CS255: Intro. to Cryptography **Winter 2024**

Assignment #3

Due: 11:59pm on Thu., Feb. 22, 2024, on Gradescope (each answer on a new page).

Problem 1. (*One-time MAC*) Recall that the one-time pad (OTP) is a semantically secure cipher that is unconditionally secure (that is, we can prove it secure without making any assumptions). In this question we build a one-time MAC that is unconditionally secure. A *one-time MAC* is a MAC that is secure against an adversary that makes at most a *single* chosen message query. The adversary chooses a message $m \in \mathcal{M}$; issues a chosen message query for m and gets back a tag t for m; and then wins the MAC game if it can output a valid message-tag pair (m^*, t^*) where $(m^*, t^*) \neq (m, t)$. The MAC is one-time unconditionally secure if no adversary can win this game with probability better than $1/|\mathcal{T}|$.

Let p be a prime and let $\mathcal{M} := \mathbb{Z}_p$, $\mathcal{K} := (\mathbb{Z}_p)^2$, and $\mathcal{T} := \mathbb{Z}_p$. Consider the following MAC (S, V) defined over $(\mathcal{M}, \mathcal{K}, \mathcal{T})$:

$$S\big((k_1, k_2), m\big) := k_1 m + k_2 \quad \text{and} \quad V\big((k_1, k_2), m, t\big) := \Big\{ \text{ accept if } t = k_1 m + k_2 \Big\}$$

Here additions and multiplications are defined in $\mathbb{Z}_p$. It is not difficult to show that (S, V) is an unconditionally secure one-time MAC (while it is not part of the homework problem, you can try to prove this for yourself). Your goal for this problem is to show that (S, V) is not two-time secure. That is, describe an adversary that can forge the MAC on some third message after issuing two chosen message queries.

Problem 2. (*Multicast MACs*) Suppose user A wants to broadcast a message to n recipients $B_1, \ldots, B_n$. Privacy is not important but integrity is: each of $B_1, \ldots, B_n$ should be assured that the message it received was sent by A. User A decides to use a MAC.

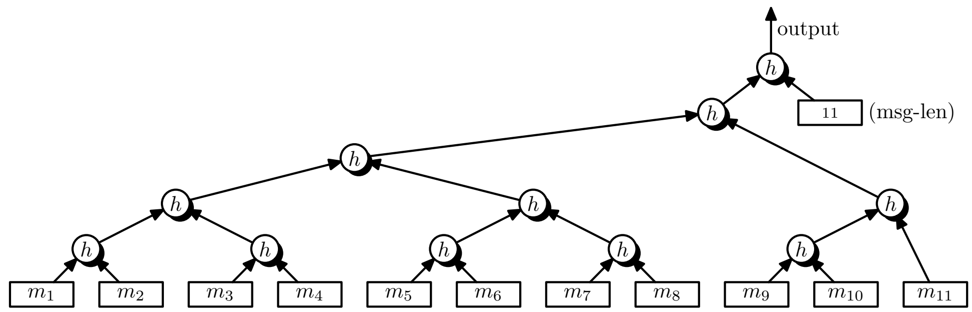
a. Suppose user A and $B_1, \ldots, B_n$ all share a secret key k. User A computes the tag for every message she sends using k. Every user B_i verifies the tag using k. Using at most two sentences explain why this scheme is insecure, namely, show that user B_1 is not assured that the messages it received are from A.

b. Suppose user A has a set $S = \{k_1, \ldots, k_\ell\}$ of ℓ secret keys. Each user B_i has some subset $S_i \subseteq S$ of the keys. When A transmits a message she appends ℓ tags to it by MACing the message with each of her ℓ keys. When user B_i receives a message it accepts the message as valid only if all tags corresponding to keys in S_i are valid. Let us assume that the users $B_1, \ldots, B_n$ do not collude with each other. What property should the sets $S_1, \ldots, S_n$ satisfy so that the attack from part (a) does not apply?

c. Show that when $n = 10$ (i.e. ten recipients) it suffices to take $\ell = 5$ in part (b). Describe the sets $S_1, \ldots, S_{10} \subseteq \{k_1, \ldots, k_5\}$ you would use.

d. Show that the scheme from part (c) is insecure if two users are allowed to collude.

Problem 3. (*Parallel Merkle-Damgård*) Recall that the Merkle-Damgård construction gives a *sequential* method for extending the domain of a CRHF. The tree construction in the figure below is a parallelizable approach: all the hash functions h within a single level can be computed in parallel. Prove that the resulting hash function defined over $(\mathcal{X}^{\leq L},\ \mathcal{X})$ is collision resistant, assuming h is collision resistant. Here h is a compression function $h : \mathcal{X}^2 \to \mathcal{X}$, and we assume the message length can be encoded as an element of $\mathcal{X}$.



More precisely, the hash function is defined as follows:

input: $m_1 \dots m_s \in \mathcal{X}^s$ for some $1 \leq s \leq L$
output: $y \in \mathcal{X}$

let $t \in \mathbb{Z}$ be the smallest power of two such that $t \geq s$ (i.e., $t := 2^{\lceil \log_2 s \rceil}$)
for $i = s+1$ to t: $m_i \leftarrow \perp$
for $i = t+1$ to $2t-1$:
  $\ell \leftarrow 2(i-t)-1, \quad r \leftarrow \ell + 1$ // *indices of left and right children*
  if $m_\ell = \perp$ and $m_r = \perp$: $m_i \leftarrow \perp$ // *if node has no children, set node to null*
  else if $m_r = \perp$: $m_i \leftarrow m_\ell$ // *if one child, propagate child as is*
  else $m_i \leftarrow h(m_\ell, m_r)$ // *if two children, hash with h*
output $y \leftarrow h\big(m_{2t-1},\ s\big)$ // *hash final output and message length*

Problem 4. (*Davies-Meyer*) In the lecture we saw that Davies-Meyer is used to convert an ideal block cipher into a collision resistant compression function. Let $E(k, m)$ be a block cipher where the message space is the same as the key space (e.g. 128-bit AES). Show that the following methods do not work:

$$f_1(x, y) = E(y, x) \oplus y \quad \text{and} \quad f_2(x, y) = E(x,\ x \oplus y)$$

That is, show an efficient algorithm for constructing collisions for f_1 and f_2. Recall that the block cipher E and the corresponding decryption algorithm D are both known to you.

Problem 5. (*Authenticated encryption*) Let (E, D) be an encryption system that provides authenticated encryption. Here E does not take a nonce as input and therefore must be a randomized encryption algorithm. Which of the following systems provide authenticated encryption? For those that do, give a short proof. For those that do not, present an attack that either breaks CPA security or ciphertext integrity.

a. $E_1(k, m) = \big[c \leftarrow E(k, m),\ \text{output}\ \ (c, c)\ \big]$ and $D_1(k,\ (c_1, c_2)\ ) = D(k, c_1)$

b. $E_2(k, m) = \big[c \leftarrow E(k, m),\ \text{output}\ \ (c, c)\ \big]$ and $D_2(k,\ (c_1, c_2)\ ) = \begin{cases} D(k, c_1) & \text{if } c_1 = c_2 \\ \mathsf{fail} & \text{otherwise} \end{cases}$

c. $E_3(k, m) = \big(\ E(k, m),\ E(k, m)\ \big)$ and $D_3(k,\ (c_1, c_2)\ ) = \begin{cases} D(k, c_1) & \text{if } D(k, c_1) = D(k, c_2) \\ \mathsf{fail} & \text{otherwise} \end{cases}$

To clarify: $E(k, m)$ is randomized so that running it twice on the same input will result in different outputs with high probability.

d. $E_4(k, m) = \big(\ E(k, m),\ H(m)\ \big)$ and $D_4(k,\ (c_1, c_2)\ ) = \begin{cases} D(k, c_1) & \text{if } H(D(k, c_1)) = c_2 \\ \mathsf{fail} & \text{otherwise} \end{cases}$

where H is a collision resistant hash function.

Problem 6. Let F be a secure PRF defined over $(\mathcal{K}, \mathcal{X}, \mathcal{Y})$ where $\mathcal{Y} := \{0,1\}^n$. Let $(E_{\text{ctr}}, D_{\text{ctr}})$ be the cipher derived from F using randomized counter mode. Let $H : \mathcal{Y}^{\leq L} \to \mathcal{Y}$ be a collision resistant hash function. Consider the following attempt at building an AE-secure cipher defined over $(\mathcal{K},\ \mathcal{Y}^{\leq L}, \mathcal{Y}^{\leq L+2})$:

$$E'(k, m) := E_{\text{ctr}}\big(k,\ (H(m), m)\big)\ ; \qquad D'(k, c) := \left\{ \begin{array}{l} (t, m) \leftarrow D_{\text{ctr}}(k, c) \\ \text{if } t = H(m) \text{ output } m, \text{ else } \mathsf{reject} \end{array} \right\}$$

Note that when encrypting a single block message $m \in \mathcal{Y}$, the output is three blocks: the random IV, a ciphertext block corresponding to $H(m)$, and a ciphertext block corresponding to m. Show that (E', D') is not AE-secure by showing that it does not have ciphertext integrity. Your attack should make a single encryption query.

At some point in the past, this type of construction was used to protect secret keys in the Android KeyStore. Your attack resulted in a compromise of the key store.

Problem 7. Alice and Bob run the Diffie-Hellman protocol in the cyclic group $\mathbb{G} = \mathbb{Z}_{101}^*$ with generator $g = 11$. What is the Diffie-Hellman secret $s = g^{ab} \in \mathbb{G}$ if Alice uses $a = 7$ and Bob uses $b = 43$? You do not need a calculator to solve this problem!

Problem 8. (*Exponentiation algorithms*) Let $\mathbb{G}$ be a finite cyclic group of order p with generator g. In class we discussed the repeated squaring algorithm for computing $g^x \in \mathbb{G}$ for $0 \leq x < p$. The algorithm needed at most $2 \log_2 p$ multiplications in $\mathbb{G}$.

In this question we develop a faster exponentiation algorithm. For some small constant w, called the window size, the algorithm begins by building a table T of size 2^w defined as follows:

$$\text{set} \;\; T[k] := g^k \;\; \text{for } k = 0, \ldots, 2^w - 1 \; . \tag{1}$$

a. Show that once the table T is computed, we can compute g^x using only $(1 + 1/w)(\log_2 p)$ multiplications in $\mathbb{G}$. Your algorithm shows that when the base of the exponentiation g is fixed forever, the table T can be pre-computed once and for all. Then exponentiation is faster than with repeated squaring.
Hint: Start by writing the exponent x base 2^w so that:

$$x = x_0 + x_1 2^w + x_2 (2^w)^2 + \ldots + x_{d-1} (2^w)^{d-1} \quad \text{where } 0 \leq x_i < 2^w \text{ for all } i = 0, \ldots, d-1.$$

Here there are d digits in the representation of x base 2^w. Start the exponentiation algorithm with x_{d-1} and work your way down, squaring the accumulator w times at every iteration.

b. Suppose every exponentiation is done relative to a different base, so that a new table T must be re-computed for every exponentiation. What is the worse case number of multiplications as a function of w and $\log_2 p$?

c. Continuing with Part (b), compute the optimal window size w when $\log_2 p = 256$, namely the w that minimizes the overall worst-case running time. What is the worst-case running time with this w? (counting only multiplications in $\mathbb{G}$)