Most results in revenue-maximizing auction design hinge on “getting the price right” — offering goods to bidders at a price low enough to encourage a sale, but high enough to garner non-trivial revenue. Getting the price right can be hard work, especially when the seller has little or no a priori information about bidders’ valuations.

A simple alternative approach is to “let the market do the work”, and have prices emerge from competition for scarce goods. The simplest-imaginable implementation of this idea is the following: first, if necessary, impose an artificial limit on the number of goods that can be sold; second, run the welfare-maximizing VCG mechanism subject to this limit.

We prove that such “supply-limiting mechanisms” achieve near-optimal expected revenue in a range of single- and multi-parameter Bayesian settings. Indeed, despite their simplicity, we prove that they essentially match the state-of-the-art in prior-independent mechanism design.

1. INTRODUCTION

1.1. A Matching Problem

Consider the problem of matching agents with unknown preferences to a set of goods for sale, with the goal of maximizing the seller’s revenue. For example, a travel website selling hotel accommodation would like to match agents to a set of hotel rooms. Each unit-demand agent only needs one room, and has a different private value for each type of room. Uncertainty about agents’ values is modeled by drawing the values from prior distributions, with one distribution per good (one distribution for a suite at the Ritz, another for a room at Best Western, and so on). The seller wishes to maximize her expected revenue, but at the same time wants to minimize the resources spent estimating the underlying distributions, as well as the risks associated with getting these distributions wrong and the complexity of the procedures involved in carrying out the sale. In addition, a far-seeing seller might also want to (approximately) maximize social welfare.

Maximizing expected revenue in the matching problem above is difficult even when the value distributions are known. The difficulty stems from the problem’s multi-parameter nature. The theory of optimal auction design stops short of solving settings in which the description of an agent’s preferences requires multiple parameters. Recent breakthrough work of Chawla et al. [2010a] circumvents this limitation by introducing approximately optimal mechanisms. These mechanisms make use of a priori knowledge of the value distributions.

Our work focuses on prior-independent mechanisms, whose description does not reference any prior distributions, yet for every set of prior distributions satisfying standard assumptions, has expected revenue close to that of the optimal mechanism tailored to the distributions. We present prior-independent mechanisms that have near-optimal expected revenue for a variety of market environments, including the multi-
parameter matching problem described above. The importance of prior-independence is discussed by Dhangwatnotai et al. [2010].

Our mechanisms are extremely simple, and are based on the natural idea of artificially limiting the supply to increase bidder competition for the goods. Previous prior-independent mechanisms are based largely on some form of random sampling to estimate the prior distributions [Balcan et al. 2005; Dhangwatnotai et al. 2010]. Ours are the first known prior-independent mechanisms for nontrivial multi-parameter settings.2

1.2. A Supply-Limiting Mechanism

Before describing our results in more detail, we motivate the approach of limiting supply and develop some intuition for how it works by considering a very simple, single-parameter context — multi-unit auctions. In this context the seller only has a single item, but may have several identical units of it, and the agents are unit-demand. A concrete example is the sale of digital goods such as software licenses. In this particular case, the supply is essentially unlimited.

Now consider the supply-limiting mechanism in Algorithm 1. This mechanism is simple and natural, and does not rely on knowing or sampling distributions. Intuitively, limiting the supply increases competition, and indeed it is not hard to show that this mechanism guarantees approximately-optimal revenue, assuming that bidder valuations are i.i.d. draws from a distribution satisfying a standard regularity assumption. In particular, although the mechanism is oblivious to the underlying distribution, in expectation it achieves at least half of the expected revenue of the optimal mechanism tailored to the distribution. We remark that by using the VCG mechanism,3 our mechanism also guarantees near-optimal social welfare, even though our weak distributional assumptions allow the revenue and welfare of other mechanism to be very far from each other.

**ALGORITHM 1:** A Generic Prior-Independent Mechanism

(1) Set a supply limit equal to half of the number of bidders.
(2) Run the VCG mechanism subject to this supply limit.

This paper shows that under minimal regularity assumptions, the simple, prior-independent mechanism above and its revenue guarantee generalize to significantly more complex settings. In other words, we identify settings in which the prior-independent VCG mechanism with limited supply is guaranteed to have near-optimal expected revenue. For the matching problem discussed above, we prove the following theorem.

**THEOREM 1.1 (PRIOR-INDEPENDENT MECHANISM FOR MATCHING (INFORMAL)).**

For every matching environment with \( n \geq 2 \) bidders, the expected revenue of the mechanism in Algorithm 1 is at least a constant fraction of the optimal expected revenue.

The constant fractions we achieve are quite good in many cases, e.g., we achieve a fraction of \( \frac{1}{4} \) when the number of bidders exceeds the number of goods.

2Simultaneously and independently, another group obtained similar results using different mechanisms; see Section 1.5 for a detailed discussion.
3We assume for now that the reader is familiar with the VCG mechanism, see the Preliminaries section for details.
1.3. Technical Approach: Reduction to Bulow-Klemperer-Type Theorems

Our technical approach to establishing approximation properties of supply-limiting mechanisms is based on a general reduction to “Bulow-Klemperer-type” theorems. To convey the basic idea, we once again illustrate our approach using the simple example of a multi-unit auction for digital goods.

A slightly generalized version of Bulow and Klemperer’s well-known result [1996] states that when selling $k$ units of an item to $n$ unit-demand bidders, whose single-parameter values are drawn i.i.d. from a regular distribution, the expected revenue of the VCG mechanism with $k$ additional bidders is at least that of the optimal mechanism without additional bidders (see [Kirkegaard 2006] for a simple proof). We briefly sketch how to use this Bulow-Klemperer theorem to analyze the mechanism in Algorithm 1 in the multi-unit auction context.

Consider first the “halved environment”, with half of the $n$ original bidders and a corresponding supply limit of $n/2$. One can show that if we were to restrict the optimal mechanism to run on this sub-environment instead of the original one, its expected revenue is at least half of that of the optimal auction for the original environment. Now we conceptually add back the removed bidders but without changing the supply, and run VCG. It follows from the Bulow-Klemperer theorem that the expected revenue is at least as high as the optimal expected revenue for the halved environment. Therefore the supply-halving mechanism guarantees at least half of the optimal expected revenue on the original environment. In fact, one can achieve other trade-offs between revenue and welfare by setting suitable supply limits.

In general, a Bulow-Klemperer-type theorem states that instead of running the optimal mechanism on the original environment, we can get approximately as much revenue in expectation by running the VCG mechanism on a suitably augmented environment with additional bidders or supply. We make explicit the connection between Bulow-Klemperer-type theorems and prior-independent mechanisms. A sketch of our reduction procedure, which applies to both single- and multi-parameter settings, appears in Reduction 2.

---

**REDUCTION 2:** From Prior-Independent Auctions to Bulow-Klemperer-Type Theorems

(1) **Restriction:** Restrict the auction environment by dropping bidders and/or limiting supply.  
**Guarantee:** Optimal expected revenue approximately maintained.  
**Proof:** By a general subadditivity property of optimal expected revenue in the bidder set.

(2) **Augmentation:** Augment the restricted environment by adding bidders while maintaining the supply limit.  
**Guarantee:** Expected revenue of VCG approximates optimal expected revenue.  
**Proof:** By a suitable Bulow-Klemperer-type theorem.

---

To instantiate the reduction in various single-parameter environments, we can use generalizations of the original Bulow-Klemperer result to matroid environments [Dughmi et al. 2009] and to non-i.i.d. bidders [Hartline and Roughgarden 2009]. For the matching problem, we need to prove the first generalization of the original Bulow-Klemperer theorem to a nontrivial multi-parameter setting.

**THEOREM 1.2 (B-K-TYPE THEOREM FOR MATCHING (INFORMAL)).** For every matching environment with $n$ bidders and $m$ goods, the expected revenue of the VCG mechanism with either (1) $m$ additional bidders or (2) $O(n)$ additional bidders and a supply limit $n$, is at least a constant fraction of the optimal expected revenue in the original environment.
Table I: Summary of results for i.i.d. regular environments.

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VCG$^{\leq \ell}$ denotes the VCG mechanism with supply limit $\ell$.
VCG$^{\leq \ell, j}$ denotes the VCG mechanism with global supply limit $\ell$ and local supply limits $\{\ell_j\}$ (see Section 6.3).

$^\dagger$The rank of the matroid is denoted by $r$, and its packing number by $\kappa$.

$^*$Results holds for multi-unit matchings as well (see Section 6.3).

Our proof is based on the following ideas (see Section 5 for details). We first observe there is a very simple upper bound on the optimal expected revenue for the matching environment — the expected revenue from selling each good separately to $n+1$ bidders. We need to show that VCG with additional bidders does just as well. Recall that in VCG, the winner of a certain good pays for the “damages” incurred by the losers, and in both cases stated in the above theorem it is guaranteed there are $O(n)$ many losers. The technical challenge is a dependency issue — by definition the losers seem likely to have lower values for the good than those of the $n+1$ bidders to which it is sold separately, so that their damages may not be enough to cover the expected revenue. We show this is not the case due to the structure of matchings, specifically the fact that the values of bidders for one particular good play a limited role in VCG’s choice of winners, so limited in fact that the only implication on the losers’ values is being lower than that of the winner. We formalize these ideas by introducing an auxiliary selling procedure, conceptually and revenue-wise half-way between selling the good separately and selling it as part of VCG, namely we run VCG but defer the sale of the good until exactly $n+1$ bidders remain, so that by construction their values are unaffected by dependency issues.

1.4. Our Results and Organization

Our main result is a collection of approximately-optimal supply-limiting mechanisms for different auction environments with i.i.d. regular bidders, as detailed in Table I. For single-parameter settings, we show supply-limiting mechanisms for $k$-unit and matroid environments. For multi-parameter settings, we show three versions of a supply-limiting mechanism for matching environments, each with a different approximation guarantee. The choice among these versions should be according to the parameters of the environment at hand. Our results generalize to multi-unit matching environments as well.
1.5. Related Work

Most related to our results are the following. Dughmi et al. investigate conditions under which VCG inadvertently yields near-optimal revenue. They use a generalized Bulow-Klemperer result to show this is guaranteed in single-parameter matroid environments with sufficient competition in the form of disjoint bases [Dughmi et al. 2009]. Our reduction encompasses and generalizes this result. Hartline and Roughgarden also study conditions for when simple Vickrey-based mechanisms achieve near-optimal revenue and in particular they derive an anonymous-reserve mechanism from one of their Bulow-Klemperer-type results [Hartline and Roughgarden 2009, Theorem 5.1]. This mechanism however is not prior-independent and is inherently limited to single items [Hartline and Roughgarden 2009, Example 5.4]. Chawla et al. use posted-price mechanisms that rely on prior distributions — i.e., mechanisms that are not prior-independent — to achieve a 6.75-approximation for the matching setting with multiple units and non-i.i.d. bidders, and also a 32/33-approximation for a more general environment than what we consider, namely a graphical matroid with unit-demands [Chawla et al. 2010a]. Bandi and Bertsimas introduce an uncertainty set based model for additive valuations with budgets and correlation, then formulate and solve the related robust optimization problem [Bandi and Bertsimas 2014]. Devanur et al. independently consider a similar set of problems as us, but using different mechanisms and analyses [Devanur et al. 2011]. Their mechanisms are more complex and less natural than the supply-limiting mechanisms studied here.

In terms of techniques, our limited-supply mechanisms are special cases of maximal-in-range-mechanisms (see, e.g., [Nisan and Ronen 2000]), which can be implemented in dominant strategies. We apply a reduction of Chawla et al. that relates single- and multi-parameter settings ([Chawla et al. 2010a], see Section 2). Some of our techniques are inspired by Chawla et al.’s analysis of the VCG mechanism’s performance in a job scheduling context [Chawla et al. 2011].

2. PRELIMINARIES

2.1. Basic Auction Environments

Auction environments are settings in which goods are sold to bidders. We distinguish between items — different kinds of goods, and units — different copies of the same good. While bidders have the same value for different units of the same item, their values for different items are independent from one another (although possibly drawn from the same distribution). We use the following notation: item \( j \) means the \( j \)-th kind of good sold in the auction, \( k_j \) denotes the number of units available of item \( j \), and \( m = \sum_j k_j \) is the total number of units available of all items.

Our results apply to two basic auction environments — in the first there are \( k \) multiple units of a single item, and in the second there are \( m \) multiple items but only one unit of each. In both cases we will assume there are at least \( n > 2 \) bidders, since prior-independence is impossible with a single bidder (in the sense that for every prior-independent mechanism there is an environment for which the mechanism’s approximation guarantee is arbitrarily poor). We now describe the two environments of interest in more detail; for extensions of these environments and generalizations of our results see Section 6.

2.1.1. Single-Parameter \( k \)-Unit Environments. In general, a single-parameter auction environment is composed of a set \( \{1, \ldots, n\} \) of bidders, and a non-empty collection \( \mathcal{I} \subseteq 2^{\{1, \ldots, n\}} \) containing subsets of bidders who can win the auction simultaneously.\(^4\)

\(^4\)This description of a single-parameter environment is sufficient for our purpose; more general definitions can be found in the literature.
Subsets in \( \mathcal{I} \) are called feasible allocations. We assume that every subset of a feasible allocation is also feasible (formally, the set system \( \{1, \ldots, n\}, \mathcal{I} \) is downward-closed). We also assume that every bidder belongs to at least one feasible allocation. Every bidder \( i \) has a private value \( v_i \) for winning, which is drawn independently at random from a distribution \( F_i \) (the environment is called single-parameter since the value for winning is described by one parameter). We say the bidders are i.i.d. if their distributions are independent and identical.

A \( k \)-unit (or multi-unit) environment is a single-parameter environment in which a subset of bidders is a feasible allocation if and only if its size is at most \( k \) (we assume \( k \leq n \)). This captures the situation where there are \( k \) identical units of the same item for sale, and every bidder is unit-demand, i.e., interested in buying at most one unit. We can also impose an additional supply limit of \( \ell \leq k \), restricting feasible allocations to size at most \( \ell \). An i.i.d. \( k \)-unit environment is one in which the bidders are i.i.d; our supply-limiting results apply to such environments.

2.1.2. Multi-Parameter Matching Environments. A (single-unit) matching environment is a multi-parameter environment with \( n \) bidders and \( m \) different items for sale. We only have one unit of each item available, and a multi-unit version of matching environment will be defined later. Bidders are unit-demand, in the sense that each bidder can only win at most one item. Feasible allocations are all matchings of items to bidders, such that each bidder wins at most one item and each item is assigned to at most one bidder. We can also impose an additional supply limit of \( \ell \leq m \), restricting the matchings to size at most \( \ell \). Bidder \( i \) has a private value \( v_{i,j} \) for winning item \( j \), which is drawn independently at random from a distribution \( F_{i,j} \). We say the bidders are i.i.d. if \( F_{i,j} \) does not depend on \( i \), which we can simply denote by \( F_j \). Our supply-limiting results apply to i.i.d. matching environments in which the bidders are i.i.d.

2.2. Mechanisms

Our work focuses on deterministic mechanisms (but applies to randomized mechanisms with only a constant-factor loss). A deterministic mechanism is comprised of

- an allocation rule \( x \), which maps a bid vector \( b \in [0, \infty)^n \) in the single-parameter case or \( b \in [0, \infty)^{nm} \) in the multi-parameter case to a feasible allocation; and
- a payment rule \( p \), which maps a bid vector \( b \) to a payment vector in \( [0, \infty)^n \).

We assume a quasi-linear utility model, in which each bidder aims to maximize her value for the chosen allocation minus her payment for it. A mechanism is truthful if given any bid profile \( b_{-i} \), bidder \( i \) maximizes her utility by being truthful, i.e., bidding \( b_i = v_i \) in the single-parameter case and \( b_{i,j} = v_{i,j} \) in the multi-parameter case. All the mechanisms we study are both truthful and individually rational — the utility for bidding truthfully is always non-negative. From now on we no longer distinguish between bids and values and use \( v_i \) or \( v_{i,j} \) to denote both.

The well-known VCG mechanism is of special interest to us [Vickrey 1961; Clarke 1971; Groves 1973]. It maximizes social welfare by choosing a feasible allocation \( x^* \) that maximizes the total value to the bidders, and charges every bidder \( i \) a payment equal to \( i \)'s “externality” — the difference between the maximum total value if \( i \) does not participate in the auction and the value of all other bidders when \( i \) does participate. In the context of single-item environments the VCG mechanism is called the Vickrey auction. We augment the VCG mechanism by adding to it a supply limit of \( \ell \), such that

\[ \text{Chawla et al. [2010b] show that for matching environments, the expected revenue from the optimal deterministic mechanism is within a constant factor of the expected revenue from the optimal randomized mechanism. Thus our results for deterministic mechanisms apply to randomized mechanisms up to a constant factor.} \]
the total number of allocated units in its chosen allocation is at most \( \ell \); we denote this supply-limiting VCG mechanism by VCG\(\leq\ell\).

2.3. Optimal Mechanism Design

2.3.1. Myerson’s Mechanism for Single-Parameter Environments. For single-parameter environments, Myerson [1981] determined the optimal mechanism that maximizes expected revenue. Given a distribution \( F \) with density \( f \), define its virtual valuation function to be \( \phi_F(v) = v - \frac{1-F(v)}{f(v)} \). Myerson showed the following.

**Lemma 2.1 (Myerson’s Lemma).** Given a single-parameter environment and a truthful mechanism \((x, p)\), for every bidder \( i \) and fixed value profile \( v_{-i} \) of the other bidders, 
\[
E_{v_i \sim F_i}[p_i(v)] = E_{v_i \sim F_i}[\phi_{F_i}(v_i) \cdot x_i(v)].
\]

Myerson’s lemma implies that maximizing expected revenue can be reduced to maximizing the expected total virtual value of the bidders, also called virtual surplus. Myerson’s mechanism maximizes virtual surplus and is optimal by the above lemma.

2.3.2. The Regularity Assumption

**Definition 2.2 (Regular Distribution).** A distribution \( F \) is regular if its virtual valuation function is monotone non-decreasing.

Most of the commonly-studied distributions are regular, including the uniform, exponential and normal distributions. We say that bidders are regular if their values are drawn from regular distributions. The assumption that bidders are regular is standard in optimal mechanism design, in particular when aiming for simplicity and/or prior-independence. In what follows we assume that all distributions are regular and possess positive smooth density functions.

2.3.3. Representative Environments for Upper-Bounding Optimal Multi-Parameter Revenue. The optimal mechanism for multi-parameter matching environments is currently unknown. Chawla et al. [2010a] introduced the concept of representative environments in order to upper-bound the optimal expected revenue in i.i.d. matching environments despite not knowing the optimal mechanism.

Given a matching environment \( \text{Env} \) with \( m \) items, \( n \) i.i.d. bidders and value-distributions \( \{F_j\}_j \), the representative environment \( \text{Env}^{\text{rep}} \) has the same \( m \) items but \( nm \) single-parameter bidders, one for every pair of original bidder and item \((i, j)\). The \( m \) bidders in \( \text{Env}^{\text{rep}} \) corresponding to original bidder \( i \) are called \( i \)'s representatives. Like bidder \( i \)'s value for item \( j \) in \( \text{Env} \), representative \((i, j)\)'s value \( v_{i,j} \) for winning in \( \text{Env}^{\text{rep}} \) is drawn independently at random from \( F_j \). Note that every subset \( S \) of representatives in \( \text{Env}^{\text{rep}} \) corresponds to an allocation in \( \text{Env} \) — if representative \((i, j)\) is in \( S \) then item \( j \) is allocated to bidder \( i \) in \( \text{Env}^{\text{rep}} \). Feasible allocations in \( \text{Env}^{\text{rep}} \) are subsets of representatives such that the corresponding allocation in \( \text{Env} \) is feasible. In particular, since every bidder \( i \) in \( \text{Env} \) is unit-demand, only one of its representatives in \( \text{Env}^{\text{rep}} \) can win simultaneously.

Given a truthful mechanism \( M \) for \( \text{Env} \), its allocation rule can be used to construct a truthful mechanism \( M^{\text{rep}} \) for \( \text{Env}^{\text{rep}} \). The following lemma relates the expected revenue of the two mechanisms. Intuitively, \( \text{Env}^{\text{rep}} \) involves more competition than \( \text{Env} \) since representatives of the same bidder compete with one another, and so the expected revenue of \( M^{\text{rep}} \) is higher.

**Lemma 2.3 ([Chawla et al. 2010a]).** The expected revenue of \( M^{\text{rep}} \) for \( \text{Env}^{\text{rep}} \) upper bounds the expected revenue of \( M \) for \( \text{Env} \).
3. WARM-UP: A SUPPLY-LIMITING MECHANISM FOR I.I.D. $k$-UNIT ENVIRONMENTS

In this section we formally prove the following theorem, discussed informally in the introduction, in order to illustrate our general reduction in a simple single-parameter environment. To simplify notation we assume that the number of bidders $n$ is even. This assumption is essentially without loss of generality since if $n$ is odd, one can first remove a bidder from the environment, losing at most a $1/n$-fraction of the optimal expected revenue.

**THEOREM 3.1 (SUPPLY-LIMITING MECHANISM FOR I.I.D. $k$-UNIT ENVIRONMENTS).**
For every $k$-unit environment with $n \geq 2$ i.i.d. regular bidders, the expected revenue of the supply-limiting mechanism $\text{VCG}^{\leq n/2}$ is a $\min\{2, \frac{n}{n-k}\}$-approximation to the optimal expected revenue.

The proof of Theorem 3.1 using the reduction requires the following slightly generalized version of the original Bulow-Klemperer theorem [1996].

**THEOREM 3.2 (GENERALIZED BULOW-KLEMPERER THEOREM).** For every $k$-unit environment with i.i.d. regular bidders and supply limit $\ell$, the expected revenue of VCG with $\min\{k, \ell\}$ additional bidders is at least as high as the optimal expected revenue.

**PROOF OF THEOREM 3.1.** We instantiate our general reduction (Reduction 2) as follows.

1. **Restriction:** Remove $\min\{\frac{n}{2}, k\}$ bidders from the environment, and if $k > \frac{n}{2}$ limit the supply to $\frac{n}{2}$ units.
2. **Augmentation:** Add back $\min\{\frac{n}{2}, k\}$ bidders.

We first claim that the restriction step reduces the expected optimal revenue by only a small constant factor. By submodularity of the expected optimal revenue in the bidder set [Dughmi et al. 2009, Theorem 3.1], removing bidders from the environment reduces the optimal expected revenue by a factor of at most $\min\{2, \frac{n}{n-k}\}$. Limiting the supply to $\frac{n}{2}$ when $k > \frac{n}{2}$ does not affect the optimal expected revenue since the bidders are unit-demand and since in this case the number of bidders in the restricted environment is $\frac{n}{2}$.

We now apply the Bulow-Klemerer theorem as stated in Theorem 3.2 to the restricted environment. If $k > \frac{n}{2}$, the restricted environment has $\frac{n}{2}$ bidders, $k$ units and supply limit $\frac{n}{2}$, while if $k \leq \frac{n}{2}$, the restricted environment has $n - k$ bidders and $k$ units. In both cases, by the Bulow-Klemerer theorem, the expected revenue of VCG with $\min\{\frac{n}{2}, k\}$ additional bidders is at least as high as the optimal expected revenue.

We have shown that running VCG with $\min\{\frac{n}{2}, k\}$ additional bidders on the restricted environment is a $\min\{2, \frac{n}{n-k}\}$-approximation to the optimal expected revenue on the original environment. But this is equivalent to running the supply-limiting mechanism $\text{VCG}^{\leq n/2}$ on the original environment (where the supply limit of $\frac{n}{2}$ is vacuous if $k \leq \frac{n}{2}$). This completes the proof. \qed

The approximation factor in the above theorem is asymptotically tight.

**PROPOSITION 3.3 (ASYMPTOTIC TIGHTNESS).** For every $0 < \rho < 1$, there exists an $n$-unit environment with $n$ i.i.d. bidders whose values are drawn from a regular distribution $F$, such that $\text{VCG}^{\leq \rho n}$ gives in expectation at most $a (\rho + o(1))$-fraction of the optimal expected revenue.

**PROOF.** First suppose that $1/n \leq \rho \leq 1/2$. Let the distribution $F$ be the uniform distribution over the support $[1, 1 + \epsilon]$, for sufficiently small $\epsilon = \epsilon(n) > 0$. The optimal
expected revenue given $F$ is roughly $n$, while VCG with supply limit $\rho m$ can extract at most $\rho m(1 + \epsilon) \leq n/2 + o(1)$.

Now suppose $1/2 < \rho \leq \frac{n-1}{n}$. Let the distribution $F$ be $F(z) = \frac{1}{1+\rho z}$ over the support $[0, H]$, for sufficiently large $H$. The optimal expected revenue is at least the expected revenue achieved by offering a posted price $H$ to every bidder, which extracts $H(1 - \frac{H}{1+\rho H}) = \frac{H}{1+\rho H} \approx 1$ from every bidder in expectation. In VCG with supply limit $\rho m$, the expected revenue comes from the $\rho m + 1$ highest bid. This bid is concentrated around $z = \frac{1-\rho}{\rho}$, the value of $z$ such that $F(z) = 1 - \rho$. So VCG achieves an expected revenue of roughly $\frac{1-\rho}{\rho} \rho m = (1 - \rho)n \leq \frac{n}{2}$. \(\square\)

4. A SUPPLY-LIMITING MECHANISM FOR I.I.D. MATCHING ENVIRONMENTS

4.1. Statement and Discussion of Main Theorems

In this section we present our main result — a supply-limiting mechanism for i.i.d. matching environments. More precisely, we present three alternative supply-limiting mechanisms, all VCG-based, with different approximation factors depending on the parameters $n$ and $m$ of the i.i.d. matching environment. The relation between the number of bidders $n$ and total number of items $m$ in the environment at hand determines which supply-limiting mechanism is most suitable for it.

We denote the revenue from the optimal mechanism for $n$ bidders by $OPT(n)$, and the revenue from the supply-limiting VCG mechanism for $n$ bidders by $VCG^{\leq \ell}(n)$, sometimes omitting $\ell$ from the notation when $\ell \geq \min\{n, m\}$. Note that $OPT(n)$ and $VCG^{\leq \ell}(n)$ are random variables over the sample space of bidder valuation profiles $v$. All expectations below are over $v$.

For simplicity of notation we will assume that $n/2$ (or $n/3$ where appropriate) is integer. If this is not the case, the approximation guarantees below hold up to a small multiplicative factor (the maximum loss in optimal expected revenue from dropping one or two bidders from the environment).

**Theorem 4.1** (2-Approximation for $m \leq n/2$). For every matching environment with $n \geq 2$ i.i.d. regular bidders and $m \leq n/2$ items, $\mathbb{E}[VCG(n)] \geq \frac{1}{2} \mathbb{E}[OPT(n)]$.

**Theorem 4.2** ($\frac{4m}{n}$-Approximation for $m \geq n/2$). For every matching environment with $n \geq 2$ i.i.d. regular bidders and $m \geq n/2$ items, $\mathbb{E}[VCG^{\leq \ell}(n)] \geq \frac{n}{4m} \mathbb{E}[OPT(n)]$.

**Theorem 4.3** (27-Approximation for General $n, m$). For every matching environment with $n \geq 3$ i.i.d. regular bidders and $m$ items, $\mathbb{E}[VCG^{\leq n/3}(n)] \geq \frac{1}{27} \mathbb{E}[OPT(n)]$.

Intuitively, achieving good approximation guarantees becomes more difficult as the number of items grows relative to the number of bidders, since the natural competition among the bidders in the environment is diversified across different items. Accordingly, when number of items is less than half of the number of bidders, we show that simply applying VCG achieves a 2-approximation to the optimal expected revenue (Theorem 4.1). When the number of items is more than half of the number of bidders but still proportional to it, applying VCG while artificially limiting the supply to half of the number of bidders achieves a $\frac{4m}{n}$-approximation, in particular a 4-approximation when $m = n$ (Theorem 4.2). Finally, when the number of items is possibly much larger than the number of bidders, limiting the supply still achieves a constant-factor approximation but with a larger constant. We find that setting the supply limit to a third of
the number of bidders guarantees a 27-approximation (Theorem 4.3). We believe this approximation factor can be further improved, and leave this as an open problem.

The remainder of the paper is largely dedicated to proving the first two of the above theorems. The proof of Theorem 4.3 is more involved, and its details appear in the Appendix B. By applying our general reduction, all proofs boil down to proving appropriate Bulow-Klemperer-type theorems. In Section 4.2 we state these theorems and in Section 4.3 we show how the main theorems reduce to them. The proofs of the Bulow-Klemperer-type theorems themselves are the main technical contribution of our work, and appear in Section 5 (for those corresponding to the first two main theorems) and in Appendix B (for that corresponding to Theorem 4.3).

4.2. Statement and Discussion of Multi-Parameter Bulow-Klemperer-Type Theorems

In order to prove Theorems 4.1 to 4.3 via our general reduction, we need the following corresponding Bulow-Klemperer-type theorems.

**Theorem 4.4 (B-K with \(m \) More Bidders).** For every matching environment with \(n\) i.i.d. regular bidders and \(m\) items, \(E[VCG(n+m)] \geq E[OPT(n)]\).

**Theorem 4.5 (\(m/n\)-Approximate B-K for \(m \geq n\) with \(n\) More Bidders).** For every matching environment with \(n\) i.i.d. regular bidders and \(m \geq n\) items, \(E[VCG^{\leq n}(2n)] \geq \frac{n}{m} E[OPT(n)]\).

**Theorem 4.6 (9-Approximate B-K with \(2n\) More Bidders).** For every matching environment with \(n\) i.i.d. regular bidders and \(m\) items, \(E[VCG^{\leq n}(3n)] \geq \frac{9}{5} E[OPT(n)]\).

For proofs see Section 5 (Theorems 4.4 and 4.5) and Appendix B (Theorem 4.6).

The first of the above Bulow-Klemperer-type theorems states that for matching environments with \(m\) items, the expected revenue of VCG with \(m\) additional bidders is at least as high as the optimal expected revenue. This generalizes the original Bulow-Klemperer theorem to the more complex multi-parameter matching setting. If \(m \gg n\) however, the required resource augmentation — adding \(m\) bidders when originally there are only \(n\) — is substantial, which will cause our reduction to give weak bounds.

Our second and third Bulow-Klemperer-type theorems address this issue by requiring the addition of \(O(n)\) bidders. This is made possible by using supply-limiting VCG, which restricts the total number of allocated items to at most \(n\) out of the \(m\) items available, and by relaxing the optimality requirement to approximate-optimality. Theorem 4.5 provides a good approximation factor when \(m\) is larger than \(n\) but proportional to it. Theorem 4.6 guarantees a constant approximation factor of 9 for any values of \(n, m\).

4.3. Proof of Main Theorems by Applying the General Reduction

We now reduce the three main theorems to the three Bulow-Klemperer-type theorems by instantiating our general reduction. The following lemma is the key and may be of independent interest. It applies to general auction environments (including multi-parameter ones) and states that the optimal expected revenue achievable from two sets of bidders separately exceeds that is achievable from the union of the two sets.

A corollary of this subadditivity lemma is that if we remove bidders from an i.i.d. environment until only a constant fraction of the bidders are left, we still maintain a constant fraction of the optimal expected revenue.

Let \(OPT(S)\) denote the optimal expected revenue achievable from bidder set \(S\).

**Lemma 4.7 (Subadditivity of Optimal Expected Revenue in Bidder Set).** For every auction environment with bidder subsets \(S\) and \(T\), \(E[OPT(S)] + E[OPT(T)] \geq E[OPT(S \cup T)]\).
PROOF. It is easy to prove that $\text{OPT}(S)$ is monotone in $S$, and therefore we can assume that $S$ and $T$ are disjoint. Let $M$ be the optimal mechanism for $S \cup T$. For every valuation profile $v_T$ of bidders in $T$, we define the following mechanism $M_{v_T}$. The mechanism $M_{v_T}$ gets bids from bidders in $S$, and simulates $M$ by using $v_T$ as the “bids” of bidders in $T$. By an averaging argument, there exists a vector $v_T$ such that mechanism $M_{v_T}$’s expected revenue $E_{\nu_T} = \text{OPT}(S)$ is at least the part of the optimal expected revenue $E_{\nu_T} = \text{OPT}(S \cup T)$ that comes from $S$, and the expected revenue of $M_{v_T}$ is upper-bounded in turn by $E(\text{OPT}(S))$. Similarly, the part of the optimal expected revenue that comes from $T$ is upper-bound by $E(\text{OPT}(T))$. Summing up we have the desired subadditivity claim. □

**Corollary 4.8.** For every auction environment with $n$ i.i.d. bidders and for every integer $c$ that divides $n$, $E(\text{OPT}(n/c)) \geq \frac{1}{c}E(\text{OPT}(n))$.

**4.3.1. Reduction Instantiations**

**Proof of Theorem 4.1.** We need to show $E(\text{VCG}(n)) \geq \frac{1}{2}E(\text{OPT}(n))$ under assumptions of i.i.d. bidders, regularity and $m \leq n/2$. We instantiate our general reduction (Reduction 2) as follows.

1. **Restriction:** Remove $m$ bidders from the environment.
2. **Augmentation:** Add back $m$ bidders.

By Corollary 4.8 of the subadditivity property and by monotonicity of the optimal expected revenue, restricting the environment does not hurt the optimal expected revenue too much, i.e., $E(\text{OPT}(n)) \leq 2E(\text{OPT}(n/2)) \leq 2E(\text{OPT}(n - m))$. Applying the appropriate Bulow-Klemperer-type theorem, Theorem 4.4, to the restricted environment with $n - m$ bidders and $m$ items, gives $E(\text{OPT}(n - m)) \leq E(\text{VCG}(n))$, completing the proof. □

**Proof of Theorem 4.2.** We need to show $E(\text{VCG}^{\leq n/2}(n)) \geq \frac{n}{4m}E(\text{OPT}(n))$ under assumptions of i.i.d. bidders, regularity and $m \geq n/2$. We instantiate the reduction as follows.

1. **Restriction:** Remove $n/2$ bidders from the environment.
2. **Augmentation:** Add back $n/2$ bidders.

As above, the proof is by the inequality chain $E(\text{OPT}(n)) \leq 2E(\text{OPT}(n/2)) \leq \frac{4m}{3}E(\text{VCG}^{\leq n/2}(n))$, where the first inequality is by Corollary 4.8, and the second inequality is by applying the appropriate Bulow-Klemperer-type theorem (Theorem 4.5) to the restricted environment with $n/2$ bidders and $m \geq n/2$ total units. □

**Proof of Theorem 4.3.** We need to show $E(\text{VCG}^{\leq n/3}(n)) \geq \frac{1}{27}E(\text{OPT}(n))$ under assumptions of i.i.d. bidders and regularity. We instantiate the reduction as follows.

1. **Restriction:** Remove $\frac{n}{3}$ bidders from the environment.
2. **Augmentation:** Add back $\frac{n}{3}$ bidders.

As above, the proof is by the inequality chain $E(\text{OPT}(n)) \leq 3E(\text{OPT}(n/3)) \leq 27E(\text{VCG}^{\leq n/3}(n))$, where the first inequality is by Corollary 4.8, and the second inequality is by applying the appropriate Bulow-Klemperer-type theorem (Theorem 4.6) to the restricted environment with $n/3$ bidders and $m$ items. □
5. PROOF OF BASIC BULOW-KLEMPERER-TYPE THEOREMS FOR I.I.D. MATCHING ENVIRONMENTS

In this section we prove two Bulow-Klemperer-type theorems for i.i.d. matching environments — Theorems 4.4 and 4.5. Theorem 4.6 is much more challenging to prove and is not dealt with in this section.

We begin with the proof of Theorem 4.4, divided into two parts. In Section 5.1 we identify an upper bound on the optimal expected revenue in the original environment, and a lower bound on the revenue of the VCG mechanism in the augmented environment. The advantage of this step is that these bounds are relatively simple to analyze and are already similar in form, though not identical. In Section 5.2 we carefully relate the two bounds, thus establishing the theorem. In Section 5.3 we show how the proof extends to establish Theorem 4.5 as well.

5.1. Basic Upper and Lower Bounds

Let $Vic_j(n+1)$ be the revenue from selling item $j$ to $n+1$ i.i.d. bidders with value-distribution $F_j$ using the Vickrey (second-price) auction. We use the concept of representative environment to show that the optimal expected revenue from selling all items to $n$ bidders in an i.i.d. matching environment is upper-bounded by the expected revenue from selling each item separately to $n+1$ single-parameter bidders.

**Lemma 5.1 (Upper Bound on Optimal Expected Revenue).** For every matching environment with $n$ i.i.d. regular bidders, $E[OPT(n)] \leq \sum_j E[Vic_j(n+1)]$.

**Proof.** Given the matching environment, consider the corresponding complete bipartite graph with bidders on one side and items on the other, and the bidders’ values for items drawn from distributions $\{F_j\}_j$ as edge weights; recall that feasible allocations correspond to matchings. By Lemma 2.3, the optimal expected revenue in the matching environment is upper-bounded by the optimal expected revenue in its single-parameter counterpart, the corresponding representative environment.

We now relax the feasibility constraints, by which we may only increase the optimal expected revenue. We define a new environment in which feasible allocations are all subsets of edges such that at most one edge is incident to an item-node (but unlike a matching, multiple edges can be incident to every bidder-node). Observe that the new environment is equivalent in terms of revenue to a collection of $m$ single-item environments, where in the $j$-th environment item $j$ is auctioned to $n$ single-parameter bidders whose values are drawn i.i.d. from the regular distribution $F_j$. By the original Bulow-Klemperer theorem (Theorem 3.2), the optimal expected revenue from the $j$-th environment is upper-bounded by $E[Vic_j(n+1)]$. Summing up over all items completes the proof. □

The revenue from the VCG mechanism is the sum of VCG payments for allocated items. We lower-bound the VCG payment for an allocated item $j$.

**Observation 5.2 (Lower Bound on VCG Revenue).** For every matching environment, the VCG payment for item $j$ is at least the value of any unallocated bidder for $j$.

**Proof.** If bidder $i$ wins item $j$, then the VCG payment for $j$ is equal to the externality that $i$ imposes on the rest of the bidders by winning $j$. Since $i$ prevents any unallocated bidder from getting $j$, the payment is at least the unallocated bidder’s value for $j$. □

In our matching context, the upper and lower bounds above turn out to share a similar form. On one hand, by definition of the Vickrey auction, the upper bound
on the expected revenue from separately auctioning item $j$ is equal to the second-highest value for $j$ among $n + 1$ bidders with values drawn independently from $F_j$. On the other hand, the lower bound on the VCG payment for item $j$ in the augmented environment is equal to the highest value for $j$ among $n$ unallocated bidders with values drawn independently from $F_j$. We are using here the fact that since the augmented environment includes $m$ more bidders, all items are allocated and exactly $m$ out of $n + m$ bidders are allocated.

From this it may appear as if we have already shown that the lower bound exceeds the upper bound. However, a dependency issue arises — conditioned on the event that a bidder in the augmented environment is unallocated by VCG, her value for item $j$ is no longer a random sample from $F_j$. We address this issue in the next section by introducing a deferred allocation selling procedure.

5.2. Relating the Upper and Lower Bounds via Deferred Allocation

Algorithm 3 describes a deferred allocation procedure for selling item $j$.

**ALGORITHM 3: Selling Item $j$ by Deferred Allocation**

Given a matching environment with $n + m$ bidders and $m$ items:

1. Find a welfare-maximizing feasible allocation (a maximum matching) of all items other than $j$ to a subset of the bidders. Let $U$ be the set of $n + 1$ bidders who remain unallocated.
2. Run the Vickrey auction to sell item $j$ to bidder set $U$.

We now show how deferred allocation resolves the dependency issue. Consider the revenue from selling item $j$ to bidder set $U$ by the deferred allocation procedure described in Algorithm 3. We use this revenue to relate the upper and lower bounds found in the previous section, as depicted in Figures 1a to 1c.

**Claim 5.3 (Relating to Upper Bound).** The revenue from selling item $j$ by deferred allocation is equal in expectation to $E[Vic_j(n+1)]$.

**Proof.** Observe that the revenue from selling item $j$ to bidder set $U$ by the Vickrey auction is the second-highest value of a bidder in $U$ for $j$. Since we exclude item $j$ in step (1) of the deferred allocation procedure and allocate it only in step (2), the allocation in step (1) does not depend on the bidders’ values for $j$. Therefore, the values of the unallocated bidders in $U$ for item $j$ are still independent random samples from $F_j$. The expected second-highest among $n + 1$ values drawn independently from $F_j$ is equal to $E[Vic_j(n+1)]$. □

To relate to the lower bound in Claim 5.2, we need the following stability property.

**Claim 5.4 (Stability).** For every value profile of the augmented matching environment, the set of bidders left unallocated by VCG is $U$ with at most one bidder removed.

**Proof.** Given the augmented matching environment, consider again the corresponding complete bipartite graph with bidders on one side and items on the other, and the bidders’ values for items as edge weights. The VCG mechanism finds the maximum-weighted matching in this graph.\(^6\) Our claim is a direct corollary of the following well-known property of matchings (see, e.g., [Wastlund 2008, Lemma 2.2]).

\(^6\)We assume there is a unique maximum-weighted matching. This holds with probability 1 as all distributions have smooth density functions.
Starting with a maximum-weighted matching of size \( m - 1 \), if we add a node (the excluded item \( j \)) to one side of the bipartite graph and find the maximum-weighted matching of size \( m \), the set of matched nodes on the other side remains the same up to a single additional node. \( \square \)

Using this claim we can lower-bound the VCG payment for item \( j \) in the augmented environment.

**Claim 5.5 (Relating to Lower Bound).** For every value profile of the augmented matching environment, the VCG payment for item \( j \) is at least the revenue from selling item \( j \) by deferred allocation.

**Proof.** The revenue from selling item \( j \) by deferred allocation is the second-highest value of a bidder in \( U \) for \( j \). Let \( i_1, i_2 \) be the two bidders in \( U \) who value item \( j \) the most. By definition, these bidders are left unallocated by the deferred allocation procedure, and by the previous claim, one of them (say \( i_1 \)) is also unallocated by the VCG mechanism. Recall that an unallocated bidder’s value for item \( j \) gives a lower bound on the VCG payment for \( j \) (Observation 5.2). So the VCG payment for \( j \) is at least \( v_{i_1,j} \), which in turn is at least the second-highest value of a bidder in \( U \) for item \( j \). \( \square \)

Putting everything together, we can now complete the proof of the Bulow-Klemperer-type theorem.

**Proof of Theorem 4.4 (B-K for Matching with \( m \) More Bidders).** We need to show that for every matching environment with \( n \) i.i.d. regular bidders and \( m \) total units, \( \mathbb{E}[VCG(n + m)] \leq \mathbb{E}[OPT(n)] \). By Claim 5.5, the VCG payment for item \( j \) in the augmented environment is at least the revenue from selling item \( j \) by deferred allocation, which by Claim 5.3 is equal in expectation to \( \mathbb{E}[Vic_j(n + 1)] \). Summing up over all items, the total expected VCG revenue in the augmented environment is at least \( \sum_j \mathbb{E}[Vic_j(n + 1)] \), and by Lemma 5.1 this upper-bounds the optimal expected revenue in the original environment. \( \square \)

### 5.3. The Case of \( m \geq n \)

**Proof of Theorem 4.5 (B-K for Matching with \( n \) More Bidders).** We need to show that for every matching environment with \( n \) i.i.d. regular bidders and \( m \geq n \) items, \( \mathbb{E}[VCG \leq n(2n)] \geq \frac{n}{m} \mathbb{E}[OPT(n)] \). The proof is similar to that of Theorem 4.4; here we highlight the necessary adjustments.

The upper bound on the optimal expected revenue remains \( \sum_j \mathbb{E}[Vic_j(n + 1)] \) (Lemma 5.1). As for the lower bound, it is no longer the case that in the augmented environment all items are allocated, and so we make use of a generalization of Observation 5.2 — the VCG payment for item \( j \) is lower-bounded not only by the value of any unallocated bidder for \( j \) itself, but also by the value of any unallocated bidder for any unallocated item. We call the highest of the latter among all unallocated bidders and items the *global* lower bound on VCG payments, and denote it by \( G \). Note that since VCG is now applied with a supply limit of \( n \), exactly \( n \) out of the \( 2n \) bidders in the augmented environment remain unallocated.

We use the modified deferred allocation selling procedure in Algorithm 4; observe that Claims 5.3, 5.4 and 5.5 hold. For Theorem 4.4 these claims were sufficient to complete the proof, by applying the following chain of arguments: all items are allocated by VCG in the augmented environment; the VCG payment for item \( j \) is at least the revenue from selling \( j \) by deferred allocation; the deferred allocation revenue is equal in expectation to \( \mathbb{E}[Vic_j(n + 1)] \); and \( \sum_j \mathbb{E}[Vic_j(n + 1)] \) is at least the optimal expected
Fig. 1: Example of relating bounds by deferred allocation \((n = 2, m = 2)\).
(a) Applying VCG to augmented matching environment (not all edges are shown). Solid edges correspond to maximum-weighted matching. Payment for item \(j = 2\) is at least \(\max\{v_{2,2}, v_{4,2}\}\) (Observation 5.2). Values of unallocated bidders are not i.i.d. samples from distribution \(F_2\).
(b) Applying deferred allocation for item \(j = 2\) to augmented matching environment. Solid edge corresponds to maximum-weighted matching excluding \(j = 2\). The set \(U\) is a superset of the unallocated bidders in (a) (Claim 5.4). Payment for \(j = 2\) is \(\max_2\{v_{1,2}, v_{2,2}, v_{4,2}\} \leq \max\{v_{2,2}, v_{4,2}\}\), where \(\max_2\) is the second-highest value (Claim 5.5). Values of bidders in \(U\) are i.i.d. samples from \(F_2\).
(c) Applying Vickrey for item \(j = 2\) to \(n + 1\) bidders, as part of the upper bound (Lemma 5.1). Values of bidders are i.i.d. samples from \(F_2\) so in expectation we get the same payment for \(j = 2\) as from deferred allocation (Claim 5.3).

**ALGORITHM 4: Selling Item \(j\) by Deferred Allocation — The Case of \(m \geq n\)**

Given a matching environment with \(2n\) bidders and \(m\) items:

1. Find a welfare-maximizing feasible allocation (a maximum matching) of \(n - 1\) items other than \(j\) to a subset of the bidders.
   Let \(U\) be the set of \(n + 1\) bidders who remain unallocated.
2. Run the Vickrey auction to sell item \(j\) to bidder set \(U\).

By the above, we can charge the VCG payments for the \(n\) allocated items against the total revenue from selling all \(m\) items by deferred allocation, which is equal in expectation to \(E[Vic_j(n + 1)]\), thus obtaining an approximation factor of \(\frac{m}{n}\). □

6. EXTENSIONS

6.1. From \(k\)-Unit to Matroid Environments

A matroid environment is a single-parameter environment in which the set system \((\{1, \ldots, n\}, \mathcal{I})\) of bidders and feasible allocations forms a matroid (for an exposition on matroids see, e.g., [Oxley 1992]). Recall that the rank of a matroid is the size of its bases, and the packing number of a matroid is its maximum number of disjoint bases.
Theorem 6.1 (Supply-Limiting Mechanism for I.i.d. Matroids). For every matroid environment with $n \geq 2$ i.i.d. regular bidders, rank $r$ and packing number $\kappa$:

1. If $\kappa \geq 2$ then the expected revenue of the VCG mechanism is a 2-approximation to the optimal expected revenue;
2. If $\kappa = 1$ then the expected revenue of the supply-limiting mechanism $\text{VCG}^{\leq \lfloor r/2 \rfloor}$ is a 4-approximation to the optimal expected revenue.

The proof below is by instantiating the general reduction, where the restriction and augmentation consist roughly of removing and adding back a basis of bidders, and applying Dughmi et al.'s Bulow-Klemperer-type result for i.i.d. matroid environments [2009, Lemma 6.1], stated here for completeness.

Theorem 6.2 (B-K for I.i.d. Matroid Environments). For every matroid environment with i.i.d. regular bidders, the expected revenue of VCG with an additional matroid-basis of bidders is at least as high as the optimal expected revenue.

Proof of Theorem 6.1. Observe that if the matroid's packing number $\kappa$ is 1, then its rank $r$ is at least 2. We instantiate the reduction as follows.

1. Restriction: If $\kappa = 1$, first intersect the original matroid with an $\lfloor r/2 \rfloor$-uniform matroid to get a new matroid environment with $\kappa' \geq 2$.
   Now that the packing number is at least 2, we can remove a basis of bidders from the environment.
2. Augmentation: Add back a basis of bidders (without changing back the supply limit).

To analyze this mechanism we first upper-bound the loss due to the restriction step. If $\kappa = 1$, the first step of the restriction incurs a loss of factor $r/\lfloor r/2 \rfloor$, and the second step incurs a loss of factor $n/(n - \lfloor r/2 \rfloor)$. Since by assumption $n$ is even, the total worst-case loss is 4. If $\kappa \geq 2$, since at most half of the bidders are removed, the loss factor is at most 2.

The expansion step is justified by the Bulow-Klemperer-type result for i.i.d. matroids (Theorem 6.2), by which running VCG on the environment augmented by an extra basis achieves expected revenue as high as the optimal expected revenue in the unaugmented environment.

6.2. From I.i.d. to Asymmetric Bidders

Consider a single-parameter $k$-unit environment, where every bidder has a publicly-observable attribute $\alpha$, say age bracket, which determines her regular distribution $F_\alpha$. In this environment, the values are i.i.d. for bidders with the same attribute, and are independent but not necessarily identically distributed for bidders with different attributes. Furthermore, assume that the environment is non-singular in the sense that there is no bidder with a unique attribute. This setting was first introduced by Dhangwatnotai et al. [2010].

We propose the following supply-limiting mechanism. For every set of $n_\alpha$ bidders with the same attribute $\alpha$ (and hence the same distribution), impose a local supply limit of $\min\{k, \lfloor \frac{n_\alpha}{2} \rfloor \}$ on the number of units that can be allocated to this bidder set, and run VCG. We remark that this supply-limiting mechanism is considerably simpler than Myerson’s optimal mechanism for this setting, which requires computing different virtual value functions for different attributes.

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7We assume here that $n_\alpha$ is even; the case of odd $n_\alpha$ can be handled with a small loss.
Theorem 6.3 (Supply-Limiting for Attribute-Based Environments). For every non-singular \( k \)-unit environment with attribute-based regular bidders, the expected revenue of the above supply-limiting VCG mechanism is a 4-approximation to the optimal expected revenue.

Proof. We instantiate our reduction as follows.

1. **Restriction:** For every \( a \), remove \( \min\{k, \frac{n_a}{\ell_a}\} \) bidders with attribute \( a \) from the environment, and if \( k \geq \frac{n_a}{\ell_a} \), limit the supply for bidders with attribute \( a \) to \( \frac{n_a}{\ell_a} \) units.

2. **Augmentation:** For every \( a \), add back \( \min\{k, \frac{n_a}{\ell_a}\} \) bidders with attribute \( a \).

In the restriction step, we remove at most half of the bidders and so lose a factor of at most 2 [Dughmi et al. 2009, Theorem 3.1]. Note that limiting the supply has no effect on the revenue, and that after the restriction we essentially have a global supply limit of \( \min\{k, \sum a \frac{n_a}{\ell_a}\} \) on the restricted environment. Therefore we can view this as a parallel multi-unit environment (see Appendix A). In particular, this is a parallel multi-unit environment with a single item, bidder sets of size \( \max\{n_a - k, \frac{n_a}{\ell_a}\} \), a global supply limit of \( \min\{k, \sum a \frac{n_a}{\ell_a}\} \), and local supply limits of \( \min\{k, \frac{n_a}{\ell_a}\} \). We can now apply Theorem A.1 to conclude that in the augmentation step we lose at most another factor of 2. We have thus shown that running VCG on the restricted environment with \( \min\{k, \frac{n_a}{\ell_a}\} \) additional bidders per attribute \( a \) is a 4-approximation to the optimal expected revenue on the original environment, and this is equivalent to running the supply-limiting mechanism on the original environment. This completes the proof. \( \square \)

6.3. From Matching to Multi-Unit Matching Environments

A \( k \)-unit (or multi-unit) matching environment is a multi-parameter matching environment with multiple units per item. There are \( k_j \) units of every item \( j \) and a total of \( m = \sum_j k_j \) units. We can also impose an additional global supply limit \( \ell \leq m \) on the total number of allocated units, and/or local supply limits \( \{\ell_j\} \) where \( \ell_j \leq k_j \) limits the number of allocated units of every item \( j \).

Two of the supply-limiting mechanisms introduced in Section 4 for i.i.d. matching environments apply directly to i.i.d. multi-unit matching environments as well. Theorems 4.1 and 4.2 hold without change for multiple units. Recall that Theorem 4.3 gives a constant approximation guarantee in the challenging case where the number of items \( m \) is much larger than the number of bidders \( n \). In order to generalize this theorem to multi-unit matching, we introduce a slightly more general supply-limiting mechanism. Let \( \text{VCG}^{\leq \ell_j \leq \ell} \) be the VCG mechanism with a global supply limit \( \ell \) on the total number of allocated units, and local supply limits \( \{\ell_j\} \) on the number of allocated units of every item \( j \). We then have the following multi-unit version of Theorem 4.3.

Theorem 6.4 (27-Approximation for Multiple Units). For every multi-unit matching environment with \( n \geq 3 \) i.i.d. regular bidders, \( m \) total units and \( k_j \) units per item \( j \), \( \mathbb{E}[\text{VCG}^{\leq n/3, \leq \lceil k_j/2 \rceil}(n)] \geq \frac{1}{27} \mathbb{E}[\text{OPT}(n)] \).

We prove this theorem via our general reduction, using the following multi-unit version of the Bulow-Klemperer-type result in Theorem 4.6.

Theorem 6.5 (9-Approximate B-K for Multiple Units). For every multi-unit matching environment with \( n \) i.i.d. regular bidders, \( m \) total units and \( k_j \) units per item \( j \), \( \mathbb{E}[\text{OPT}(n)] \leq 9 \mathbb{E}[\text{VCG}^{\leq n, \leq \lceil k_j/2 \rceil}(3n)] \).

Proofs appear in Appendix C.
REFERENCES


APPENDIX

A. BULOW-KLEMPERER-TYPE THEOREM FOR PARALLEL MULTI-UNIT ENVIRONMENTS

The theorem developed in this section has applications in proving supply-limiting results for multi-unit matching environments (Section 6.3 and Appendix C) and for asymmetric environments (Section 6.2).

A single-parameter parallel $k_j$-unit (or parallel multi-unit) environment consists of multiple items and a $k_j$-unit auction for each item $j$. These auctions are related by a global supply limit $\ell$. For each item $j$ there are $k_j \leq n$ units available, and $n$ interested unit-demand bidders, whose values are drawn i.i.d. from a regular distribution $F_j$. There is a total of $m = \sum_j k_j$ units, of which at most $\ell$ can be allocated at the same time. A feasible allocation is thus a set of bidders containing at most $\min\{n, \ell\}$ bidders overall, and at most $k_j$ bidders per item $j$. We can also impose a local supply limit $\ell_j$ on the number of units allocated of item $j$.

A parallel multi-unit environment is a particular case of a matroid environment, the underlying matroid being the intersection of a partition matroid with an $\ell$-uniform matroid. As such, the Bulow-Klemperer-type theorem for non-i.i.d. matroid environments by Hartline and Roughgarden [2009, Theorem 4.4] applies to it. However, this theorem requires augmenting the environment with an additional “duplicate” bidder for every original bidder, and adding the constraint that at most one of each pair of
duplicates wins simultaneously. This is wasteful as many of the original bidders in a parallel multi-unit environment are i.i.d., namely all bidders interested in the same item $j$.

The following theorem shows that it is sufficient to augment the environment with only $k_j$ additional bidders per item $j$.

**Theorem A.1 (B-K for Parallel Multi-Unit Environments).** For every parallel $k_j$-unit environment with local supply limits $\{\ell_j\}$, the expected revenue of VCG with $\min\{k_j, \ell_j\}$ additional bidders per item $j$ is a 2-approximation to the optimal expected revenue.

We remark that this theorem can be used to derive a corresponding supply-limiting mechanism for parallel multi-unit environments.

**Proof.** We need to show that the expected revenue of VCG in the parallel multi-unit environment after augmentation is a 2-approximation to the optimal expected revenue in the original environment. As shown by Hartline and Roughgarden [2009, Lemma 4.5], a 2-approximation would follow if we prove the following commensuration conditions:

(C1). $E_v[\sum_{i \in VCG(v) \setminus OPT(v)} \phi_i] \geq 0$, and

(C2). $E_v[\sum_{i \in VCG(v) \setminus OPT(v)} p_i(v)] \geq E_v[\sum_{i \in OPT(v) \setminus VCG(v)} \phi_i]$ 

where $\phi_i$ is the virtual value of bidder $i$, $v$ denotes a valuation profile of both original and augmenting bidders, and $OPT(v), VCG(v)$ denote the winning bidders chosen by the optimal mechanism in the original environment and VCG in the augmented environment, respectively.

The proof of (C2) in Hartline and Roughgarden [2009] can be directly applied to our setting. However, proving (C1) in our setting turns out to be more technically challenging. In particular, the random sets $VCG(v)$ and $OPT(v)$ are dependent in more complicated ways. To handle such dependency issues, we introduce an auxiliary allocation procedure as an intermediary for the analysis, and make careful use of the FKG inequality along the way (see [Alon and Spencer 1992]). The proof relies on the fact that in our setting, the VCG mechanism (as well as the auxiliary procedure) is a simple greedy process, which takes the top bidders up to the local supply limit for each item, and then sorts this set and take the best ones up to the global supply limit.

In the remainder of this section we prove (C1). This is sufficient to complete the proof of the theorem.

Fix an item $j$. We let $B_j$ contain the $n$ original bidders for item $j$ as well as the $k_j$ augmented bidders for item $j$. We aim to prove a stronger version of (C1) where $i$ ranges only over $B_j$. We condition the rest of the analysis on fixed values of all bidders (including augmenting bidders) for all items except $j$, and fixed values of the original bidders for item $j$. Now only the values of the augmenting bidders for item $j$, which we denote by $v'$, are still random.

As the optimal mechanism does not rely on the augmenting bidders, the winning set of $OPT$ is now fully determined, which we denote by $OPT$ as well. Let $O_j$ denote those bidders from $B_j$ that win in $OPT$. To analyze the VCG mechanism, we consider the auxiliary procedure in Algorithm 5.

Let $T(v')$ be the bidders for item $j$ that win in this procedure. (So $O_j \subseteq T(v') \subseteq B_j$.) Let $VCG_j(v')$ be the bidders for item $j$ that win in VCG. The following claim relates the auxiliary procedure to VCG.
Claim A.2. For all values $v'$ of augmenting bidders for item $j$, $VCG_j(v')\setminus T(v')$ always have nonnegative virtual values.

Proof of Claim A.2. Compare the auxiliary procedure to VCG. In VCG, we have the additional freedom of replacing bidders in $O_j$ by others. By a property of the greedy process of VCG, each replacement will be either from $B_j$ with even higher values and therefore higher virtual values, or from bidders for other items. As all bidders in $O_j$ already had nonnegative virtual values, the replacements that come from $B_j$ (i.e., bidders in $VCG_j(v')\setminus T(v')$) will also have nonnegative virtual values.

The following claim relates the auxiliary procedure to OPT.

Claim A.3. $T(v')\setminus O_j$ have nonnegative total virtual value in expectation over all values $v'$ of augmented bidders for item $j$.

Proof of Claim A.3. Let $\psi_i(v')$ for $i = 1, \ldots, k_j$ be the $i$-th highest virtual value (or value) of a bidder in $B_j\setminus O_j$. Let 0-1 variable $1_i(v')$ indicate whether the $i$-th highest bidder in $B_j\setminus O_j$ wins in the auxiliary procedure, and let $p_i = \mathbb{E}_{v'}[1_i(v')]$ be the corresponding probability. It follows that the total virtual value of bidders in $T(v')\setminus O_j$ is $\sum_{i=1}^{k_j} \mathbb{E}_{v'}[\psi_i(v') \cdot 1_i(v')]$.

By the greedy nature of the auxiliary procedure, one can verify that if the $i$-th highest bidder in $B_j\setminus O_j$ wins in the auxiliary procedure, and some bidders in $B_j\setminus O_j$ increase their values, then the $i$-th highest bidder in $B_j\setminus O_j$ still wins. Formally, for two valuation profiles $v'$ and $v''$ of the augmenting bidders for item $j$ with $v'_i \leq v''_i$ for all $i$, we have $1_i(v') \leq 1_i(v'')$. This positive correlation allows us to apply the FKG inequality [Alon and Spencer 1992], and we have:

$$\sum_{i=1}^{k_j} \mathbb{E}_{v'}[\psi_i(v') \cdot 1_i(v')] \geq \sum_{i=1}^{k_j} \mathbb{E}_{v'}[\psi_i(v')] \cdot \mathbb{E}_{v'}[1_i(v')]$$

$$= \sum_{i=1}^{k_j} \mathbb{E}_{v'}[\psi_i(v')] \cdot p_i$$

$$= \sum_{i=1}^{k_j} \left( \mathbb{E}_{v'}[\sum_{i'=1}^{i} \psi_{i'}(v')] \cdot (p_i - p_{i+1}) \right)$$

where we let $p_{k+1} = 0$.

It is easy to see that $p_i$ is decreasing in $i$. Therefore it suffices to prove that $\sum_{i=1}^{k} \mathbb{E}_{v'}[\psi_i(v')] \geq 0$ for all $i$. Fixing $i$, this is the total virtual value from the top $i$ bidders from $B_j\setminus O_j$. The expected total virtual value of the first $i$ augmented bidders (sorted by identity) for item $j$ exactly equals 0. It follows that $\psi_1, \ldots, \psi_i$ as the top $i$ virtual values can only have a nonnegative total sum.

Now by Claims A.2 and A.3, $\mathbb{E}_{v'}[\sum_{i \in VCG_j(v')\setminus O_j} \phi_i] \geq 0$. Summing over all $v'$, all values of bidders for other items, and all items $j$, we have $\mathbb{E}_{v'}[\sum_{i \in VCG(v)\setminus OPT(v)} \phi_i] \geq 0$, which verifies condition (C1) and completes the proof of Theorem A.1.
B. BULOW-KLEMPERER-TYPE THEOREM FOR MATCHING

In this appendix we prove Theorem 4.6, a Bulow-Klemperer-type theorem for (single-unit) matching with $2n$ additional bidders and supply limit $n$. In previous theorems we added $m$ additional bidders, in which case a simple item-by-item charging scheme sufficed to carry out the proof. Adding $m$ bidders simplifies the analysis since the VCG mechanism can sell every unit, and so there is little loss in accounting for the optimal expected revenue item-by-item. By contrast, when only $2n$ bidders are added and $m$ is much larger than $n$, our analysis needs to take into account that, for any given valuation profile, the VCG mechanism is extracting revenue from at most $3n$ out of the $m$ units. This requires a number of additional technical ideas which we describe below.

In Section B.1 we give an overview of our proof; details of the analysis appear in subsequent sections. Throughout we assume for simplicity that $m \geq 2n$ and $n > 1$. In this appendix we may use “unit” and “item” interchangeably (there will be no confusion since here every item has a single unit).

B.1. Proof Overview

B.1.1. Upper Bound. Towards an upper bound on the optimal expected revenue for the original matching environment $Env$, we consider a single-parameter setting $\hat{Env}$ with $m(n+1)$ bidders and $m$ items. $\hat{Env}$ is a parallel 1-unit environment as defined in Appendix A: for each item $j$ separately, there are $n+1$ bidders who are interested in $j$ with values drawn i.i.d. from $F_j$, and we constrain that at most $2n$ out of the $m$ total units can be allocated. The expected VCG revenue for $\hat{Env}$ upper-bounds the optimal expected revenue for the single-parameter representative environment $Env_{rep}$ of the matching environment up to a constant factor, by applying a suitable Bulow-Klemperer-type theorem (Theorem A.1). This in turn upper-bounds the optimal expected revenue for $Env$ by Lemma 2.3.

The VCG revenue for $\hat{Env}$ has an explicit form that consists of the maximum among two types of terms. Let $A$ be the set of allocated items. The first type corresponds to the second-highest value $S_j$ for each item $j \in A$, and the second type corresponds to the highest value $H_j$ for an unallocated item $j \not\in A$. We refer to these terms as the local and global upper bounds respectively on the VCG revenue. (It might appear that the local term is similar to the upper bound we use in the $m \leq n$ case, which is sum of Vickrey revenues. However, sum of $S_j$ over $j \in A$ is in fact harder to compete with, as $A$ is optimizing for total $H_j$ value over all unit subsets with size $2n$.)

B.1.2. Lower Bound. On the other hand, consider the augmented version of the original multi-parameter environment $Env$, which has $3n$ bidders and $m$ total units. Denote this augmented environment by $Env'$.

Let $B$ be the set of items allocated by the supply-limiting mechanism $VCG_{\leq n}$ over $Env'$. The payment we get from each $j \in B$ has local and global lower bounds. The local lower bound is the highest value of an unallocated bidder for item $j$ (we used this bound in the analysis of the $m \leq n$ case), and the global lower bound is the highest value of an unallocated bidder for an unallocated item.

B.1.3. Charging Arguments. We compare the expected revenues from running the two mechanisms on $\hat{Env}$ and $Env'$ respectively, where expectations are over the probability space of $3mn$ random values (in $\hat{Env}$ only $m(n+1)$ of these are used). We charge the local and global upper bounds on the revenue of VCG for $\hat{Env}$ to the local and global lower bounds on the revenue of $VCG_{\leq n}$ for $Env'$.

We remark that the difference between the number of values in $\hat{Env}$ and $Env'$, i.e. $m(n+1)$ vs. $3n$, is because in $Env'$ we need there to be a gap between the number of
bidders and the supply limit, which is set to \( n \); the precise quantity \( 3n \) was chosen to optimize the approximation ratio.

**B.1.4. Relating the Global Bounds.** First, global bounds are charged against each other. Consider the bidder set in \( \hat{Env} \) that contains, for every item \( j \) in the set of \( 2n \) allocated items \( A \), the bidder with the highest value for \( j \). By the i.i.d. assumption, for every \( j \) this bidder is distributed uniformly among the \( n + 1 \) bidders who are interested in \( j \). Now "project" the bidder set found in \( \hat{Env} \) onto \( Env' \) — because several bidders in \( \hat{Env} \) can be attributed to the same bidder in \( Env' \), the cardinality of the bidder set can decrease. However, by a balls and bins argument, the cardinality of the bidder set in \( Env' \) will still be larger than \( n \) with constant probability. When this is the case, there is some bidder \( i \) (the bidder with highest value for item \( j \), let's say) who is unallocated by \( VCG \leq 3n \) due to the supply limit, and furthermore \( j \) must be unallocated as well. It follows that \( v_{i,j} \) is a global lower bound on the VCG payments for allocated items in \( Env' \), and at the same time is also at least as high as the global upper bound on the VCG payments for allocated items in \( \hat{Env} \).

**B.1.5. Relating the Local Upper Bound to the Lower Bounds.** Second, we charge the local upper bound against the local and global lower bounds. In particular, whenever possible we use previous arguments to charge \( S_j \) of an item \( j \in A \) against the payment we get from a possibly different item \( j' \in B \). Since the ratio between the sets of allocated items \( |A| \) and \( |B| \) is at most 2, we are guaranteed not to overcharge. Whenever \( S_j \) cannot be charged against an item in \( B \), it means that for this particular valuation profile, \( S_j \) in \( \hat{Env} \) comes from the value of some bidder who is allocated by \( VCG \leq n \) in \( Env' \). If we permute the values for \( j \) randomly among the bidders, as is justified by the i.i.d. assumption, we can use a deferred allocation argument to show that with probability at least \( \frac{4}{9} \), one of the top two bids for \( j \) comes from an unallocated bidder. So the good case in which we are able to charge occurs with high probability, and the argument follows by taking expectation.

This completes the overview of the proof; a more detailed description follows.

**B.2. Notation**

We formally define the notation introduced in the overview of the proof above, and present several additional definitions.

**Environments.** Let \( Env \) be a matching environment with \( n \) i.i.d. regular bidders and \( m \) total units, and let \( Env' \) be the augmented matching environment with \( 2n \) additional bidders. Let \( Env^{rep} \) be the representative environment corresponding to \( Env \). Environment \( \hat{Env} \) is obtained from \( Env^{rep} \) by adding one additional bidder per item, and replacing the unit-demand constraint by the more relaxed constraint of limiting the supply to \( 2n \) items that can be allocated in total.\(^8\) So \( \hat{Env} \) is a parallel 1-unit environment with \( n + 1 \) bidders per item and global supply limit \( 2n \).

**Valuation profiles** \( V \), \( v \). Let \( V = \{V^1, \ldots, V^m\} \) be a collection of \( m \) sets, each containing \( n + 1 \) random values. The values in set \( V^j \) are i.i.d. samples from \( F_j \). The collection \( V \) corresponds to a valuation profile in environment \( \hat{Env} \) up to naming of the bidders, where \( V^j \) contains the values of the bidders interested in item \( j \). Let \( v \) be a vector of \( 3nm \) random values. Values \( v_{1,j}, \ldots, v_{3n,j} \) are i.i.d. samples from \( F_j \). The vector \( v \) cor-

\(^8\)Note there is deliberate slackness in this relaxation — to obtain a matroid environment it would have been sufficient to replace the unit-demand constraint by a supply limit of \( n \); by further relaxing the supply limit to \( 2n \) we aid later analysis.
responds to a valuation profile in environment $Env'$, where $v_{i,j}$ is the value of bidder $i$ for item $j$.

Random variables over $V$. Over the sample space of $V$ we define the following. For every $j$, random variables $H_j, S_j$ are the highest and second-highest values for item $j$. Random variable $N$ is the $(2n+1)$-highest among $\{H_1, \ldots, H_m\}$ if $m > 2n$ and 0 otherwise. Random set $A$ contains every item $j$ such that $H_j$ is within the min{$2n, m$} highest among $\{H_1, \ldots, H_m\}$. Let $a_1, \ldots, a_{|A|}$ denote the items in $A$ ordered by their $H_j$ value from high to low. Observe that if VCG runs on $Env$ with valuation profile $V$, the set of min{$2n, m$} allocated items is equal to $A$.

Random variables over $v$. Over the sample space of $v$ we define the following. Consider running VCG$^{\leq n}$ on $Env'$ with valuation profile $v$. Random set $B$ contains every item $j$ such that $j$ is allocated by VCG$^{\leq n}$. Random variable $G$ (for global) is the highest value of an unallocated bidder for an unallocated item, and for every $j$, random variable $L_j$ (for local) is the highest value of an unallocated bidder for item $j$.

B.3. Upper and Lower Bounds

Lemma B.1 (Upper Bound on Optimal Expected Revenue). $E[OPT(Env)] \leq 2E[VCG(Env)]$, i.e., the expected revenue of the optimal mechanism for $Env$ is upper bounded by twice the expected revenue of the VCG mechanism for $Env$.

Proof. We know that $E[OPT(Env)] \leq E[OPT(Env^{rep})]$ (Lemma 2.3). Relaxing the unit-demand constraint in $Env^{rep}$ while maintaining a supply limit of $2n$ only increases the optimal revenue. The result is a matroid environment (a parallel 1-unit environment with $n$ bidders per item and global supply limit $2n$, to be precise), to which we add one bidder per item and apply the Bulow-Klemperer-type result in Theorem A.1,\(^9\) stating that the expected revenue of VCG on the resulting environment $Env$ is a 2-approximation to the optimal expected revenue. This completes the proof. \(\square\)

Observation B.2 (Global and Local Upper Bounds). Given a valuation profile $V$ for environment $Env$, the VCG payment for every allocated item $j \in A$ is the maximum among $N$ (global upper bound) and $S_j$ (local upper bound).

Proof. The VCG revenue from every allocated item $j$ is upper bounded by the highest value for any unallocated item, and by the second price for $j$ among the $n+1$ bidders interested in $j$. Due to the supply limit of $2n$ in $Env$ and by definitions of $N$ and $S_j$, we can write this as $\max\{N, S_j\}$. \(\square\)

Observation B.3 (Global and Local Lower Bounds). Given a valuation profile $v$ for environment $Env'$, the VCG payment of the supply-limiting VCG$^{\leq n}$ mechanism for every allocated item $j \in B$ is at least the maximum among $G$ (global lower bound) and $L_j$ (local lower bound).

Proof. The observation is based on VCG payments reflecting externalities. Say bidder $i$ wins item $j$, then if $i$ were absent from the auction, a currently unallocated bidder could have won a currently unallocated item, without violating the supply limit of $n$ and without interfering with the rest of the current allocation. So by definition of $G$, it gives a lower bound on bidder $i$’s payment for item $j$. Similarly, if bidder $i$ were

\(^9\)We can’t use the classic Bulow-Klemperer theorem here since representative bidders who are interested in different items are not i.i.d. Alternatively with some modification we could have used the Bulow-Klemperer theorem of Hartline and Roughgarden for non-i.i.d. bidders [2009, Theorem 4.4].
absent from the auction, then any currently unallocated bidder could have won item $j$, and $L$ is also a lower bound on $i$'s payment for $j$. 

**B.4. Relating the Upper and Lower Bounds**

In this section we relate the upper and lower bounds to show the following.

**Lemma B.4 (Main).** $\mathbb{E}[\text{VCG}^{\leq n}(\text{Env}')] \geq \frac{2}{3}\mathbb{E}[\text{VCG}(\text{Env})]$.

We then use this lemma to complete the proof of the Bulow-Klemperer-type theorem for the general $n,m$ case.

The first step in relating the bounds is to fix the valuation profile $V$ for environment $\text{Env}$. This completely determines the outcome of the VCG mechanism over $\text{Env}$, and fixes the set of allocated items $A = \{a_1, \ldots, a_{|A|}\}$, the global upper bound $N$, and the local upper bound $S_j$ for every item $j$.

Now consider a valuation profile for environment $\text{Env}'$ that's compatible with $V$. For every item $j$, $n+1$ out of the $3n$ values for $j$ are fixed, but the rest of the values as well as the attribution of values to bidders remain random. We denote such a valuation profile by $v(V)$. From now on, our probabilistic arguments are all over the remaining randomness in $v(V)$.

**B.4.1. Relating the Global Bounds**

**Lemma B.5.** $\Pr[|G| \geq N] \geq \frac{2}{3}$.

**Corollary B.6.** Since $G$ is non-negative, $\mathbb{E}[G] \geq \frac{2}{3}N$.

**Proof of Lemma B.5.** If $m \leq 2n$, then by definition $N = 0$ and the claim holds trivially. Assume from now on $m > 2n$.

Recall that $A$ contains the $2n$ most-valued items according to the fixed valuation profile $V$. By definition of $N$, it is upper-bounded by the highest value in $V$ for any item in $A$. Clearly it is also upper-bounded by the highest value in $v(V)$ for any item in $A$. It remains to relate $G$ to this bound.

In environment $\text{Env}'$ with valuation profile $v(V)$, consider the random subset of bidders that contains, for every $j \in A$, the bidder with the highest value for $j$ among all $3n$ bidders. Denote this random subset of bidders by $A'$. Notice that for every $j$, the bidder with the highest value for $j$ is distributed uniformly at random among the $3n$ bidders. Therefore, $A'$ corresponds to the random subset of bins chosen by throwing $2n$ balls into $3n$ bins uniformly at random. The following claim formalizes the intuition that the balls are likely to occupy many of the bins.

**Claim B.7.** $\Pr[|A'| \leq n] \leq \frac{1}{3}$.

**Proof.** We upper bound the probability that $2n$ balls thrown uniformly at random into $3n$ bins all land in a subset of at most $n$ bins. The following is a loose upper bound: $(\binom{3n}{n})n^{2n}/(3n)^{2n}$, where $(\binom{3n}{n})$ is the number of subsets of $n$ out of $3n$ bins, $n^{2n}$ is the number of possibilities to arrange $2n$ balls in $n$ bins, and $(3n)^{2n}$ is the number of possibilities to arrange $2n$ balls in $3n$ bins (this upper bound is not tight due to over-counting in the numerator). Simplifying we get the expression $(\binom{3n}{n})/3^{2n}$, which is at most $\frac{1}{3}$ for every $n$. 

We remark that as $n \to \infty$, the cardinality of $A'$ becomes concentrated around its expectation, and $\Pr[|A'| \leq n] \to 0$.

In the likely event that $|A'| > n$, there is at least one bidder $i \in A'$ that is unallocated by the supply-limiting mechanism $\text{VCG}^{\leq n}$ due to its supply limit. Let $j$ be an item in $A$ such that $i$ has the highest value for $j$ among all bidders. Then $j$ is necessarily...
unallocated by VCG, as otherwise the VCG allocation is not welfare-maximizing.
We conclude that if $|A'| > n$ then $v_{i,j}$ is the value of an unallocated bidder for an
unallocated item and so $G \geq v_{i,j}$. Since $v_{i,j} \geq N$, the probability that $G \geq N$ is at least
$\frac{2}{3}$, as required.

B.4.2. Relating the Local Upper Bound to the Lower Bounds. For every item $j \in A$ allocated
by VCG in $\hat{\text{Env}}$, we can relate between the local upper bound $S_j$ and the lower bounds
$G$ and $L_j$ provided that item $j$ has the following property.

**Definition B.8 (Good Item).** Given a valuation profile $v(V)$ for environment $\text{Env}'$, item $j$ is good if when $j$ is removed from $\text{Env}'$ and VCG $\leq n - 1$ is applied, the two bidders
with the highest and second-highest values for $j$ remain unallocated.

When item $j$ is good we can show the following relation among the relevant bounds,
depending on whether $j$ is allocated by VCG $\leq n$ on $\text{Env}'$ and so $j \in B$, or not.

**Lemma B.9.** For every good item $j$, if $j \in B$ then $L_j \geq S_j$, otherwise $G \geq S_j$.

Furthermore we argue that an item is good with high probability.

**Observation B.10.** For every item $j$, $\Pr[j \text{ is good}] \geq \frac{4}{9}$.

We now begin to prove Lemma B.9, which is based on the claim that one of the two
bidders in $\text{Env}'$ with highest and second-highest values for a good item $j$ remains un-
allocated by VCG $\leq n$. This follows from the definition of a good item together with a
stability property of VCG allocations in a matching environment, by which the allocations
of VCG $\leq n$ and VCG $\leq n - 1$ without item $j$ are almost the same. We start by showing
this stability property.

**Lemma B.11 (Extended Stability Property of Matching).** For every
matching environment with a bipartite graph of bidders and items, and every parame-
ter $n$ that is at most the number of bidders, let $M$ be the maximum weighted matching
of size at most $n$, and let $M_{-j}$ be the maximum weighted matching of size at most $n - 1$
that doesn’t include item $j$. Then the set of matched bidders in $M_{-j}$ is a subset of the
matched bidders in $M$.

**Proof.** We assume no coincidences, i.e., that no two distinct matchings have the
same weight.\(^\text{10}\)

First note that if $n = 1$ the claim becomes trivial, and if item $j$ is not matched in
$M$ then it follows from a well-known property in matching theory (see, e.g., [Wastlund
2008, Lemma 2.2]). From now on assume that $n > 1$ and item $j$ is matched in $M$.

Let subgraph $H$ be the symmetric difference $M \Delta M_{-j}$. By the no coincidences as-
sumption, $H$ can’t contain an even-length path that doesn’t include item $j$. In particu-
lar, since all cycles are even-length and don’t include item $j$, $H$ can’t contain a cycle. We
proceed by case analysis, showing that in cases 1 and 2 the lemma holds while cases 3
and 4 lead to contradictions.

1. If $H$ contains a single odd-length path including item $j$, then $M$ matches a single
   extra bidder.
2. If $H$ contains an even-length path including item $j$ and a single additional odd-
   length path, again $M$ matches a single extra bidder.

\(^\text{10}\)In the matching environments we’re concerned with, all distributions have smooth densities and so the
assumption holds with probability 1. In general, this assumption is without loss of generality via a standard
perturbation argument.
(3) If $H$ contains an odd-length path including item $j$ and several additional paths, the total number of edges in $H$ is odd, so the lengths of the additional paths sum up to an even number, contradicting the no coincidences assumption.

(4) If $H$ contains an even-length path including item $j$ and several additional paths, the additional paths must include either an even-length path or two odd-length paths, contradicting the no coincidences assumption.

\[ \square \]

We can now use the stability property to prove Lemma B.9.

**Proof of Lemma B.9.** Let $U$ be the set of bidders who who are left unallocated by the deferred allocation procedure of removing item $j$ from Env' and running VCG$^{\leq n-1}$. Since $j$ is good, $U$ includes the two bidders with highest and second-highest values $H_j, S_j$ for $j$. By the above stability property, the set of bidders who remain unallocated by VCG$^{\leq n}$ is either exactly $U$ or $U$ with one bidder removed. We conclude that there's an unallocated bidder whose value for $j$ is at least $S_j$. By definition of $G$ and $L_j$ as the highest value of an unallocated bidder for an unallocated item and for item $j$, respectively, depending on whether item $j$ is allocated either $G \geq S_j$ or $L_j \geq S_j$, as required. \[ \square \]

To show that item $j$ is good with constant probability for every $j$, we use a deferred allocation argument and utilize the ratio between the number of bidders in Env' and the supply limit of VCG$^{\leq n}$.

**Proof of Observation B.10.** Running VCG$^{\leq n-1}$ on environment Env' after removing item $j$ leaves at least $2n+1$ out of the $3n$ bidders unallocated, so $|U| \geq 2n+1$. Since item $j$ does not take part in this deferred allocation procedure, its values are distributed uniformly among all bidders, and the probability that a certain value is attributed to a bidder in $U$ is $|U|/3n \geq 2/3$. The probability that the two bidders with highest and second-highest values for item $j$ are both in $U$ is therefore at least $\frac{4}{9}$. \[ \square \]

**B.4.3. Proof of Main Lemma and Main Theorem.** Using the relations we established among the various bounds, we are now ready to prove our main lemma and theorem. We assume for simplicity that $m > n$ (otherwise the proof reduces to that of Theorem 4.4). Our proof is based on a subtle charging argument – we need to charge the payments for items $a_1, \ldots, a_{|A|}$ in $A$, which were allocated by VCG on Env, against the payments for the $n$ items in $B$, which were allocated by VCG$^{\leq n}$ on Env'. We use a somewhat different argument for items in $A \cap B$ and in $A \setminus B$.

**Proof of Lemma B.4 (Main Lemma).** For every $k \in [2n]$, we define an auxiliary random variable $X_k$ as follows.

\[
\begin{aligned}
\text{if } k \leq |A| \text{ and } a_k \in B \text{ then } X_k = \max\{G, L_{a_k}\} \\
\text{if } k \leq |A| \text{ and } a_k \notin B \text{ then } X_k = G \\
\text{otherwise } X_k = 0
\end{aligned}
\]

We also define a 2-to-1 mapping $\beta$ from the set $\{1, \ldots, 2n\}$ to the set of $n$ items $B$, such that for every $k \leq |A|$, if $a_k \in B$ then $\beta(k) = a_k$. We now show that $X_k$ serves as an intermediary between the upper and lower bounds.

For the lower bounds, observe that $X_k \leq \max\{G, L_{\beta(k)}\}$, since if $k \leq |A|$ and $a_k \in B$ then $X_k = \max\{G, L_{a_k}\} = \max\{G, L_{\beta(k)}\}$, and the other cases clearly hold as well.
Summing up over $k$ we get
\begin{equation}
\sum_{k=1}^{2n} X_k \leq 2 \sum_{k=1}^{2n} \max\{G, L_\beta(k)\} \leq 2 \sum_{j \in B} \max\{G, L_j\} \leq 2 \text{VCG} \leq n \tag{1}
\end{equation}

where the second inequality uses the property that $\beta$ is a 2-to-1 mapping.

For the upper bounds, let $k \leq |A|$. Since $X_k \geq G$ we know that $E[X_k] \geq E[G] \geq \frac{2}{9} N$ (Lemma B.5). Now combining the probability that $a_k$ is a good item (Observation B.10), with the fact that if $a_k$ is good then $X_k \geq S_{a_k}$ (Lemma B.9), we also have that $E[X_k] \geq \frac{4}{9} S_{a_k}$. Summing up over $k \leq |A|$ we get
\begin{equation}
\sum_{k=1}^{|A|} E[X_k] \geq \frac{4}{9} \sum_{k=1}^{|A|} \max\{N, S_{a_k}\} \geq \frac{4}{9} E[\text{VCG}(\hat{\text{Env}})] \tag{2}
\end{equation}

Combining Inequalities 1 and 2 completes the proof that $E[\text{VCG} \leq n \leq \max\{G, L\}] \geq \frac{4}{9} E[\text{VCG}(\hat{\text{Env}})]$. \hfill \Box

PROOF OF THEOREM 4.6 (B-K FOR MATCHING WITH 2n MORE BIDDERS. Follows directly from the main lemma and from the bounds we established (Lemma B.1). \hfill \Box

C. BULOW-KLEMPERER-TYPE THEOREM FOR MULTI-UNIT MATCHING

In this section we prove Theorem 6.5, the multi-unit version of Theorem 4.6, i.e., a Bulow-Klemperer-type theorem for multi-unit matching with $2n$ additional bidders, global supply limit $n$, and local supply limits $\ell_j = \lceil k_j / 2 \rceil$. In particular we show that
\begin{equation}
E[\text{OPT}(n)] \leq 9 E[\text{VCG} \leq \max\{G, L\} (3n)]
\end{equation}

The proof is similar to that of Theorem 4.6 which appears in Appendix B. The main novel component is the single-parameter Bulow-Klemerer-type result for parallel multi-unit environments, which is required to upper bound the optimal expected revenue, and this result has already been stated in Theorem A.1 and proved in Appendix A. In what follows we highlight the remaining differences from the proof of Theorem 4.6.

For simplicity, we assume throughout that $m \geq 2n$ (otherwise, one can always use the Bulow-Klemperer-type results in Theorems 4.4 and 4.5 instead of Theorem 6.5). This assumption simplifies the analysis since it guarantees that despite the local supply limits, there are enough units such that $\text{VCG} \leq \max\{G, L\}$ allocates to exactly $n$ bidders. We also assume for simplicity that $k_j < n$ for every $j$ (the case where $k_j = n$ is only simpler).

C.1. Upper and Lower Bounds

C.1.1. Notation. We use the same notation as in the proof of Theorem 4.6, but in some cases to denote somewhat different objects. We now state the differences.

Environments. Let $\text{Env}$ be a multi-unit matching environment with $n$ i.i.d. regular bidders, $m \geq 2n$ total units, and $k_j \leq n$ units per item $j$. Let $\text{Env}', \text{Env}^{\text{rep}}$ be the augmented and representative environments, respectively. Environment $\hat{\text{Env}}$ is obtained from $\text{Env}^{\text{rep}}$ by adding $n$ additional bidders per item, and replacing the unit-demand constraint by a supply limit of $2n$. So $\hat{\text{Env}}$ is a $k_j$-unit, $m$-item environment with $2n$ bidders per item and supply limit $2n$.

Valuation profiles $V, v$. Let $V$ be a collection of sets $V_j$, each containing $2n$ i.i.d. samples from $F_j$. There is not change in $v$. 

Random variables over V. Let \( S_j \) be the \((k_j + 1)\)-th-highest value for item \( j \). If \( m > 2n \), random variable \( N \) is the \((2n + 1)\)-st-highest among a set containing the \( k_j \) highest values for every item \( j \). Otherwise, \( N = 0 \). Let \( A \) be a multiset containing the \( 2n \) items (with repetitions) with the highest values among a set containing the \( k_j \) highest values for every item \( j \). Let \( a_1, \ldots, a_{2n} \) denote the items in \( A \) ordered by their value from high to low. Observe that if VCG runs on \( \text{Env} \) with valuation profile \( V \), the set of \( 2n \) allocated items is equal to \( A \).

Random variables over \( v \). Consider running VCG\([^n,\leq[k_j/2]]\) on \( \text{Env}' \) with valuation profile \( v \). Let \( B \) be a multiset containing a copy of \( j \) for every allocated unit of \( j \). Random variable \( G \) (for \( \text{global} \)) is the highest value of an unallocated bidder for an unallocated unit, and for every \( j \), random variable \( L_j \) (for \( \text{local} \)) is the highest value of an unallocated bidder for item \( j \).

C.1.2. Bounds. Bounds do not change. Note that the proof of the upper bound on the optimal expected revenue critically uses the single-parameter Bulow-Klemperer-type result in Theorem A.1 to justify the augmentation of the environment by adding \( n \geq k_j \) bidders per item, with no restrictions of the form “at most one of every pair of duplicates can be allocated”.

C.2. Relating the Upper and Lower Bounds

C.2.1. Relating the Global Bounds

Lemma C.1. \( \Pr[G \geq N] \geq \frac{2}{3} \).

Corollary C.2. Since \( G \) is non-negative, \( \mathbb{E}[G] \geq \frac{2}{3} N \).

Proof of Lemma C.1. In environment \( \text{Env}' \) with random valuation profile \( \nu(V) \), denote by \( A' \) the random subset of bidders that contains the highest bidders for items in \( A \). Subset \( A' \) corresponds to the random subset of bins chosen by throwing \( 2n \) balls into \( 3n \) bins uniformly at random, under the restriction that some balls cannot fall in the same bins, because they correspond to values of different bidders for the same item. The probability that \( |A'| \leq n \) is thus at most the probability calculated in Claim B.7, i.e., \( \Pr[|A'| \leq n] \leq \frac{1}{3} \). So with high probability, there exists a bidder \( i \in A' \) that is unallocated by the supply-limiting mechanism VCG\([^n,\leq[k_j/2]]\) due to the global supply limit.

Let \( j \) be one of the items in \( A \) for which bidder \( i \) is in \( A' \), i.e., \( i \) has one of the highest values for \( j \). At least one unit of \( j \) is unallocated by VCG\([^n,\leq[k_j/2]]\). We conclude that if \( |A'| > n \) then \( v_{i,j} \) is the value of an unallocated bidder for an unallocated unit and so \( G \geq v_{i,j} \). Since \( v_{i,j} \geq N \), the probability that \( G \geq N \) is at least \( \frac{2}{3} \), as required.

C.2.2. Relating the Local Upper Bound to the Lower Bounds. We redefine a \( \text{good} \) item as follows.

Definition C.3 (Good Item). Given a valuation profile \( \nu(V) \) for environment \( \text{Env}' \), item \( j \) is \( \text{good} \) if when all units of \( j \) are removed from \( \text{Env}' \) and VCG with global supply constraint of \( n - [k_j/2] \) and local supply constraints \( \ell_j' \neq j \) is applied, then \( [k_j/2] + 1 \) bidders whose values for \( j \) are among the \( k_j + 1 \) overall highest values for \( j \) remain unallocated.

When item \( j \) is good we can show the following.

Lemma C.4. For every good item \( j \), if \( j \in B \) then \( L_j \geq S_j \), otherwise \( G \geq S_j \).

Furthermore we argue that an item is good with high probability.
Observation C.5. For every item $j$, $\Pr[j \text{ is good}] \geq \frac{4}{9}$.

We first prove Lemma C.4, using the following stability property.

Lemma C.6 (Generalized Stability Property). Consider a matching environment with a bipartite graph of bidders and items in which every item $j$ can be matched up to $\lceil \frac{k_j}{2} \rceil$ times. For every parameter $n$ that is at most the number of bidders, let $M$ be the maximum weighted matching of size at most $n$, and let $M_{-j}$ be the maximum weighted matching of size at most $n - \lfloor \frac{k_j}{2} \rfloor$ that doesn’t include item $j$. Then the set of matched bidders in $M_{-j}$ is a subset of the matched bidders in $M$.

Proof. By iterative application of the stability property in Lemma B.11. □

Proof of Lemma C.4. Let $U$ be the set of bidders who are left unallocated by the deferred allocation procedure of removing item $j$ from $Env'$ and running VCG with global supply constraint of $n - \lfloor \frac{k_j}{2} \rfloor$ and local supply constraints $\ell'_j \neq j$. Since $j$ is good, $U$ includes $\lceil \frac{k_j}{2} \rceil + 1$ bidders whose values for $j$ are among the $k_j + 1$ overall highest values for $j$. By the above stability property, the set of bidders who remain unallocated by VCG $\leq n \leq \lceil \frac{k_j}{2} \rceil$ is $U$ with at most $\lfloor \frac{k_j}{2} \rfloor$ bidders removed. We conclude that there’s an unallocated bidder whose value for $j$ is at least $S_j$. By definition of $G$ and $L_j$ as the highest value of an unallocated bidder for an unallocated unit and for item $j$, respectively, depending on whether item $j$ is allocated either $G \geq S_j$ or $L_j \geq S_j$, as required. □

We show that item $j$ is good with constant probability for every $j$ by a deferred allocation argument.

Proof of Observation C.5. Running VCG with global supply constraint of $n - \lceil \frac{k_j}{2} \rceil$ and local supply constraints $\ell'_j \neq j$ on environment $Env'$ after removing item $j$ leaves at least $2n + \lceil \frac{k_j}{2} \rceil$ out of the $3n$ bidders unallocated, i.e., $|U| \geq 2n + \lceil \frac{k_j}{2} \rceil$. Since item $j$ does not take part in this deferred allocation procedure, its $k_j + 1$ highest values are distributed uniformly among all bidders. The probability that at least $\lceil \frac{k_j}{2} \rceil + 1$ of these values are in $U$ is at least $\frac{4}{9}$. □

C.2.3. Proof of Main Lemma and Main Theorem. The statement of the main lemma and its proof do not change. Note that to define the 2-to-1 mapping from $\{1, \ldots, 2n\}$ to $B$ used in the proof we apply the assumption that $m \geq 2n$ and so $|B| = n$.

Proof of Theorem 6.5 (B-K — Multi-Unit Version). Follows directly from the main lemma and from the established bounds. □