Lecture 3. LLL application: Ramsey numbers

As an application of the asymmetric LLL (Theorem 2.8) we will prove a lower bound for Ramsey numbers $R(3, l)$.

For integers $k, l \geq 2$ the Ramsey number $R(k, l)$ is the minimal positive integer $N$, such that for every edge-coloring of the complete graph $K_N$ there exists a red clique of size $k$ or a blue clique of size $l$. By induction on $k$ and $l$, one can easily prove an upper bound

$$R(k, l) \leq \binom{k + l - 2}{k - 1}.$$

We will now consider the case $k = 3$ and $l \geq 3$. Then the above inequality reads

$$R(3, l) \leq \frac{1}{2} l(l + 1).$$

We would like obtain a lower bound for $R(3, l)$ from the LLL.

In order to do so, we consider a complete graph on $n$ vertices and a random coloring of its edges with the colors red and blue: We color each edge red with probability $p$ and blue with probability $1 - p$; independently for all edges. Here the number of vertices $n$ and the probability $p$ will be specified later (depending on the value of $l$). Our goal is to obtain with positive probability a coloring without a red triangle and without a blue $l$-clique, since this would establish the lower bound $R(3, l) > n$.

For each 3-element subset $T$ of the vertex set let $A_T$ be the event that the three vertices in $T$ form a red triangle. Note that for each such $T$ we have

$$\mathbb{P}(A_T) = p^3$$

and the number of these events $A_T$ is $\binom{n}{3}$. Furthermore, for each $l$-element subset $S$ of the vertex set let $B_S$ be the event that the $l$ vertices in $S$ form a blue clique. Note that for each such $S$ we have

$$\mathbb{P}(B_S) = (1 - p)^{\binom{l}{2}}$$

and the number of these events $B_S$ is $\binom{n}{l}$.

Let us now define a dependency graph for these events. We join two events of the form $A_T$ or $B_S$, if the corresponding sets $S$ or $T$ intersect in at least 2 vertices (i.e. if they share an edge). It is clear that an event $A_T$ or $B_S$ is mutually independent of all events with which it does not share an edge. Hence the graph we just defined is indeed a dependency graph.

Let us now bound the degrees in this graph. First consider a vertex $A_T$. It is connected to at most $3n$ events $A_{T'}$. (In order to be connected, the triangles $T$ and $T'$ need to share an edge. $T$ has 3 edges and for each of them there are at most $n$ choices of a third vertex to form a triangle $T'$). Also, trivially $A_T$ is connected to at most $\binom{n}{3}$ events $B_S$. (We use a trivial bound here since the actual number of dependencies is not much less.) Now, let us consider a vertex $B_S$. It is connected to at most $\binom{l}{2}n$ events $A_T$ (In order to have a connection, the sets $S$ and $T$ need to intersect in at least 2 elements. There are $\binom{l}{2}$ choices of two elements in $S$ and for each of them at most $n$ choices for the third vertex in $T$). Also, trivially $B_S$ is connected to at most $\binom{n}{l}$ events $B_{S'}$.

So in order to apply the LLL we need to find positive real numbers $x, y \in (0, 1)$ with

$$p^3 = \mathbb{P}(A_T) \leq x(1 - x)^{3n}(1 - y)^{\binom{l}{2}}$$
and 

\[(1 - p)^{(1)} = P(B_S) \leq y(1 - x)^{(1)}n(1 - y)^{(1)}.\]

Here \(x\) and \(y\) shall be the values \(x_i\) in Theorem 2.8. Note that we choose the same value \(x\) for all events \(A_T\) (since the situation is symmetric for all choices of \(T\)) and the same value \(y\) for all events \(B_S\) (since the situation is symmetric for all choices of \(S\)).

Let us assume that for some value of \(n\) we can find positive real numbers \(p, x, y \in (0, 1)\) such that both of the above inequalities are fulfilled. Then the LLL yields that with positive probability none of the events \(A_T\) and \(B_S\) happen. This means that there is a coloring of the edges of a complete graph on \(n\) vertices such that there is no red triangle and no blue \(l\)-clique. Therefore we have \(R(3, l) > n\), if we can find \(p, x, y \in (0, 1)\) as above.

Let us now try to find such \(p, x, y \in (0, 1)\) for \(n\) as large as we can. We guess that \(y = \frac{1}{(\frac{1}{l})}\) would be a smart choice, because then the term \((1 - y)^{(1)}\) is roughly constant (around \(e^{-1}\)). Furthermore, we observe that \(p\) and \(x\) need to fulfill the following inequalities:

\[p^3 \leq x(1 - x)^{3n}(1 - y)^{(1)} \leq x\]

and

\[e^{-p^{(1)}} \approx (1 - p)^{(1)} \leq y(1 - x)^{(1)}n(1 - y)^{(1)} \leq (1 - x)^{(1)}n \approx e^{-x(1)}.\]

Hence we need \(p \geq xn \geq p^3n\) and therefore \(p \leq \frac{1}{\sqrt{n}}\). Finally, to guess the dependence of \(n\) on \(l\) we note

\[e^{-p^{(1)}} \approx (1 - p)^{(1)} \leq y(1 - x)^{(1)}n(1 - y)^{(1)} \leq y = \frac{1}{(\frac{1}{l})} \approx e^{-l \log n},\]

hence \(pl^2 \geq p^{(1)} \geq l \log n\) and therefore \(l \geq \frac{1}{p} \log n \geq \sqrt{n} \log n\).

Motivated by this we assume \(l \geq 20\sqrt{n} \log n\) and choose \(y = \frac{1}{(\frac{1}{l})}, x = \frac{1}{(2n)^{3/2}}\) and \(p = \frac{1}{(3 \sqrt{n})}\) (the constants here are not really important, they are just chosen in such a way that the inequalities work out). Then we have

\[(1 - y)^{(1)} = \left(1 - \frac{1}{(\frac{1}{l})}\right)^{(1)} \geq e^{-1.01}\]

if \(n\) is sufficiently large (since \((1 - \frac{1}{m})^m \to e^{-1}\) as \(m \to \infty\)). Furthermore for sufficiently large \(n\) we also have

\[(1 - x)^{3n} = \left(1 - \frac{1}{(9n)^{3/2}}\right)^{3n} \geq 1 - \frac{1}{3 \sqrt{n}} \geq e^{-0.01}\]

for sufficiently large \(n\). Thus,

\[p^3 = \frac{1}{(27n)^{3/2}} \leq \frac{1}{(9n)^{3/2}} \cdot e^{-1.02} \leq x(1 - x)^{3n}(1 - y)^{(1)},\]

which establishes the first desired inequality.

For the second inequality note that for sufficiently small \(h > 0\) we have \(1 - h \geq e^{-2h}\). So for sufficiently large \(n\) we get

\[(1 - x)^{(1)m} \geq e^{-2xn^{(1)}} = e^{-\frac{2}{3 \sqrt{n}}(\frac{1}{l})}.\]
Furthermore, using \( l \geq 20 \sqrt{n \log n} \),

\[
y = \frac{1}{n^l} \geq e^{-l \log n} \geq e^{-l/20 \sqrt{n}} \geq e^{-l(l-1)/19 \sqrt{n}} \geq e^{-l(l-1)/19 \sqrt{n} + 1.01} = e^{-\frac{1}{9 \sqrt{n}} \binom{l}{2} + 1.01}.
\]

Hence

\[
(1 - p) \binom{l}{2} \leq e^{-\frac{1}{9 \sqrt{n}} \binom{l}{2}} = e^{-\frac{1}{9 \sqrt{n}} \binom{l}{2} + 1.01} e^{-\frac{2}{9 \sqrt{n}} \binom{l}{2} e^{-1.01}} \leq y(1 - x)^{\binom{l}{2}}(1 - y)^{\binom{l}{2}},
\]

which verifies the second desired inequality.

Thus, for \( l \geq 20 \sqrt{n \log n} \) we can find \( p, x, y \in (0, 1) \) with the two desired inequalities. This implies (by the LLL, as described above) that \( R(3, l) > n \) whenever \( l \geq 20 \sqrt{n \log n} \) and \( n \) sufficiently large.

Note that \( n \leq \frac{l^2}{(40 \log l)^2} \) implies

\[
20 \sqrt{n \log n} \leq 20 \frac{l}{40 \log l} \log(l^2) = l.
\]

Therefore we have \( R(3, l) \geq \frac{l^2}{(40 \log l)^2} \) for all sufficiently large \( l \). This proves the following theorem.

**Theorem 3.1** There is a positive constant \( c > 0 \), such that \( R(3, l) \geq c \frac{l^2}{\log^2 l} \) for all \( l \geq 3 \).

So we have proved \( c \frac{l^2}{\log^2 l} \leq R(3, l) \leq \frac{1}{2} l(l + 1) \). We conclude this section by remarking that the true answer is \( R(3, l) = \Theta \left( \frac{l^2}{\log l} \right) \). (The upper bound was proved by Ajtai, Komlós, Szemerédi and the lower bound by Jeong Han Kim.)