Math 108 Problem Set 4 Solutions

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Problem 1. We begin with a way of traversing all strings in \(\{0,1\}^{n-1}\) changing one bit at a time. (This may be done by using the Gray code, for example.) Let’s call the strings in this list \(S_1, S_2, \ldots, S_{2^n-1}\). Given a string \(S \in \{0,1\}^{n-1}\), let \(\tilde{S}\) denote the string obtained by switching all of the bits in \(S\). Then I claim that the following sequence of strings in \(\{0,1\}^n\) solves the problem: 

\[0S_1, 1\tilde{S}_1, 0S_2, 1\tilde{S}_2, 0S_3, 1\tilde{S}_3, \ldots, 0S_{2^n-1}, 1\tilde{S}_{2^n-1}.\]

Indeed, first note that going from \(0S_i\) to \(1\tilde{S}_i\) switches all \(n\) bits. Going from \(1\tilde{S}_i\) to \(0S_{i+1}\), however, switches all but one of the bits, namely, the bit that gets switched when going from \(S_i\) to \(S_{i+1}\). So we are switching the right number of bits at each step. Finally, as \(S_i\) ranges over all strings in \(\{0,1\}^{n-1}\), so does \(\tilde{S}_i\). Therefore, the strings \(0S_i\) consist of all strings in \(\{0,1\}^n\) beginning with a 0, while the strings \(1\tilde{S}_i\) consist of all strings in \(\{0,1\}^n\) beginning with a 1. The above sequence therefore runs through each string in \(\{0,1\}^n\) exactly once.

Problem 2. A clique in \(G\) is precisely a set of elements of \(P\) that are pairwise comparable, i.e., a chain in \(P\). By Homework 3, Problem 3, therefore, \(P\) may be decomposed into \(k\) antichains. If we simply assign one of our \(k\) colors to each antichain (that is, each vertex in the antichain is colored with that color; so the first antichain receives color 1, the second color 2, and so forth), then we obtain a \(k\)-coloring of \(G\), because any two neighbors are comparable, hence not in the same antichain (by definition of an antichain), hence a different color.

Problem 3. We use a probabilistic argument, which is similar to a nice (and well-known) argument for proving Sperner’s bound on the largest antichain in \(2^{[n]}\). Let \(\mathcal{F}\) be a family as in the problem. Consider a random permutation \(\sigma : [n] \to [n]\). We write this permutation as an ordering of the elements of \([n]\): \(\sigma = \{\sigma(1), \sigma(2), \sigma(3), \ldots, \sigma(n)\}\). A prefix of \(\sigma\) is the set corresponding to an initial string in \(\sigma\). That is, a prefix is a set of the form \(\{\sigma(1), \sigma(2), \ldots, \sigma(m)\}\) for some \(0 \leq m \leq n\) (\(m = 0\) corresponds to the empty set). We will compute the expected number of prefixes of \(\sigma\) that lie in \(\mathcal{F}\). On the one hand, by linearity....
of expectation, this number is the sum over all \( A \in \mathcal{F} \) of the probability that \( A \) is a prefix of \( \sigma \). For any such \( A \), this probability is \( 1 / \binom{n}{|A|} \), since each of the \( \binom{n}{|A|} \) subsets of \([n]\) of size \(|A|\) is equally likely to be the prefix of \( \sigma \) of size \(|A|\). Thus, the expected value of the number of prefixes lying in \( \mathcal{F} \) is

\[
\sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}}
\]

On the other hand, the set of prefixes of a given \( \sigma \) forms a chain in \( 2^{[n]} \), since clearly we have \( \{\sigma(1), \sigma(2), \ldots, \sigma(m)\} \subset \{\sigma(1), \sigma(2), \ldots, \sigma(k)\} \) if \( m < k \). By assumption, however, \( \mathcal{F} \) contains no chains of size 3, hence the expected number of prefixes of \( \sigma \) lying in \( \mathcal{F} \) is \( \leq 2 \). We therefore obtain

\[
\sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}} \leq 2
\]

Now \( \binom{n}{k} \leq \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} \) for all \( k \), so the left side above is \( \geq |\mathcal{F}| / \binom{n}{\lfloor n/2 \rfloor} \). We deduce that

\[
|\mathcal{F}| \leq 2 \left( \frac{n}{\lfloor n/2 \rfloor} \right) = \left( \frac{n}{\lfloor n/2 \rfloor} \right) + \left( \frac{n}{\lceil n/2 \rceil} \right).
\]

Note by the way that this inequality is sharp, at least when \( n \) is odd. Indeed, it is attained if we choose \( \mathcal{F} \) to be the set of subsets of \([n]\) of size either \( \lfloor n/2 \rfloor \) or \( \lceil n/2 \rceil \).