Please try to solve the homework on your own. Discussions are okay but make your own effort.

Problem 1. (a) Prove by induction that the number of trees on vertices \( \{1, 2, \ldots, t\} \) with degrees \( d_1, \ldots, d_t \geq 1 \) such that \( \sum_{i=1}^{t} d_i = 2(t-1) \) is the multinomial coefficient

\[
\binom{t-2}{d_1-1, d_2-1, \ldots, d_t-1}.
\]

(b) Let \( F \) be a fixed forest in \( K_n \) with connected components of vertex-sizes \( f_1, f_2, \ldots, f_t \). Prove that the number of spanning trees containing \( F \) is

\[
n^{t-2} \prod_{i=1}^{t} f_i.
\]

(c) Show that this implies

\[
\Pr[A \subseteq T & B \subseteq T] = \Pr[A \subseteq T] \cdot \Pr[B \subseteq T]
\]

where \( A, B \) are fixed vertex-disjoint sets of edges in \( K_n \) and \( T \) is a uniformly random spanning tree in \( K_n \).

Problem 2. Let \( G \) be a \( d \)-regular bipartite graph of girth (minimum cycle length) at least \( g \), such that \( d \leq 2^{g/2} \), and let’s say \( g \geq 16 \) (sufficiently large constant). Prove that there is an “acyclic edge coloring” with \( 2d \) colors: incident edges get different colors and every cycle gets at least 3 different colors.

Hint: Start with the fact that every \( d \)-regular bipartite graph has an edge coloring with \( d \) colors.

Problem 3. Prove that for any fixed \( p \in (0,1)^n \),

(a) the \( \tilde{q}_S(p) \) polynomials are log-submodular in \( S \): \( \tilde{q}_{S \cup T}(p) \tilde{q}_{S \cap T}(p) \leq \tilde{q}_S(p) \tilde{q}_T(p) \).

Hint: Prove that \( \tilde{q}_S / \tilde{q}_{S-a} \geq \tilde{q}_T / \tilde{q}_{T-a} \), whenever \( a \in S \subset T \).

(b) the \( q_S(p) \) polynomials are also log-submodular: \( q_{S \cup T}(p) q_{S \cap T}(p) \leq q_S(p) q_T(p) \).

Hint: Use part (a).

Problem 4. Let \( G = (V, E) \) be a cycle on \( tn \) vertices, and \( V = V_1 \cup \ldots \cup V_n \) a partition of the vertices into (not necessarily consecutive) sets of size \( |V_i| = t \). Prove for \( t = 11 \) and every \( n \geq 1 \) that there is an independent set containing one vertex from each set \( V_i \).

Bonus question: What is the smallest \( t \) for which you can prove this?