Problem 1. Let $A_1, \ldots, A_n$ be finite sets.
(a) Prove that there is a set $T = \{t_1, t_2, \ldots, t_n\}$ such that $t_i \in A_i$ are distinct, if and only if $|\bigcup_{j \in J} A_j| \geq |J|$ for all $J \subseteq [n]$. (Don’t just say it’s Hall’s theorem - prove it using some statement we had in class.)
(b) Define $I = \{I \subseteq [n] : \forall J \subseteq I; |\bigcup_{j \in J} A_j| \geq |J|\}$. Show that $M = ([n], I)$ is a matroid. Also, prove that we can test whether $S \in I$ for a given set $S$ efficiently - this is not clear from the definition. What is the description of the matroid polytope $P(M)$?

Problem 2. An edge cover is a set of edges $F$ such that every vertex is incident to some $e \in F$.
(a) Prove that the minimum cardinality of an edge cover + the maximum cardinality of a matching = the number of vertices (even in non-bipartite graphs).
(b) Prove that a minimum-weight edge cover can be found by reduction to maximum-weight matching. (Hint: think about $w'_{uv} = w_u + w_v - w_{uv}$ where $w_u = \min\{w_{uv} : (u,v) \in E\}$.)

Problem 3. Consider the matroid base polytope $P_{\text{base}}(M) = \text{conv}\{\chi_B : B \text{ is a base of } M\}$. Prove that two vertices $\chi_B, \chi_{B'}$ form an edge of $P_{\text{base}}(M)$ if and only if $|B \setminus B'| = |B' \setminus B| = 1$. (You can use the results of homework problems as known facts.)

Problem 4. Recall the Generalized Assignment Problem, with machines $M$, jobs $J$ and allowable machine-jobs pairs $E$. The polytope of fractional solutions that we considered was the following:

$$P = \{x \in \mathbb{R}_+^E : \forall j \in J; \sum_{i \in M : (i,j) \in E} x_{ij} = 1;$$

$$\forall i \in M; \sum_{j \in J : (i,j) \in E} p_{ij} x_{ij} \leq T\}$$

We assume here that $p_{ij} \leq T$ for all $(i,j) \in E$. Prove that if we want to convert a fractional solution $x$ into an integer one, we need to lose at least a factor of 2 in terms of $T$. In other words, for every $\lambda < 2$ find an instance such that there is a fractional solution for $T = 1$, but there is no integer solution for $T = \lambda$.

Problem 5. Suppose we have a connected graph $G = (V, E)$ and we would like to allocate some edges to $k$ agents with monotone submodular functions $w_i : 2^E \rightarrow \mathbb{R}_+$. We would like to allocate disjoint subsets of edges $E_1, \ldots, E_k$ to the agents so as to maximize the total value $\sum_{i=1}^k w_i(E_i)$, under the additional constraint that $E \setminus \bigcup_{i=1}^k E_i$ is still a connected subgraph of $G$. Show that this problem can be approximated within a factor of $1 - 1/e$. 