Linearly Homomorphic Signatures over Binary Fields and New Tools for Lattice-Based Signatures

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PKC 2011
Taormina, Italy
7 March 2011

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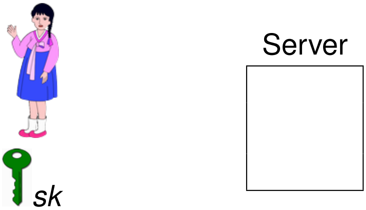

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$\mathbf{v} \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$
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- Security: no adversary can authenticate any vector $\mathbf{v}^*$ outside $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$.

Motivation: Network Coding

Network coding routing mechanism [ACLY00]:

- Interpret data as vectors in $\mathbb{F}_p^n$.
- Routers send random linear combinations of received vectors, along with coefficients.
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Current solutions authenticate vectors over $\mathbb{F}_p$ for large p.
For efficiency, we want to use vectors defined over $\mathbb{F}_2$.

Our Contributions

- Linearly homomorphic signatures over $\mathbb{F}_2$.
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  - New k-SIS assumption; reduction to worst-case lattice assumptions (used for security result).
  - Result on distributions of sums of discrete Gaussian samples (used for privacy result).
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  - Tight length bounds for discrete Gaussian samples.
- k-time signature scheme without random oracles.
  - Application of new k-SIS assumption.

Building Block: GPV Trapdoor Function

- $\Lambda \subset \mathbb{Z}^m$ a lattice (full-rank subgroup), defined by matrix $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$ $(n < m)$:

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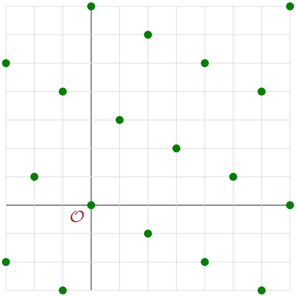

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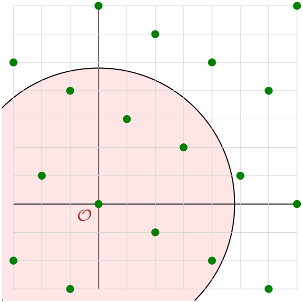

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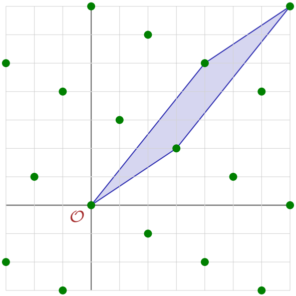

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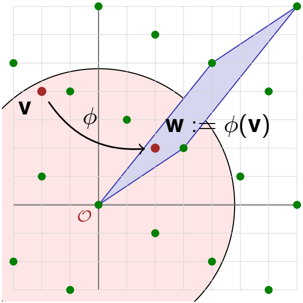

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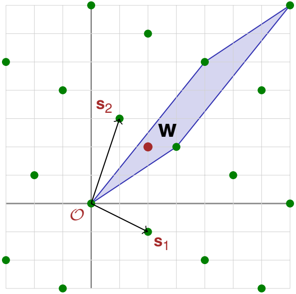

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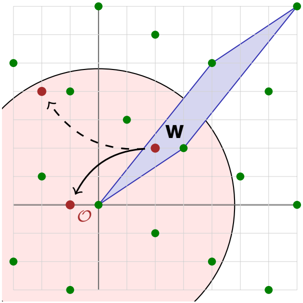

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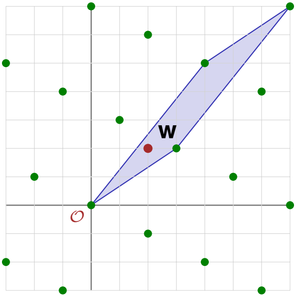

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- Sampling short vectors in $\Lambda + \mathbf{w}$ without short basis is hard.

Linearly Homomorphic Signatures: Key Ideas

GPV sign/verify algorithms: $H\colon \{0,1\}^* \to \mathbb{Z}_q^n$

$pk = \mathbf{A} \in \mathbb{Z}_q^{n\times m}$ $sk =$ short basis of $\Lambda_q^\perp(\mathbf{A})$

Sign($\mathbf{v}$) := short vector in $(\Lambda_q^\perp(\mathbf{A}) + H(\mathbf{v}))$

Verify(σ) := 1 iff σ is short, $\mathbf{A}\cdot\sigma \bmod q = H(\mathbf{v})$

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- Define signature on $\mathbf{v}_1 + \mathbf{v}_2$ to be $\sigma := \sigma_1 + \sigma_2$.
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Goal: Reduce system's security to the following problem.

$\text{SIS}_{q,m,\beta}$ Problem

Given random $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$,
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- Problem: signatures are already short vectors in $\Lambda_q^{\perp}(\mathbf{A})$, so can't simulate in a reduction.
- Solution: Make a new assumption!
  (and then reduce it to a standard assumption).

Goal: Reduce system's security to the following problem.

k-$\text{SIS}_{q,m,\beta}$ Problem

Given random $\mathbf{A} \in \mathbb{Z}_q^{n\times m}$ and k short vectors $\mathbf{e}_1, \ldots, \mathbf{e}_k \in \Lambda_q^\perp(\mathbf{A})$ find an $\mathbf{v}^* \in \Lambda_q^\perp(\mathbf{A})$ with $\|\mathbf{v}^*\| < \beta$ and $\mathbf{e}^* \notin \mathbb{Q}\text{-span}(\mathbf{e}_1, \ldots, \mathbf{e}_k)$.

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Sadly, the k-SIS-to-SIS reduction is exponential in k:

$$\beta' \approx k! \cdot n^{k/2} \cdot \beta.$$

But this is OK if $k = O(1)$.

Idea of the k-SIS-to-SIS Reduction

Given SIS challenge $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$, do:

- Choose $\mathbf{e}_1, \ldots, \mathbf{e}_k$ from Gaussians over $\mathbb{Z}^{m+k}$.
- Define $\mathbf{B}$ by appending k random columns to $\mathbf{A}$ such that

$$\underbrace{\left(\begin{array}{c|ccc} \mathbf{A} & \mathbf{b}_1 & \cdots & \mathbf{b}_k \end{array} \right)}_{\mathbf{B}} \cdot \begin{pmatrix} | \\ \mathbf{e}_i \\ | \end{pmatrix} = 0 \bmod q \quad \text{for all } i$$

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Theorem

$(\mathbf{B}, \mathbf{e}_1, \ldots, \mathbf{e}_k)$ *produced in this way is statistically indistinguishable from a k-SIS challenge in dimension $m + k$.*

Real k-SIS challenge: fix $\mathbf{B}$, then choose $\mathbf{e}_i \in \Lambda_q^{\perp}(\mathbf{B})$.

k-SIS-to-SIS Reduction, Continued

Given simulated k-SIS challenge $(\mathbf{B}, \mathbf{e}_1, \ldots, \mathbf{e}_k)$

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- First m entries of $\mathbf{v}^*$ are in $\Lambda_q^{\perp}(\mathbf{A})$ — solves SIS problem!

k-SIS-to-SIS Reduction, Continued

Given simulated k-SIS challenge $(\mathbf{B}, \mathbf{e}_1, \ldots, \mathbf{e}_k)$

$$\underbrace{\left(\begin{array}{c||ccc} \mathbf{A} & \mathbf{b}_1 & \cdots & \mathbf{b}_k \end{array} \right)}_{\mathbf{B}} \cdot \begin{pmatrix} \mathbf{v}^* \\ 0 \\ 0 \end{pmatrix} = 0 \bmod q$$

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Gaussian elimination blows up length by a factor $\approx k! \cdot n^{k/2}$.

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Specifically: given two vector spaces

$$V = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k), \qquad W = \text{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$$

and a set of coefficients $\{c_i\}$ with

$$\sum c_i \mathbf{v}_i = \sum c_i \mathbf{w}_i,$$

even unbounded adversary cannot distinguish derived signature on $\sum c_i \mathbf{v}_i$ from derived signature on $\sum c_i \mathbf{w}_i$.

New Tool Used to Prove Privacy

Theorem

Let $\mathbf{e}_i \in \mathbb{Z}^m$ be sampled from a discrete Gaussian over $\Lambda + \mathbf{t}_i$ with parameter σ. Let $c_i \in \{0, 1\}$. Then for sufficiently large σ, the distribution of $\sum c_i \mathbf{e}_i$ is a discrete Gaussian over $\Lambda + \sum c_i \mathbf{t}_i$.*

*up to negligible statistical distance

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Corollary: Linearly homomorphic signatures over $\mathbb{F}_2$ are private.

Proof idea:

- Sigs on $\mathbf{v}_i$ sampled from discrete Gaussian distribution, derived sigs are linear combinations.
- By theorem, distribution of derived signature on $\mathbf{v} = \sum c_i \mathbf{v}_i$ depends only on $\{c_i\}$ and $\mathbf{v}$, not on the $\mathbf{v}_i$.
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Theorem generalizes to tuples of discrete Gaussians.

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- Sign/Verify algorithms same as in homomorphic scheme:

$$\begin{aligned} \mathrm{Sign}(\mathbf{v}) &:= \text{Gaussian sample from} \quad (\Lambda_{2q}^{\perp}(\mathbf{A}) + q \cdot \mathbf{v}) \\ \mathrm{Verify}(\sigma) &:= 1 \quad \text{iff} \quad \|\sigma\| < \beta, \; \mathbf{A} \cdot \sigma = q \cdot \mathbf{v} \bmod 2q. \end{aligned}$$

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Theorem

Let $\mathbf{e} \in \mathbb{Z}^n$ be sampled from a discrete Gaussian with parameter σ. Then for any $\epsilon > 0$ we have w.h.p.

$$(1 - \epsilon) \cdot \sigma \sqrt{n/2\pi} \leq \|\mathbf{e}\| \leq (1 + \epsilon) \cdot \sigma \sqrt{n/2\pi}.$$

Best previous result was $\|\mathbf{e}\| \leq \sigma\sqrt{n}$.

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Thank you!