and $\exists \overrightarrow{x}. \exists \overrightarrow{y}.\, wit_s(\phi)$ are $(\mathcal{T})_{\vee}$-equivalent.

So, suppose $\mathcal{C}$ is a $(\mathcal{T})_{\vee}$-interpretation that satisfies $wit_s(\phi)$, meaning $\mathcal{A}$, differing from $\Theta(\mathcal{C})$ at most on $y_{i,j} \in \overrightarrow{y}$, where $y_{i,j}^{\mathcal{A}} = (s^{\mathcal{C}})^j(z_i^{\mathcal{C}})$, satisfies

$$\phi_{\mathcal{C}}^{\dagger} \wedge \psi(vars(\phi_{\mathcal{C}}^{\dagger}) \cup \overrightarrow{y}) \wedge \Psi_{\vee}(\overrightarrow{y}) \wedge Fun(\overrightarrow{y}) = wit(\phi_{\mathcal{C}}^{\dagger} \wedge \Psi_{\vee}(\overrightarrow{y}) \wedge Fun(\overrightarrow{y})).$$

There is, therefore, a $\mathcal{T}$-interpretation $\mathcal{B}$ that satisfies $wit(\phi_{\mathcal{C}}^{\dagger} \wedge \Psi_{\vee}(\overrightarrow{y}) \wedge Fun(\overrightarrow{y}))$ with $\sigma^{\mathcal{B}} = vars_{\sigma}(wit_s(\phi))^{\mathcal{B}}$, for each $\sigma \in S$, the set $vars_{\sigma}(wit_s(\phi))$ happening to be the same as $vars_{\sigma}(wit(\phi_{\mathcal{C}}^{\dagger} \wedge \Psi_{\vee}(\overrightarrow{y}) \wedge Fun(\overrightarrow{y})))$. We then build a $(\mathcal{T})_{\vee}$-interpretation $\mathcal{D}$ with: $\sigma^{\mathcal{D}} = \sigma^{\mathcal{B}}$ for every $\sigma \in S$; $x^{\mathcal{D}} = x^{\mathcal{B}}$ for every variable x; and $s^{\mathcal{D}}(y_{i,j}^{\mathcal{D}}) = y_{i,j+1}^{\mathcal{D}}$ for every $1 \leq i \leq n$ and $0 \leq j \leq M_i + 1$, and $s^{\mathcal{D}}(a) = a$ for all other elements of $\sigma_1^{\mathcal{D}}$. We indeed have that $s^{\mathcal{D}}$ is a function since $\mathcal{B}$ satisfies $Fun(\overrightarrow{y})$, and $\mathcal{D}$ satisfies $\psi_{\vee}$ because $\mathcal{B}$ satisfies $\Psi_{\vee}(\overrightarrow{y})$; it follows that, not only $\mathcal{D}$ satisfies $wit_s(\phi)$, but also has the property that $\sigma^{\mathcal{D}} = vars_{\sigma}(wit_s(\phi))^{\mathcal{D}}$, proving wit_s is indeed a witness for $(\mathcal{T})_{\vee}$.

(b) We assume $(\mathcal{T})_{\vee}$ is finitely witnessable, with witness wit, and then state that

$$wit_0(\phi) = \phi \wedge wit(\phi)^{\dagger} \wedge \bigwedge_{i=1}^{n}(y_{i,0} = z_i) \wedge \Psi_{\vee}(\overrightarrow{y}) \wedge Fun(\overrightarrow{y})$$

is a witness for $\mathcal{T}$, where: $vars_{\sigma_1}(wit(\phi)) = \{z_1, \ldots, z_n\}$; M_i is the maximum of j such that $s^j(z_i)$ appears in $wit(\phi)$; $\overrightarrow{y} = \{y_{i,j} : 1 \leq i \leq n, 1 \leq j \leq M_i + 2\}$ are fresh variables of sort σ_1; $wit(\phi)^{\dagger}$ is obtained from $wit(\phi)$, in this case by replacing $s^j(z_i) = s^q(z_p)$ for $y_{i,j} = y_{p,q}$, where $1 \leq i \leq n$ and $0 \leq j \leq M_i$;

$$\Psi_{\vee}(\overrightarrow{y}) = \bigwedge_{i=1}^{n} \left[(y_{i,2} = y_{i,1}) \vee (y_{i,2} = y_{i,0})\right]$$

codifies the validity of $\psi_{\vee}$ among the values $y_{i,j}$, and

$$Fun(\overrightarrow{y}) = \bigwedge_{i=1}^{n} \bigwedge_{p=1}^{n} \bigwedge_{j=0}^{M_i+1} \bigwedge_{q=0}^{M_p+1} \left[(y_{i,j} = y_{p,q}) \rightarrow (y_{i,j+1} = y_{p,q+1})\right]$$

codifies the fact that s should be a function among these same values. We start by taking a quantifier-free Σ_n formula ϕ and a $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies ϕ, and then $\hat{s}(\mathcal{A})$ satisfies ϕ and thus $\exists \overrightarrow{x} \,.\, wit(\phi)$, where

$$\overrightarrow{x} = vars(wit(\phi)) \setminus vars(\phi) = vars(wit_0(\phi)) \setminus [vars(\phi) \cup \overrightarrow{y}].$$

There must exist a $(\mathcal{T})_{\vee}$-interpretation $\mathcal{C}$, differing from $\hat{s}(\mathcal{A})$ at most on $\overrightarrow{x}$, that satisfies $wit(\phi)$ and in addition ϕ, since $\hat{s}(\mathcal{A})$ does satisfy ϕ and $\mathcal{C}$ does not change the value of the variables in ϕ. Taking then the $\mathcal{T}$-interpretation $\mathcal{B}$ that differs from $\Theta(\mathcal{C})$ at most on $\overrightarrow{x} \cup \overrightarrow{y}$ such that, for x in $\overrightarrow{x}$ one has $x^{\mathcal{B}} = x^{\mathcal{C}}$, and for $y_{i,j}$ in $\overrightarrow{y}$ one has $y_{i,j}^{\mathcal{B}} = (s^{\mathcal{C}})^j(z_i^{\mathcal{C}})$, it is easy to see that: $\mathcal{B}$ satisfies $\phi \wedge wit(\phi)^{\dagger}$, since $\mathcal{C}$ satisfies $\phi \wedge wit(\phi)$; and $\mathcal{B}$ also satisfies $\Psi_{\vee}(\overrightarrow{y})$ and $Fun(\overrightarrow{y})$ given that $z_i^{\mathcal{B}} = y_{i,0}^{\mathcal{B}} = \cdots = y_{i,M_i+2}^{\mathcal{B}}$ for each $1 \leq i \leq n$. Hence, $\mathcal{A}$ satisfies $\exists \overrightarrow{x} \,.\, \exists \overrightarrow{y} \,.\, wit_0(\phi)$. Now, if the $\mathcal{T}$-interpretation $\mathcal{A}$ satisfies instead $\exists \overrightarrow{x} \,.\, \exists \overrightarrow{y} \,.\, wit_0(\phi)$, given that the variables in $\overrightarrow{x}$ and $\overrightarrow{y}$ do not occur in ϕ, it follows that $\mathcal{A}$ satisfies ϕ, and so ϕ and $\exists \overrightarrow{x} \,.\, \exists \overrightarrow{y} \,.\, wit_0(\phi)$ are $\mathcal{T}$-equivalent.

Now, suppose $\mathcal{A}$ is a $\mathcal{T}$-interpretation that satisfies $wit_0(\phi)$; $\hat{s}(\mathcal{A})$ must then satisfy $wit(\phi)$, and so there exists a $(\mathcal{T})_{\vee}$-interpretation $\mathcal{C}$ that satisfies $wit(\phi)$ with $\sigma^{\mathcal{C}} = vars_{\sigma}(wit(\phi))^{\mathcal{C}}$ for each $\sigma \in S$. We then take the $\mathcal{T}$-interpretation $\mathcal{B}$, differing from $\Theta(\mathcal{C})$ at most on $\overrightarrow{y}$, such that $y_{i,j}^{\mathcal{B}} = (s^{\mathcal{C}})^j(z_i^{\mathcal{C}})$. This way, $\mathcal{B}$ satisfies ϕ because $\mathcal{C}$ does so; it satisfies $wit(\phi)^{\dagger}$ and $\bigwedge_{i=1}^{n} y_{i,0} = z_i$ by its very definition; and it satisfies $\Psi_{\vee}(\overrightarrow{y})$ and $Fun(\overrightarrow{y})$ because, respectively, $\mathcal{C}$ satisfies $\psi_{\vee}$ and $s^{\mathcal{C}}$ is a function. Furthermore, for any $\sigma \in S$, $vars_\sigma(wit(\phi)) \subset vars_\sigma(wit_0(\phi))$, and so $\sigma^{\mathcal{B}} = vars_\sigma(wit_0(\phi))^{\mathcal{B}}$.

4. (a) Suppose $\mathcal{T}$ is strongly finitely witnessable, with a witness $wit(\phi) = \phi \wedge \psi(vars(\phi))$, where $\psi(vars(\phi))$ is a formula that depends only on the variables of ϕ, and we state that then $(\mathcal{T})_\vee$ is also strongly finitely witnessable, with witness wit_s as defined in item 3(b) above: of course, we already know ϕ and $\exists \overrightarrow{x}.\exists \overrightarrow{y}.\, wit_s(\phi)$ are $(\mathcal{T})_\vee$-equivalent, where $\overrightarrow{x} = vars(wit(\phi)) \setminus vars(\phi)$. So, take a set of variables V, an arrangement δ_V over V, and a $(\mathcal{T})_\vee$-interpretation $\mathcal{C}$ that satisfies $wit_s(\phi) \wedge \delta_V$; let W be $V \cup \overrightarrow{x} \cup \overrightarrow{y} \cup vars(\phi)$, δ_W be the arrangement on W such that x is related to y iff $x^{\mathcal{C}} = y^{\mathcal{C}}$, and we have that $\mathcal{C}$ satisfies $wit_s(\phi) \wedge \delta_W$. Then $\Theta(\mathcal{C})$ satisfies

$$\phi^\dagger_{\mathcal{C}} \wedge \psi(vars(\phi^\dagger_{\mathcal{C}}) \cup \overrightarrow{y}) \wedge \bigwedge_{i=1}^{n}(y_{i,j} = z_i) \wedge \Psi_\vee(\overrightarrow{y}) \wedge Fun(\overrightarrow{y}) \wedge \delta_W =$$
$$wit(\phi^\dagger_{\mathcal{C}} \wedge \bigwedge_{i=1}^{n}(y_{i,j} = z_i) \wedge \Psi_\vee(\overrightarrow{y}) \wedge Fun(\overrightarrow{y})) \wedge \delta_W,$$

so there must exist a $\mathcal{T}$-interpretation $\mathcal{A}$, given that wit is a strong witness, with $\sigma^{\mathcal{A}} = vars_\sigma(wit_s(\psi) \wedge \delta_V)^{\mathcal{A}}$ for each $\sigma \in S$,[6] and such that $\mathcal{A}$ satisfies

$$wit(\phi^\dagger_{\mathcal{C}} \wedge \bigwedge_{i=1}^{n}(y_{i,j} = z_i) \wedge \Psi_\vee(\overrightarrow{y}) \wedge Fun(\overrightarrow{y})) \wedge \delta_V.$$

We finally define a $(\mathcal{T})_\vee$-interpretation $\mathcal{D}$ such that: $\sigma^{\mathcal{D}} = \sigma^{\mathcal{A}}$ for every $\sigma \in S$; $x^{\mathcal{D}} = x^{\mathcal{A}}$ for every variable x; and $s^{\mathcal{D}}(y^{\mathcal{D}}_{i,j}) = y^{\mathcal{D}}_{i,j+1}$ for every $1 \leq i \leq n$ and $0 \leq j \leq M_i + 1$, and $s^{\mathcal{D}}(a) = a$ for all other elements of $\sigma_1^{\mathcal{D}}$ (again, notice $s^{\mathcal{D}}(y^{\mathcal{D}}_{i,j+2})$, for $j = M_i$, must have been defined under these conditions). We indeed have that $s^{\mathcal{D}}$ is a function, since $\mathcal{A}$ satisfies $Fun(\overrightarrow{y})$, and $\mathcal{D}$ satisfies $\psi_\vee$, because $\mathcal{A}$ satisfies $\Psi_\vee(\overrightarrow{y})$. Of course $\mathcal{D}$ satisfies $wit_s(\phi) \wedge \delta_V$, and in addition has the property that $\sigma^{\mathcal{D}} = vars_\sigma(wit_s(\phi) \wedge \delta_V)^{\mathcal{D}}$ for each $\sigma \in S$, proving wit_s is a strong witness for $(\mathcal{T})_\vee$.

(b) Assume now that $(\mathcal{T})_\vee$ is strongly finitely witnessable, with a strong witness wit, and we will prove that wit_0 from item 3(a) above is also a strong witness; as we already know, ϕ and $\exists \overrightarrow{x}.\exists \overrightarrow{y}.\, wit_0(\phi)$ are $\mathcal{T}$-equivalent, for $\overrightarrow{x} = vars(wit(\phi)) \setminus vars(\phi)$. So take a set of variables V, an arrangement δ_V on V, and a $\mathcal{T}$-interpretation $\mathcal{A}$ that satisfies $wit_0(\phi) \wedge \delta_V$. It follows that $\hat{s}(\mathcal{A})$ satisfies $wit(\phi) \wedge \delta_W$, where $W = V \cup vars(wit_0(\phi))$ and δ_W is the arrangement induced on W by making x related to y iff $x^{\hat{s}(\mathcal{A})} = y^{\hat{s}(\mathcal{A})}$; since wit is a strong witness for $(\mathcal{T})_\vee$, there exists a $(\mathcal{T})_\vee$-interpretation $\mathcal{C}$ that satisfies $wit(\phi) \wedge \delta_W$ and $\sigma^{\mathcal{C}} = vars_\sigma(wit(\phi) \wedge \delta_W)^{\mathcal{C}}$ for each $\sigma \in S$. Now, consider the $(\mathcal{T})_\vee$-interpretation $\mathcal{D}$ with $\sigma^{\mathcal{D}} = \sigma^{\mathcal{C}}$ for each $\sigma \in S$, $x^{\mathcal{D}} = x^{\mathcal{C}}$ for each variable x, and $s^{\mathcal{D}}$ equal to $s^{\mathcal{C}}$ except on the elements

$$\{y^{\mathcal{B}}_{i,j} : 1 \leq i \leq n, 1 \leq j \leq M_i + 2\} \cup \{(s^{\mathcal{C}})^j(z_i^{\mathcal{C}}) : 1 \leq i \leq m, 1 \leq j \leq M_i + 2\},$$

where we define instead: $s^{\mathcal{D}}(y^{\mathcal{C}}_{i,j}) = y^{\mathcal{C}}_{i,j+1}$, for $1 \leq i \leq n$ and $j \leq M_i + 1$; and, if $s^{\mathcal{D}}((s^{\mathcal{C}})^j(z_i^{\mathcal{C}}))$ has not yet been defined, we simply make it equal to $(s^{\mathcal{C}})^{j+1}(z_i^{\mathcal{C}})$ (notice $s^{\mathcal{D}}(y^{\mathcal{C}}_{i,j+2})$, for $j = M_i$, must have, under these conditions, already been

[6] Notice the variables in $wit(\phi^\dagger_{\mathcal{C}} \wedge \Psi_\vee(\overrightarrow{y}) \wedge Fun(\overrightarrow{y})) \wedge \delta_W$ are the same as the ones in $wit_s(\phi) \wedge \delta_W$, which in turn are the same as the ones in $wit_s(\phi) \wedge \delta_V$

defined, since either $y^{\mathcal{C}}_{i,j+2} = y^{\mathcal{C}}_{i,j}$ or $y^{\mathcal{C}}_{i,j+2} = y^{\mathcal{C}}_{i,j+1}$). Since $\hat{s}(\mathcal{A})$ satisfies $Fun(\overrightarrow{y})$, if $\mathcal{C}$ satisfies $y_{i,j} = y_{p,q}$ it also satisfies $y_{i,j+1} = y_{p,q+1}$, meaning $s^{\mathcal{D}}$ is indeed a well-defined function; furthermore, given that $\hat{s}(\mathcal{A})$ also satisfies $\Psi_{\vee}(\overrightarrow{y})$, we have that $\mathcal{C}$ either satisfies $y_{i,2} = y_{i,1}$ or $y_{i,2} = y_{i,1}$, and so $\mathcal{D}$ satisfies $\psi_{\vee}$, being therefore a $(\mathcal{T})_{\vee}$-interpretation. It is a routine exercise to show $\mathcal{D}$ satisfies ϕ, $\bigwedge_{i=1}^{n} y_{i,0} = z_i$, $Fun(\overrightarrow{y})$ and $\Psi_{\vee}(\overrightarrow{y})$, and we have to prove that is satisfies $wit(\phi)^{\dagger}$, so let $s^{j}(z_i) = s^{q}(z_p)$ be one of its atomic formulas; we have $(s^{\mathcal{D}})^{j}(z^{\mathcal{D}}_{i}) = (s^{\mathcal{D}})^{j}(y^{\mathcal{D}}_{i,0}) = y^{\mathcal{D}}_{i,j}$, and $(s^{\mathcal{D}})^{q}(z^{\mathcal{D}}_{i}) = (s^{\mathcal{D}})^{q}(y^{\mathcal{D}}_{p,0}) = y^{\mathcal{D}}_{p,q}$ from the definition of $s^{\mathcal{D}}$. Furthermore, $\hat{s}(\mathcal{A})$ satisfies $y_{i,j} = y_{p,q}$ iff $\mathcal{C}$, and thus $\mathcal{D}$, satisfy the same formula; this means $\hat{s}(\mathcal{A})$ satisfies $s^{j}(z_i) = s^{q}(z_p)$ iff $\mathcal{D}$ does so, and therefore $\mathcal{D}$ satisfies $wit(\phi)^{\dagger}$, since $\hat{s}(\mathcal{A})$ does so and this formula is quantifier-free. Given that $\sigma^{\mathcal{C}} = vars_{\sigma}(wit(\phi) \wedge \delta_W)^{\mathcal{C}}$, and $\sigma^{\mathcal{D}} = vars_{\sigma}(wit(\phi) \wedge \delta_W)^{\mathcal{D}} = vars_{\sigma}(wit_0(\phi))^{\mathcal{D}}$, wit_0 is thus indeed a strong witness for $\mathcal{T}$.

Finally, we deal with convexity. Consider the conjunction of literals $\phi = (y = s(x)) \wedge (z = s(y))$, where x, y and z are of sort σ_1, and we have that

$$\vdash_{(\mathcal{T})_{\vee}} \phi \rightarrow (x = y) \vee (x = z) \vee (y = z).$$

To see that, suppose we have a $(\mathcal{T})_{\vee}$-interpretation $\mathcal{C}$ that satisfies ϕ, but neither $x = y$ nor $x = z$; therefore, $s^{\mathcal{C}}(a) \neq a$ and $s^{\mathcal{C}}(s^{\mathcal{C}}(a)) \neq a$, where $a = x^{\mathcal{C}}$. Since $s^{\mathcal{C}}(s^{\mathcal{C}}(a)) \neq a$, we must have $s^{\mathcal{C}}(s^{\mathcal{C}}(a)) = s^{\mathcal{C}}(a)$, meaning that $\mathcal{C}$ satisfies $y = z$.

However, we do not have that $(\mathcal{T})_{\vee}$ entails $\phi \rightarrow (x = y)$, $\phi \rightarrow (x = z)$ or $\phi \rightarrow (y = z)$, as we assume $\mathcal{T}$ possesses a model $\mathcal{A}$ with $|\sigma_1^{\mathcal{A}}| \geq 2$; say $a, b \in \sigma_1^{\mathcal{A}}$. To understand why, consider the interpretation $\mathcal{C}$ and $\mathcal{D}$ of $(\mathcal{T})_{\vee}$ with: $\sigma^{\mathcal{C}} = \sigma^{D} = \sigma^{\mathcal{A}}$ for every $\sigma \in S$; $s^{\mathcal{C}}(a) = b$, $s^{\mathcal{C}}(b) = b$, $s^{\mathcal{D}}(a) = b$, $s^{\mathcal{D}}(b) = a$ and $s^{\mathcal{C}}(c) = s^{\mathcal{D}}(c) = c$ for each $c \in \sigma_1^{\mathcal{A}} \setminus \{a, b\}$; and

$$x^{\mathcal{C}} = a \quad \text{and} \quad y^{\mathcal{C}} = z^{\mathcal{C}} = b, \quad \text{and} \quad x^{\mathcal{D}} = z^{\mathcal{D}} = a \quad \text{and} \quad y^{\mathcal{D}} = b.$$

We see that both $(\mathcal{T})_{\vee}$-interpretations satisfy ϕ; however, $\mathcal{C}$ does not satisfy $x = y$ nor $x = z$, while $\mathcal{D}$ does not satisfy $y = z$ (nor also $x = y$). □

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Declarations

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