Lecture 12. The Heilmann-Lieb Theorem

Recall that earlier in the course, we appealed to the matching polynomial and an upper bound on its (real) roots. Let us remind ourselves of the definition.

**Definition 12.1** For a graph $G = (V,E)$ with edge weights $w_e$, the matching defect polynomial is

$$
M_G(x) = \sum_{\text{matching } M} (-1)^{|M|} x^{n-2|V|} \prod_{e \in M} w_e.
$$

**Theorem 12.2 (The Heilmann-Leib Theorem)** The matching defect polynomial for a graph $G$ with edge weights $w_{uv} > 0$ satisfies:

- $M_G$ is real-rooted.
- For a connected graph $G$ with $|V(G)| \geq 3$: Let $W_u = \sum_{v \in \Gamma(u)} w_{uv} - \min_{u' \in \Gamma(u)} w_{u'u}$ and $B = \max_{u \in V} W_u$. Then the maximum root of $M_G$ is at most $2\sqrt{B}$.

Note that the result does not hold when $G$ is a single edge, since then $M_G(x) = x^2 - w$ has a positive root while $W_u = W_v = 0$. Since the matching polynomial of a disconnected graph is a product of the matching polynomials of the connected components, the theorem also holds for graphs $G$ that do not contain isolated edges.

First, we prove the first bullet point for $G$ complete, $|V(G)| \geq 1$, and $w_{uv} > 0 \ \forall u \neq v$.

**Proof:** We have the following recursion: $M_\emptyset(x) = 1$, and

$$
M_G(x) = xM_{G\setminus\{u\}}(x) - \sum_{v \in V \setminus \{u\}} w_{uv}M_{G\setminus\{u,v\}}(x).
$$

(The first term corresponds to matchings avoiding $u$ and the second corresponds to matchings covering $u$.)

We prove by induction on $|V(G)|$ that: (*)

$M_G$ is real-rooted, with distinct simple roots and for all $u \in V$, $M_{G\setminus\{u\}}$ strictly interlaces $M_G$.

For the base case: $G = \{u\}$, $M_G(x) = x$ and $M_{G\setminus\{u\}}(x) = M_\emptyset(x) = 1$ so (*) holds.

Consider $|V(G)| = n$ assume that (*) holds for $|V(G')| \leq n - 1$. Let $\lambda_{n-1} < \lambda_{n-2} < \ldots < \lambda_1$ be the roots of $M_{G\setminus\{u\}}$ (real and distinct by the inductive hypothesis). We have

$$
M_G(\lambda_i) = \lambda_i M_{G\setminus\{u\}}(\lambda_i) - \sum_{v \neq u} w_{uv}M_{G\setminus\{u,v\}}(\lambda_i) = -\sum_{v \neq u} w_{uv}M_{G\setminus\{u,v\}}(\lambda_i).
$$

(12.1)
By the inductive hypothesis, $M_{G\setminus\{u,v\}}$ strictly interlaces $M_{G\setminus\{u\}}$. This means that $M_{G\setminus\{u,v\}}(\lambda_i)$ alternates signs for $i = 1, 2, 3, \ldots$ (as between two consecutive values of $\lambda_i$ there is exactly one simple root of $M_{G\setminus\{u,v\}}$). Moreover, there is no root of $M_{G\setminus\{u,v\}}$ greater than $\lambda_1$ and $M_{G\setminus\{u,v\}}$ has a positive highest coefficient, so $M_{G\setminus\{u,v\}}(\lambda_1) > 0$. This implies that $(-1)^{i-1}M_{G\setminus\{u,v\}}(\lambda_i) > 0$.

By (12.1), as $w_{uv} > 0 \forall u \neq v$, we have $(-1)^{n-1}M_G(\lambda_i) > 0$. Thus, $M_G$ has a root in each interval $(\lambda_{i-1}, \lambda_i)$ for $i = 1, 2, ..., n$. Moreover, $M_G(\lambda_1) < 0$ and $\lim_{x \to \infty} M_G(x) = \infty$ so $M_G$ has a root larger than $\lambda_1$.

Similarly $(-1)^{n-1}M_G(\lambda_{n-1}) > 0$ while $\lim_{x \to -\infty} M_G(x) = (-1)^{n} \infty$ since $\deg M_G = n$, we see that there is a root of $M_G$ below $\lambda_{n-1}$. Hence, as $M_G$ has at most $n$ roots, it has exactly $n$ roots interlacing $\{\lambda_{n-1}, ..., \lambda_1\}$.

Note that non-edges in $G$ can be thought of as zero weights $w_{uv} = 0$. Hence it is enough to consider the complete graph with weights $w_{uv} \geq 0$. Our approach is to take a limit of graphs with $w'_{uv} > 0$ and apply the above result. We use the following general fact.

**Lemma 12.3** For a sequence of polynomials $f_1(z), f_2(z), ...$ in complex variable $z$, suppose that the degrees of $f_m$ are uniformly bounded, $\Omega \subset \mathbb{C}$ is an open set, and $f_m \to f \in \mathbb{C}[z]$ coefficient-wise.

If $f_m$ has no roots in $\Omega$ for all $m$ then either $f$ has no root in $\Omega$ or $f \equiv 0$.

**Proof:** Assume that $f \not\equiv 0$ but $f$ has a root $z_0 \in \Omega$. Choose $\rho > 0$ so that

$$B_\rho(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq \rho\} \subset \Omega$$

and there is no other root of $f$ in $\overline{B}_\rho(z_0)$. (This can be done since $f$ has finitely many roots.)

Since $f_m \to f$ pointwise and $\overline{B}_\rho(z_0)$ is compact, $f_m \to f$ uniformly on $\overline{B}_\rho(z_0)$.

Let $c = \min_{z \in \partial \overline{B}_\rho(z_0)} |f(z)|$. We have $c > 0$ because $f$ has no roots in $\overline{B}_\rho(z_0)$. Take $n_0$ such that for all $n \geq n_0$, and for all $z \in \partial \overline{B}_\rho(z_0)$, we have $|f_m(z)| \geq \frac{c}{2}$. This can be done since $f_m \to f$ uniformly on $\partial \overline{B}_\rho(z_0)$. Since $f_m \to f$ coefficient-wise, we also have $f'_m \to f'$ coefficient-wise (note that the degree of the polynomials is uniformly bounded). Hence, $f'_m \to f'$ uniformly on $\partial \overline{B}_\rho(z_0)$.

As $|f_m(z)| \geq \frac{c}{2} > 0$ for $z \in \partial \overline{B}_\rho(z_0)$, we have $\int_{\partial \overline{B}_\rho(z_0)} \frac{f'_m(z)}{f_m(z)}dz \to \int_{\partial \overline{B}_\rho(z_0)} \frac{f'(z)}{f(z)}dz$ uniformly. Hence,

$$\int_{\partial \overline{B}_\rho(z_0)} \frac{f'_m(z)}{f_m(z)}dz \to \int_{\partial \overline{B}_\rho(z_0)} \frac{f'(z)}{f(z)}dz. \quad (12.2)$$

Let $f(z) = \prod_{i=1}^n (z - \lambda_i)$. Then $\frac{f'(z)}{f(z)} = \sum_{i=1}^n \frac{1}{z - \lambda_i}$. Hence, $\int_{\partial \overline{B}_\rho(z_0)} \frac{f'(z)}{f(z)}dz = 2m\pi i$ where $m$ is the multiplicity of root $z_0$ which is nonzero. On the other hand, for every $n$, we have $\int_{\partial \overline{B}_\rho(z_0)} \frac{f'_m(z)}{f_m(z)}dz = 0$ since there is no root of $f_m$ in $\overline{B}_\rho(z_0) \subset \Omega$ so $\frac{f'_m(z)}{f_m(z)}$ is holomorphic in a neighborhood of $\overline{B}_\rho(z_0)$. This contradicts (12.2).\[\square\]

Now let us finish the proof of Theorem 12.2.

**Proof:** Given $w_{uv} \geq 0$, we can take graphs $G^{(m)}$ with weights $w_{uv}^{(m)} = w_{uv}$ if $w_{uv} > 0$ and $w_{uv}^{(m)} = \frac{1}{m}$ if $w_{uv} = 0$. The sequence $M_{G^{(m)}}(x)$ satisfies the conditions in the theorem above for $\Omega = \mathbb{C}\setminus \mathbb{R}$, by the proof above for strictly positive weights. By Lemma 12.3, $M_G(x)$ has no zeros in $\mathbb{C}\setminus \mathbb{R}$ (since it is not identically 0) which means it is a real-rooted polynomial.
Now to the second bullet point. Let us assume now that $G$ is not necessarily complete and all edges have positive weights. We prove by induction on $|V(G')|$ the following claim (with no assumption on connectedness and size of $G$):

For every proper induced subgraph $G' \subseteq G$, and $u \in V(G')$ which has a neighbor in $V(G) \setminus V(G')$, for all $x > 2\sqrt{B}$, $\mathcal{M}_{G'}(x) > 0$ and $\frac{\mathcal{M}_{G'}(x)}{\mathcal{M}_{G' \setminus \{u\}}(x)} \geq \sqrt{B}$. (**)

For the base case, $G' = \{u\}$. We have $\mathcal{M}_{G'}(x) = x, \mathcal{M}_{\emptyset}(x) = 1$ and $\frac{\mathcal{M}_{G'}(x)}{\mathcal{M}_{\emptyset}(x)} = x \geq \sqrt{B}$ for all $x > 2\sqrt{B}$.

For the inductive step, assume that $|V(G')| = k$, $u \in V(G')$ has a neighbor $z$ outside of $G'$, and (** is true for all subgraphs of at most $k - 1$ vertices. We have

$$\mathcal{M}_{G'}(x) = x\mathcal{M}_{G' \setminus \{u\}}(x) - \sum_{v \in V(G') \setminus \{u\}} w_{uv}\mathcal{M}_{G' \setminus \{u,v\}}(x).$$

Hence,

$$\frac{\mathcal{M}_{G'}(x)}{\mathcal{M}_{G' \setminus \{u\}}(x)} = x - \sum_{v \in \Gamma(u) \cap G'} w_{uv} \frac{\mathcal{M}_{G' \setminus \{u,v\}}(x)}{\mathcal{M}_{G' \setminus \{u\}}(x)}.$$

We apply the inductive hypothesis to $G' \setminus u$ and each $v \in \Gamma(u) \cap G'$. We have $\frac{\mathcal{M}_{G' \setminus \{u\}}(x)}{\mathcal{M}_{G' \setminus \{u,v\}}(x)} \geq \sqrt{B}$ for all $x \geq 2\sqrt{B}$. Hence,

$$\frac{\mathcal{M}_{G'}(x)}{\mathcal{M}_{G' \setminus \{u\}}(x)} \geq x - \sum_{v \in \Gamma(u) \cap G'} \frac{1}{\sqrt{B}} w_{uv} \geq x - \left( \sum_{v \in \Gamma(u) \cap G'} w_{uv} - w_{uz} \right) \cdot \frac{1}{\sqrt{B}} \geq x - \frac{W_u}{\sqrt{B}} \geq x - \frac{B}{\sqrt{B}} \geq 2\sqrt{B} - \sqrt{B} = \sqrt{B}$$

where $z$ is a neighbor of $u$ in $G \setminus G'$. Hence we also have $\mathcal{M}_{G'}(x) > 0$.

To finish the proof, for $G$ connected with at least 3 vertices, pick a vertex $u$ with $\deg u \geq 2$. By the above, $\mathcal{M}_{G \setminus \{u\}}(x) > 0$ for $x > 2\sqrt{B}$. Also, we have $B \geq W_u = \sum_{v \in \Gamma(u)} w_{uv} - \min w_{uv} \geq \frac{1}{2} \sum_{v \in \Gamma(u)} w_{uv}$. Hence $\frac{\mathcal{M}_{G}(x)}{\mathcal{M}_{G \setminus \{u\}}(x)} \geq x - \sum_{v \in \Gamma(u)} w_{uv} \cdot \frac{1}{\sqrt{B}} > 2\sqrt{B} - \frac{2B}{\sqrt{B}} = 0$ and $\mathcal{M}_{G}(x) > 0$. \qed