You can discuss the problems but try to solve them on your own. If you collaborate on a problem, please acknowledge it on your homework. Don't use any books or the internet.

**Problem 1.** Prove that the edges of a $d$-regular bipartite graph (every vertex has degree $d$) can be colored with $d$ colors so that every vertex is incident to all $d$ colors. (Use König’s theorem.)

**Problem 2.** Let $G$ is a bipartite graph with parts $A, B$. Let $S \subseteq A$ be such that there is a matching covering $S$, and $T \subseteq B$ such that there is a matching covering $T$. Prove that there is a matching covering both $S$ and $T$.

**Problem 3.** Let $P_{\text{match}}(G) = \text{conv}\{\chi_M : M \text{ is a matching in } G\}$ denote the matching polytope of $G$. Prove that $P_{\text{match}}(G) \cap \{x : 1^T x = k\}$ is the convex hull of all matchings of size exactly $k$.

**Problem 4.** Consider the perfect matching polytope, $P_{\text{perf}}(G)$. An edge is a line segment between two vertices $s = [\chi_M, \chi_N]$ such that $s = H \cap P_{\text{perf}}(G)$ for some hyperplane $H = \{x : w^T x = \lambda\}$ and $w^T x \leq \lambda$ for all $x \in P$. Prove that $[\chi_M, \chi_N]$ is an edge if and only if $M \Delta N$ is a single cycle. What is the certifying hyperplane?

**Bonus problem.** Let $D = (V, A)$ be a directed graph and $T = (V, B)$ a directed tree (some orientation of a spanning tree on $V$). Define a matrix $C \in \mathbb{R}^{A \times B}$, defined as follows. For an edge $a = (u, v)$, let $P_a$ be the unique path from $u$ to $v$ in $T$. For $b \in B$, we define

- $C_{a,b} = 1$ if $b$ occurs in forward direction on $P_a$,
- $C_{a,b} = -1$ if $b$ occurs in backward direction on $P_a$,
- $C_{a,b} = 0$ if $b \notin P_a$.

Prove that $C$ is a totally unimodular matrix.