Lecture 20. The Kadison-Singer problem, part II

Our goal in this lecture is to prove the following theorem, which will complete the solution of the Kadison-Singer problem.

**Theorem 1.** For $A_i \succeq 0$, $Tr (A_i) \leq \alpha$, $\sum_{i=1}^n A_i = I$, the maximum root of the (real-rooted) polynomial

$$\chi(x) = \left(1 - \frac{\partial}{\partial z_m}\right) \ldots \left(1 - \frac{\partial}{\partial z_1}\right) \left(\det \left(\sum_{i=1}^m z_i A_i\right)\right) \bigg|_{z_i=x, \forall i}$$

is at most $(1 + \sqrt{\alpha})^2$.

Let’s first introduce some notation. For a stable polynomial $Q(z_1, \ldots, z_m)$, the barrier function in direction $j$ is

$$\phi^j_Q(z) = \frac{\partial}{\partial z_j} \log (Q(z)).$$

For $i, j \in [n]$, freeze $z_k$ for $k \neq i, j$ and regard $\phi^j_Q(z)$ as a polynomial in $z_i$, for which we denote $\lambda_1, \ldots, \lambda_n$ the roots in $z_i$. Then we can write

$$\phi^j_Q(z) = \sum_{l=1}^n \frac{1}{z_j - \lambda_l}.$$

We first show the following lemma. Recall that $z$ being above the roots of $Q$ means that $Q(z+t) > 0$ for any $t \geq 0$. We follow a simplified proof from Terrence Tao’s blog.

**Lemma 2.** For any stable polynomial $Q$, if $z$ is above the roots of $Q$, then $\forall i, j \in [n]$, $\phi^j_Q(z)$ is monotone decreasing and convex in $z_i$.

**Proof.** We can assume that $Q$ is a monic polynomial. (The leading coefficient must be positive and hence we can do this by scaling.) Denote $\phi^j_Q(x_i, x_j)$ to be the polynomial on $\mathbb{R}^2$ where the other variables ($z_\ell : \ell \notin \{i,j\}$) are frozen in $\mathbb{R}$. We claim that for any positive integer $k$,

$$(-1)^k \frac{\partial^k}{\partial x_j^k} \phi^j_Q(x_i, x_j) \geq 0.$$

To see this, (1) when $i = j$, it is directly from the expression $\phi^i_Q(x_i) = \sum_{l=1}^n \frac{1}{x_i - \lambda_l}$. (2) when $i \neq j$, for any fixed $x_i$, regarding $Q$ as a polynomial in $x_j$, we have real roots $\lambda_l(x_i)$ for $l \in [n]$. We can write

$$Q(x_i, x_j) = \prod_{l=1}^n (x_j - \lambda_l(x_i)).$$
Since $\phi_Q^i(x_i, x_j) = \frac{\partial}{\partial x_i} \log (Q(x_i, x_j))$, we have that

$$(-1)^k \frac{\partial^k}{\partial x_i^k} \phi_Q^i(x_i, x_j) = (-1)^k \frac{\partial}{\partial x_i} \frac{\partial^k}{\partial x_j^k} \log (Q(x_i, x_j)) = (-1)^{k-1} \frac{1}{(k-1)!} \sum_{i=1}^{n} \frac{1}{(x_j - \lambda_l(x_i))^k}.$$

Note that $\lambda_l(x_i)$ is continuous in $x_i$. It is known that it is also differentiable as a complex function in $x_i$, except for points of measure 0 (we will accept this claim without proof). We next claim that $\lambda_l(x_i)$ is non-increasing. To see this, if it is increasing, at some point the derivative is positive, let’s say at $x_i'$. Then for a complex $z_i$ close to $x_i'$ with $Im (z_i) > 0$, we have $Im (\lambda_l(z_i)) > 0$, and $Q(z_i, \lambda_l(z_i)) = 0$. It is impossible since $z_i, \lambda_l(z_i)$ are both on the upper half plane, contradicting the fact that $Q$ is stable. Therefore, $\lambda_l(x_i)$ is non-increasing, and

$$(-1)^k \frac{\partial}{\partial x_i} \frac{\partial^k}{\partial x_j^k} \log (Q(x_i, x_j)) \geq 0.$$

\[\square\]

Next we show the following lemma.

**Lemma 3.** If $Q$ is stable and $z \in R^n$ is above its roots, with $\phi_Q^i(z) < 1$, then $z$ is above the roots of $(1 - \frac{\partial}{\partial z_i})Q$.

**Proof.** For any $t \geq 0$, we have $\phi_Q^i(t + z) < 1$ by monotonicity. $\phi_Q^i(z) < 1$ implies that $Q'_{z_i}(z)/Q(z) < 1$. Since $Q(z) > 0$, we have that $Q'_{z_i}(z) < Q(z)$, which means that

$$(1 - \frac{\partial}{\partial z_i})Q(z) > 0.$$

\[\square\]

**Lemma 4.** If $Q$ is stable, $z$ is above its roots, with $\phi_Q^i(z) \leq 1 - \frac{1}{\delta}$ for $\delta > 1$. Then

$$\phi^i_{(1 - \frac{\partial}{\partial z_j})Q} (z + \delta e_j) \leq \phi^i_Q (z).$$

**Proof.** We have that

$$\phi^i_{(1 - \frac{\partial}{\partial z_j})Q} = \frac{\partial_i}{Q - Q'_{z_j}} (Q - Q'_{z_j}) = \frac{\partial_i Q}{Q} + \frac{\partial_i (1 - \phi^i_Q)}{1 - \phi^i_Q} = \phi^i_Q - \frac{\partial_j \phi^i_Q}{1 - \phi^i_Q},$$

where we write $\partial_i$ for $\frac{\partial}{\partial z_i}$ for short. Note that

$$\partial_j \phi^i_Q (z + \delta) \geq \partial_j \phi^i_Q (z_j) \geq \frac{\phi^i_Q (z_j + \delta) - \phi^i_Q (z_j)}{\delta}.$$

By (1) we have that

$$\phi^i_{(1 - \frac{\partial}{\partial z_j})Q} (z_j + \delta) = \phi^i_Q (z_j + \delta) - \frac{\partial_j \phi^i_Q (z_j + \delta)}{1 - \phi^i_Q (z_j + \delta)}.$$

(2)
By our condition we have that
\[ 1 - \phi^j_Q (z_j + \delta) \geq 1 - \phi^j_Q (z_j) \geq \frac{1}{\delta}, \]
and thus
\[ \frac{\partial_j \phi^j_Q (z_j + \delta)}{1 - \phi^j_Q (z_j + \delta)} \geq \delta \partial_j \phi^j_Q (z_j + \delta) \geq \phi^j_Q (z_j + \delta) - \phi^j_Q (z_j). \] (3)
Substituting (3) into (2) we see that
\[ \phi^i_{(1 - \frac{\partial}{\partial z_j})} Q (z_j + \delta) \leq \phi^i_Q (z_j). \]

Now we are ready to prove Theorem 1.

**Proof of Theorem 1** Let
\[ Q_k (z_1, \ldots, z_m) = \left( 1 - \frac{\partial}{\partial z_k} \right) \cdots \left( 1 - \frac{\partial}{\partial z_1} \right) \left[ \det \left( \sum_{i=1}^{m} z_i A_i \right) \right]. \]
We claim that barrier functions are bounded by \( 1 - \frac{1}{\delta} \) for \( \delta = 1 + \sqrt{\alpha} \), if with each \( \left( 1 - \frac{\partial}{\partial z_k} \right) \) operation, we increase \( t \) in the respective coordinate by \( \delta \). We use induction to show this. In the base case we have that
\[ Q_0 (z_1, \ldots, z_m) = \det \left( \sum_{i=1}^{m} z_i A_i \right), \]
and thus
\[ Q_0 (t, \ldots, t) = \det \left( \sum_{i=1}^{m} t A_i \right) = t^n > 0. \]
Recall that
\[ \frac{d}{dt} (\det (A + tB)) |_{t=0} = \det (A) \text{Tr} \left( A^{-1} B \right). \]
Using the above identity we see that
\[ \phi^i_{Q_0} (z_1, \ldots, z_m) = \text{Tr} \left( \left( \sum_{i=1}^{m} z_i A_i \right)^{-1} A_i \right), \]
which implies that
\[ \phi^i_{Q_0} (t, \ldots, t) = \frac{1}{t} \text{Tr} (A_i) \leq \alpha/t, \]
since we assumed that \( \text{Tr} (A_i) < \alpha \). Letting \( t = \alpha + \sqrt{\alpha} \), we further see that
\[ \phi^i_{Q_0} (t, \ldots, t) \leq 1 - \frac{1}{1 + \sqrt{\alpha}} = 1 - \frac{1}{\delta}. \]
Now assume that the claim holds for $k$. We have that

$$
\phi^i_{Q_{k+1}} \left( t + \delta, \ldots, t + \delta, t, \ldots, t \right)_{k+1}
$$

$$
= \phi^i_{(1-\partial_{k+1})Q_k} \left( t + \delta, \ldots, t + \delta, t, \ldots, t \right)_{k+1}
$$

$$
\leq \phi^i_{Q_k} \left( t + \delta, \ldots, t + \delta, t, \ldots, t \right)_{k}
$$

$$
\leq 1 - \frac{1}{\delta}.
$$

Therefore we see that $\chi(x) = Q_m(x, \ldots, x)$ has maximum root at most $t + \delta = (1 + \sqrt{\alpha})^2$. 
