You should solve the homework on your own. Don’t use any books or the internet.

**Problem 1.** Find the generating function of the sequence \((1, 1, 0, 1, 1, 0, 1, 1, 0, \ldots)\) (in a closed form, without infinite summations).

**Problem 2.**

- Prove that if \(A(x)\) is the generating function for the sequence \((a_0, a_1, a_2, \ldots)\), then \(\frac{A(x)}{1-x}\) is the generating function of the sequence \((a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots)\).
- Use this formula to compute \(\sum_{k=0}^{m} (-1)^k \binom{n}{k}\) for \(0 \leq m \leq n\).

**Problem 3.** Prove by suitable bijections that the following numbers are equal:

- The number of sequences \(\sigma \in \{+1, -1\}^{2n}\) such that \(\sum_{i=1}^{k} \sigma_i \geq 0\) for every \(1 \leq k \leq 2n\), and \(\sum_{i=1}^{2n} \sigma_i = 0\).
- The number of ways to arrange the numbers \(\{1, 2, \ldots, 2n\}\) in a \(2 \times n\) array so that each row and each column is increasing.
- The number of paths from \((0, 0)\) to \((n, n)\), where the steps are in the direction of either \((+1, 0)\) or \((0, +1)\), and the path must never drop below the diagonal connecting \((0, 0)\) and \((n, n)\).

What are these numbers called?

**Bonus problem.** Prove that the number \((6 + \sqrt{37})^{999}\) has at least 999 zeros after the decimal point.