You should solve the homework on your own. Don’t use any books or the internet.

Problem 1. Call two permutations $\sigma_1, \sigma_2 \in S_n$ conjugate ($\sigma_1 \sim \sigma_2$) if there is a permutation $\pi$ such that $\sigma_2 = \pi^{-1} \sigma_1 \pi$.

- Show that $\sigma_1 \sim \sigma_2$ is an equivalence relation (symmetric, reflexive and transitive).
- Show that $\sigma_1 \sim \sigma_2$ if and only if $\sigma_1$ and $\sigma_2$ have the same cycle structure (lengths of cycles, written down as a monotone sequence).
- Show that the number of equivalence classes of $\sim$ ("conjugacy classes" in $S_n$) is exactly $p(n)$, the number of partitions of $n$ into positive integers (here, $3 = 2 + 1$ is considered the same partition as $3 = 1 + 2$).

Problem 2. Let $a_k(n)$ denote the number of partitions of an integer $n$ into at most $k$ summands (as above, $3 = 2 + 1$ is considered the same partition as $3 = 1 + 2$). Let’s define $a_k(0) = 1$.

- Prove that $a_k(n)$ is equal to the number of partitions of $n$ into a sum of integers in $\{1, 2, \ldots, k\}$.
- Find a simple form for the generating function $A_k(x) = \sum_{n=0}^{\infty} a_k(n)x^n$.

Problem 3. Let $b_k(n)$ denote the number of ordered partitions of $n$ into $k$ parts, or equivalently the number of solutions to $a_1 + a_2 + \ldots + a_k = n$ such that each $a_i$ is a positive integer. (Here, we consider $3 = 1 + 2$ different from $3 = 2 + 1$.)

- Prove that $b_k(n) = \binom{n-1}{k-1}$.
- Using this formula, compute the number of all ordered partitions of $n$ (without fixing $k$).

Bonus problem. How many ways are there to triangulate a polygon with $n$ sides, without adding new vertices? (i.e., partitions into triangles by drawing non-intersecting line-segments between its vertices)