Math 108 Problem Set 6 Solutions

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**Problem 1.** The generating function of \((1, 1, 1, 1, \ldots)\) is the geometric series \(\sum_{n=0}^{\infty} X^n = 1/(1-X)\). To obtain the generating function that we want, we must subtract the generating function of the sequence \((0, 0, 1, 0, 1, \ldots)\). This is \(\sum_{n=0}^{\infty} X^{3n+2} = X^2 \sum_{n=0}^{\infty} (X^3)^n = X^2/(1-X^3)\). Thus, the generating function of the sequence in the problem is:

\[
\frac{1}{1-X} - \frac{X^2}{1-X^3} = 1 + X
\]

**Problem 2.**

- We have \(A(X)/(1-X) = (a_0 + a_1X + a_2X^2 + a_3X^3 + \ldots) \cdot (1 + X + X^2 + X^3 + \ldots)\). The \(X^n\) term in this product is \(\sum_{k=0}^{n} (a_k X^k)(X^{n-k}) = (\sum_{k=0}^{n} a_k) X^n\), so the product is indeed the generating function of the partial sums of the \(a_i\).

- Let \(S_m := \sum_{k=0}^{m} (-1)^k \binom{n}{k}\). We will first compute the generating function for the \(S_m\) \((0 \leq m \leq n)\). By the first part of the problem, this generating function is \(B(X)/(1-X)\), where \(B(X) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} X^k\) is the generating function of the sequence whose \(k\)th term is \((-1)^k \binom{n}{k}\). By the binomial theorem, \(B(X) = (1-X)^n\). Thus, the generating function of the \(S_m\) is \((1-X)^{n-1} = \sum_{m=0}^{n-1} (-1)^m \binom{n-1}{m} X^m\), the last equality coming once again from the binomial theorem. Thus, \(S_m = (-1)^m \binom{n-1}{m}\).

**Problem 3.** Let us first provide a bijection between the first and third sets. Given a sequence \(\sigma \in 1, -1^{2n}\), we get a path as in the third set by declaring that the \(k\)th step will be up if \(\sigma_k = 1\) and to the right if \(\sigma_k = -1\). The condition that \(\sum_{k=1}^{2n} \sigma_k = 0\) says precisely that we take \(n\) steps up and \(n\) to the right; that is, we wind up at \((n, n)\). The condition that \(\sum_{k=1}^{m} \sigma_k \geq 0\) for each \(1 \leq m \leq n\) says that at any point in the path, the number of
steps that have been taken up is at least the number to the right. That is, we never go below the diagonal.

Next we provide a bijection between the first and second sets (which will finish the problem, as we then automatically get one between the second and third thanks to the first paragraph). Suppose that we want to fill out the $2 \times n$ array such that the rows and columns are increasing. One can do this by first placing 1 somewhere in the array, then placing 2, then 3, etc. At step $k$, the symbol $k$ has to be placed either in the leftmost open spot of the top row or in the leftmost open spot in the bottom row. This is because if it were not placed in the leftmost open spot in one of the rows, then there would be an open spot to the left of $k$ which would have to be filled later with something larger than $k$, so the row in which $k$ lies would not be increasing. So the only question at each step is whether $k$ goes in the top or bottom row. The bijection between the first and second sets is then defined as follows. If $\sigma_k = 1$, then we place $k$ in the leftmost open spot in the top row, and if $\sigma_k = -1$, then we place it in the leftmost open spot in the bottom row. The fact that $\sum_{k=1}^{2n} \sigma_k = 0$ corresponds to the fact that there are $n$ symbols in the top row and $n$ in the bottom. The fact that $\sum_{k=1}^{m} \sigma_k \geq 0$ for all $1 \leq m \leq n$ corresponds to the fact that the number of entries in the top row is always at least as large as the number in the bottom row, which ensures that if $k$ gets put in the bottom row, then the spot just above it is already filled (with a necessarily smaller entry), hence the column containing $k$ is increasing. So all of the columns are increasing.

These are the Catalan numbers. The easiest way to see this depends on how one defines the Catalan numbers. Let us take the common definition that $C_n$ is the number of ways of writing $n$ pairs of nested parentheses. Then the first set may be seen to be in bijection with the number of such arrangements of parentheses, by having a $+1$ correspond to a left parenthesis and a $-1$ a right parenthesis.

**Bonus problem.** Define

$$A(x) = \frac{1}{1 - (6 + \sqrt{37})x} + \frac{1}{1 - (6 - \sqrt{37})x}. $$

On the one hand, finding a common denominator, we get

$$A(x) = \frac{(1 - (6 - \sqrt{37})x) + (1 - (6 + \sqrt{37})x)}{(1 - (6 + \sqrt{37})x)(1 - (6 - \sqrt{37})x)} = \frac{2 - 12x}{1 - 12x + x^2}. $$

This is a power series with integer coefficients:

$$A(x) = (2 - 12x) \sum_{k=0}^{\infty} (12x - x^2)^k. $$
On the other hand, by expanding the original formula,

\[ A(x) = \sum_{n=0}^{\infty} ((6 + \sqrt{37})^n + (6 - \sqrt{37})^n)x^n. \]

Therefore, the 999-th coefficient, \((6 + \sqrt{37})^{999} + (6 - \sqrt{37})^{999}\), is an integer. Since \(-1/10 < 6 - \sqrt{37} < 0\), the second term is a negative number, smaller than \(1/10^{999}\) in absolute value. Therefore, the first term is an integer plus a positive number smaller than \(1/10^{999}\).