You should solve the homework on your own. Don’t use any books or the internet.

**Problem 1.** [Knuth p. 408, variant of problem 11]
Let \( c_n \) denote the number of ways that an amount of \( n \) cents can be paid in coins of value 5, 10 and 25 cents.

- Find a simple form for the generating function \( C(x) = \sum_{n=0}^{\infty} c_n x^n \).
- Expand the power series explicitly up to \( x^{25} \); how many ways are there to pay 25 cents?

**Problem 2.** Let \( a_k(n) \) denote the number of partitions of an integer \( n \) into at most \( k \) parts. (Let’s define \( a_k(0) = 1 \).)

- Prove that \( a_k(n) \) is equal to the number of partitions of \( n \) into a sum of integers in \( \{1, 2, \ldots, k\} \).
- Find a simple form for the generating function \( A_k(x) = \sum_{n=0}^{\infty} a_k(n) x^n \).

**Problem 3.** [Knuth p. 408, problem 9]
If \((a_1, \ldots, a_m)\) and \((b_1, \ldots, b_m) = (a_1, \ldots, a_m)^T\) are conjugate partitions (i.e., \(m\) is the number of rows and columns in both, and \(a_1 \geq a_2 \geq \ldots a_m, \ b_1 \geq b_2 \geq \ldots b_m\) are the row/column lengths), prove that the multisets \(\{a_1 + 1, \ldots, a_m + m\}, \{b_1 + 1, \ldots, b_m + m\}\) are equal (i.e., they contain each integer the same number of times).

**Problem 4.** Call two permutations \(\sigma_1, \sigma_2 \in S_n\) conjugate \((\sigma_1 \sim \sigma_2)\) if there is a permutation \(\pi\) such that \(\sigma_2 = \pi^{-1} \sigma_1 \pi\).

- Show that \(\sigma_1 \sim \sigma_2\) is an equivalence relation (symmetric, reflexive and transitive).
- Show that \(\sigma_1 \sim \sigma_2\) if and only if \(\sigma_1\) and \(\sigma_2\) have the same cycle structure (lengths of cycles, written down as a monotone sequence).
- Show that the number of equivalence classes of \(\sim\) ("conjugacy classes" in \(S_n\)) is exactly \(p(n)\), the number of partitions of \(n\) into positive integers.