Lecture 6. Symmetric Shearer’s Lemma

Here we discuss a corollary of Shearer’s Lemma that considers the symmetric case, in which all events are given the same probability bound.

**Theorem 6.1 (Symmetric Shearer’s Lemma)** Suppose there is a collection of events \( \{E_i\}_{i=1}^n \) such that each \( E_i \) is independent of all but \( d \) other events \( (d \geq 2) \), and

\[
\mathbb{P}(E_i) \leq \frac{(d-1)^{d-1}}{d^d} =: p_{\text{Shearer}} \quad \forall \, i = 1, 2, \ldots, n.
\] (6.1)

Then

\[
\mathbb{P}\left( \bigcap_{i=1}^n \overline{E_i} \right) > 0.
\]

**Proof:** Let \( G \) be a dependency graph for \( \{E_i\} \) with maximum degree \( d \), and let \( p = (d-1)^{d-1}/d^d \).

We may assume that \( G \) is connected, since otherwise the problem reduces to a collection of independent problems. For \( G \) connected, we can find an ordering of vertices \( (v_1, \ldots, v_n) \) such that each \( v_i \), \( i \geq 2 \), has degree at most \( d-1 \) among \( \{v_i, \ldots, v_n\} \). (However, this is not possible to arrange for \( v_1 \) if \( G \) is \( d \)-regular. Therefore, we need to handle this case separately later.)

By induction on \( |S| \) we claim that

\[
\frac{\tilde{q}_S}{\tilde{q}_{S-a}} > 1 - \frac{1}{d} \quad \text{for } a \in S \text{ where } |S \cap \Gamma(a)| \leq d - 1.
\]

The base case of the induction is satisfied as

\[
\frac{\tilde{q}_a}{\tilde{q}_\emptyset} = \frac{\tilde{q}_a}{1} = 1 - p = 1 - \frac{(d-1)^{d-1}}{d^d} > 1 - \frac{d^{d-1}}{d^d} = 1 - \frac{1}{d^2}.
\]

For the general case, we will use an identity established in the proof of the asymmetric case (see Lecture 5):

\[
\frac{\tilde{q}_S}{\tilde{q}_{S-a}} = 1 - p \cdot \frac{\tilde{q}_{S \setminus \Gamma^+(a)}}{\tilde{q}_{S-a}}. \tag{6.2}
\]

Assume that \( a \in S \) is such that \( |S \cap \Gamma(a)| \leq d - 1 \), and write \( S \cap \Gamma^+(a) = \{a, a_1, a_2, \ldots, a_k\} \), \( k \leq d - 1 \). Since each \( a_i \) has degree at most \( d - 1 \) inside \( S \setminus \{a, \ldots, a_{i-1}\} \), the inductive hypothesis gives

\[
\frac{\tilde{q}_{S \setminus \Gamma^+(a)}}{\tilde{q}_{S-a}} = \frac{\tilde{q}_{S \setminus \Gamma^+(a)}}{\tilde{q}_{(S \setminus \Gamma^+(a)) + a_k}} \cdot \frac{\tilde{q}_{(S \setminus \Gamma^+(a)) + a_k + a_{k-1}}}{\tilde{q}_{S-a}} < \frac{1}{(1 - 1/d)^{d-1}}.
\]

at most \( d - 1 \) terms

\[
\frac{\tilde{q}_{S-a}}{\tilde{q}_{S-a}} < \frac{1}{(1 - 1/d)^{d-1}}.
\]

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\]
From (6.2), we get
\[
\frac{\tilde{q}_S}{\tilde{q}_{S-a}} = 1 - p \cdot \frac{\tilde{q}_{S\setminus \Gamma(a)}}{\tilde{q}_{S-a}} > 1 - p \cdot \frac{d^{d-1}}{(d-1)^{d-1}} = 1 - \frac{1}{d}
\]
which finishes the inductive claim.

Finally let us handle the case where \( S = \{v_1, \ldots, v_n\} \) and \( v_1 \) has degree \( d \). We can still use (6.2), but now the telescoping product in (6.3) may involve \( d \) terms, giving
\[
\frac{\tilde{q}_{[n]}}{\tilde{q}_{[n]-v_1}} = 1 - p \cdot \frac{\tilde{q}_{[n] \setminus \Gamma(v_1)}}{\tilde{q}_{[n]-v_1}} > 1 - p \cdot \frac{d^d}{(d-1)^d} = 1 - \frac{1}{d-1}.
\]
We note that \( 1 - \frac{1}{d-1} \) could be 0 (for \( d = 2 \)) but the strict inequality ensures that the ratio is still positive. We conclude that
\[
P\left( \bigcap_{i=1}^n E_i \right) \geq \frac{\tilde{q}_{[n]}}{\tilde{q}_{[n]-a_1}} \cdot \frac{\tilde{q}_{[n]-v_1}}{\tilde{q}_{[n]-v_2}} \cdots \frac{\tilde{q}_{v_n}}{\tilde{q}_{\emptyset}} > \left( 1 - \frac{1}{d-1} \right)^{n-1} \geq 0,
\]
completing the proof. \( \square \)

Let us compare Symmetric Shearer’s Lemma to the Lovász Local Lemma. In the LLL, assuming that all events get the same parameter \( x \), it is required that
\[
p \leq x(1 - x)^d \quad \text{(6.4)}
\]
for some \( x \in (0,1) \). The optimal choice here can be shown to be \( x = \frac{1}{d+1} \), which gives
\[
p \leq \frac{d^d}{(d+1)^{d+1}} =: p_{\text{LLL}}. \quad \text{(6.5)}
\]
Comparing (6.5) to (6.1), we see that the threshold probability in Shearer’s lemma, \( p_{\text{Shearer}} := \frac{(d-1)^{d-1}}{d^d} \), has the benefit of 1 additional dependency over the LLL. Further, the inequalities
\[
\frac{(d+1)^d}{d^d} < e < \frac{d^d}{(d-1)^d}
\]
show that
\[
\frac{1}{e(d+1)} < p_{\text{LLL}} < \frac{1}{ed} < p_{\text{Shearer}} < \frac{1}{e(d-1)}.
\]
Of course, as \( d \) grows large, \( p_{\text{LLL}} \) and \( p_{\text{Shearer}} \) are asymptotically the same.

### 6.1 Worst instance: \( d \)-regular trees

We would like to demonstrate that \( p_{\text{Shearer}} \) is optimal, in the sense that Theorem 6.1 fails if \( p_{\text{Shearer}} \) is taken any larger. The extreme case is when each \( E_i \) is dependent on exactly \( d \) other events and moreover the dependency graph is a (large) \( d \)-regular tree. Begin with a root vertex \( r \), by itself called \( T_0 \). A root with \( d-1 \) children is called \( T_1 \). Constructed recursively, \( T_\ell \) is the tree obtained by taking a root with \( d-1 \) subtrees, each of which is \( T_{\ell-1} \). Note that all vertices in levels 1 through
Figure 1: A binary tree \((d = 3)\). Here, \(T_3\) is shown (levels 0 through 3).

\(\ell - 1\) have degree \(d\); the root has degree \(d - 1\), and level \(\ell\) consists of leaves. We call this a \(d\)-regular tree of depth \(\ell\) (ignoring the slightly different degree at the root). For consistency, we also define \(T_{-1}\) to be the empty tree.

Suppose that the probability of each event is \(p\). From (6.2), we have

\[
\hat{q}_{T_\ell} = \hat{q}_{T_\ell} \setminus r - p \cdot \hat{q}_{T_\ell} \setminus \Gamma^+(r).
\]

But \(T_\ell \setminus r\) is the union of \(d - 1\) disjoint copies of \(T_{\ell - 1}\). Similarly, \(T_\ell \setminus \Gamma^+(r)\) is the union of \((d - 1)^2\) disjoint copies of \(T_{\ell - 2}\). Hence

\[
\hat{q}_{T_\ell} = (\hat{q}_{T_{\ell - 1}})^{d - 1} - p (\hat{q}_{T_{\ell - 2}})^{(d - 1)^2}.
\]

Let us define

\[
b_\ell := \frac{\hat{q}_{T_\ell}}{(\hat{q}_{T_{\ell - 1}})^{d - 1}} = 1 - p \left( \frac{(\hat{q}_{T_{\ell - 2}})^{d - 1}}{\hat{q}_{T_{\ell - 1}}} \right)^{d - 1}.
\]

That is,

\[
b_\ell = 1 - p \left( \frac{1}{b_{\ell - 1}} \right)^{d - 1}.
\]

If Shearer’s positivity conditions are satisfied for an arbitrarily large \(d\)-regular tree, then \(b_\ell > 0\) for all \(\ell \geq 0\), and also the sequence is decreasing by induction: \(b_0 = 1 - p\), \(b_1 = 1 - \frac{p}{(1 - p)^{d - 1}} \leq b_0\), and if \(b_\ell \leq b_{\ell - 1}\), then \(b_{\ell + 1} = 1 - p/b_{\ell - 1}^{d - 1} \leq 1 - p/b_{\ell - 1}^{d - 1} = b_\ell\). Hence there is a limit,

\[
\lambda := \lim_{\ell \to \infty} b_\ell,
\]

which must satisfy \(\lambda = 1 - \frac{p}{\lambda^{d - 1}}\), and hence \(p = \lambda^{d - 1} - \lambda^d\). The maximum is attained at \(\lambda = \frac{d - 1}{d}\), which gives

\[
p \leq \left( \frac{d - 1}{d} \right)^{d - 1} - \left( \frac{d - 1}{d} \right)^d = \frac{1}{d} \left( \frac{d - 1}{d} \right)^{d - 1} = p_{\text{Shearer}}.
\]

Indeed, \(p_{\text{Shearer}}\) is optimal.

### 6.2 Application of Shearer’s Lemma: the multipartite Turán problem

Consider an \(r\)-partite graph \(G\) on \(V_1 \cup V_2 \cup \cdots \cup V_r\). Suppose we have at least a certain density \(\rho\) between any two parts:

\[
e(V_i, V_j) \geq \rho |V_i||V_j| \quad \forall i \neq j.
\]
How large must $\rho$ be to guarantee the existence of a clique $K_r$ in $G$? More generally, given a graph $H$ on $r$ vertices, assume

$$\{i, j\} \in E(H) \Rightarrow e(V_i, V_j) \geq \rho |V_i||V_j|.$$ 

How large must $\rho$ be to guarantee the existence of a copy of $H$ in $G$? Following (Csikvári and Nagy, 2012), we show how to apply Shearer’s Lemma.

Pick $x_i \in V_i$ independently and uniformly at random. For each $(i, j) \in E(H)$, define an event $E_{ij} = \{\{x_i, x_j\} \notin E(G)\}$, so that if all $E_{ij}$ are avoided, then a copy of $H$ is present. Note that the probability of each event is at most $1 - \rho$ by assumption. A dependency graph for the events $E_{ij}$ is the line graph of $H$, which we call $D$: The vertices of $D$ are the edges in $H$, and two of these vertices are adjacent if and only if the corresponding edges in $H$ share a vertex. So independent sets in $D$ are exactly matchings in $H$.

First, consider Symmetric Shearer’s Lemma. The degrees in $D$ are at most $2(\Delta(H) - 1)$ where $\Delta(H)$ is the maximum degree in $H$. Hence, if the probability of each event is at most $\frac{1}{2e(\Delta(H) - 1)}$, then by Theorem 6.1 $P[\bigcap_{(i,j) \in E(H)} E_{ij}] > 0$. Equivalently, if $\rho \geq 1 - \frac{1}{2e(\Delta(H) - 1)}$ then $G$ contains a copy of $H$.

This problem is actually a rare setting where we can apply Shearer’s Lemma directly and obtain a stronger result. Consider the polynomial

$$q_\emptyset(p) = \sum_{I \in \text{Ind}(D)} (-1)^{|I|} p^I = \sum_{M \subset H \text{ matching}} (-1)^{|M|} p^{|M|}.$$ 

This last sum is a variant of the matching polynomial of $K_r$. It is most commonly defined in the following form, which we refer to as the matching defect polynomial:

$$M_H(x) = \sum_{M \subset H \text{ matching}} (-1)^{|M|} x^{r-2|M|}.$$ 

(Recall that $r = |V(H)|$.) A simple calculation gives

$$M_H(x) = x^r q_\emptyset \left( \frac{1}{x^2} \right).$$ 

It is useful in this setting to consider Property 4 of Shearer’s Lemma, stated in Lecture 5. In particular, we ask, for which $p$ is it true that

$$q_\emptyset(\lambda p) > 0 \quad \forall \lambda \in [0, 1]?$$

To answer this question, it suffices to locate the minimum positive root of $q_\emptyset$, or equivalently the maximum positive root of $M_H$. Here we appeal to the following theorem (which we will prove later in this course).

**Theorem 6.2 (Heilmann-Lieb)** For any graph $H$, the roots of the matching defect polynomial are all real and the maximum root is at most $2\sqrt{\Delta(H) - 1}$. 

It follows that the minimum positive root of $q_0(p)$ for $H$ is at least $\frac{1}{4(\Delta(H) - 1)}$. Consequently, if $\rho \geq 1 - \frac{1}{4(\Delta(H) - 1)}$ then $G$ contains a copy of $H$, which improves the bound from above ($2e$ has been improved to 4).

For $H = K_r$, which is perhaps the most interesting special case here, we obtain that density $\rho \geq 1 - \frac{1}{4(r-2)}$ is sufficient to guarantee a copy of $K_r$. In fact, here we can go one step further and obtain a slightly tighter bound. $M_{K_r}$ is known to be the Hermite polynomial of degree $r$. Recall that the Hermite polynomials are defined recursively, corresponding to the recursion in the context of matchings:

$$H_0(x) = 1, \quad H_{r+1}(x) = xH_r(x) - rH_{r-1}(x).$$

For this special case, more accurate bounds are known. In particular, the maximum root of $M_{K_r}$ is known to be $2\sqrt{r} - \Theta(r^{-1/6})$. Hence, the minimum positive root of $q_0$ is

$$\frac{1}{(2\sqrt{r} - \Theta(r^{-1/6}))^2} = \frac{1}{4r - \Theta(r^{1/3})}.$$ 

Consequently, if $\rho \geq 1 - \frac{1}{4(r-2)}$ then $G$ contains a copy of $K_r$, a slight improvement over the bound of $1 - \frac{1}{4(r-2)}$ from the Heilmann-Lieb theorem.

We conclude by mentioning that it is easy to construct an $r$-partite graph of density $\rho = 1 - \frac{1}{r-1}$ which does not contain a $K_r$ (an exercise). A better counterexample which can be found in [Csikvári-Nagy’12] implies that $\rho = 1 - \frac{1}{(2+o(1))r}$ is not sufficient to guarantee a copy of $K_r$. The gap between $1 - \frac{1}{(2+o(1))r}$ and $1 - \frac{1}{(4-o(1))r}$ remains open.

References