1 The greedy algorithm for matroids

The following algorithm finds the maximum weight base in a matroid $\mathcal{M} = (E, \mathcal{I})$

Algorithm 1 Greedy algorithm for selecting the max-weight base of a matroid

Input: a matroid $\mathcal{M} = (E, \mathcal{I})$, where $E = \{1, 2, \ldots, n\}$ is the ground set, and weight of $i$ is $w_i$.

Output: A base $B \in \mathcal{I}$ such that $w(B) = \max_{B \in \mathcal{I}} w(B)$.

1: Relabel the elements of the matroid so that $w_1 \geq w_2 \geq \ldots \geq w_n$.
2: $S \leftarrow \emptyset$.
3: for $i \leftarrow 1$ to $n$ do
4: if $S + i \in \mathcal{I}$ then
5: $S \leftarrow S + i$.
6: end if
7: end for
8: return $S$

Theorem 1 (Rado/Gale) For any ground set $E = \{1, 2, \ldots, n\}$, and a family of subsets $\mathcal{I} \subset 2^E$, Algorithm 1 returns the maximum-weight base for any set of weights $w : E \to \mathbb{R}$ if and only if $\mathcal{M} = (E, \mathcal{I})$ is a matroid.

We prove this theorem in two parts.

Claim 2 ($\Leftarrow$ part) Suppose that $(E, \mathcal{I})$ is a matroid. For any set of weights assigned to the elements of $E$, Algorithm 1 returns the maximum-weight base.

Proof: Wlog assume that $w_1 \geq w_2 \geq \ldots \geq w_n$. We prove that at any point of the execution of the algorithm, there exists an optimal base $B$ such that $S \subseteq B$ and $B \setminus S$ is among the remaining elements.

In particular, let $S_i$ be the set $S$ after observing the first $i$ elements (e.g. $S_0 = \emptyset$). We use induction to show that for any $S_i$, there exists an optimal base $B_i$ such that $S \subseteq B_i$, and $B_i \setminus S_i \subseteq \{i+1, \ldots, n\}$. Note this certainly implies the claim, since we get $S_n = B_n$ is an optimal base. The base case of the induction is trivially satisfied for $S_0 = \emptyset$.

Suppose before the $i$-th iteration we have $S_{i-1} \subseteq B_{i-1}$ where $B_{i-1}$ is a max-weight base. If the algorithm does not add $i$ to $S_{i-1}$, it means that $S_{i-1} + i \notin \mathcal{I}$ and therefore $i$ cannot be in $B_{i-1}$ (by the downward closed-ness property of the matroid). Therefore, $B_i = B_{i-1}$ satisfies the induction statement.

Now assume that the algorithm adds $i$ to $S_{i-1}$. By the induction hypothesis, if $i \in B_{i-1}$, then $S_i = S_{i-1} + i \subseteq B_{i-1}$, and we can set $B_i = B_{i-1}$. Otherwise, by the extension axiom of the matroid $\mathcal{M}$, the set $S_i$ can be extended from $B_{i-1}$ until it becomes a base, say $B'$. Since $i \notin B_{i-1}$, we have
Moreover, since $B'$ is also a base, we have $|B_{i-1} \setminus B'| = 1$, so let $B_{i-1} \setminus B' = \{j\}$. Therefore we can write $B' = B_{i-1} + i - j$.

Since $B_{i-1} \setminus S_i \subset \{i + 1, \ldots, n\}$, we have $j \in \{i + 1, \ldots, n\}$ is one of the remaining elements. Therefore, since the algorithm orders the element decreasingly according to their weights, we have $w_j \leq w_i$. But this means that $w(B') = w(B_{i-1}) + w_i - w_j \geq w(B_{i-1})$. By the optimality assumption of $B_{i-1}$, we have $w(B') = w(B_{i-1})$, hence $B_i = B'$ satisfies the induction statement.

**Claim 3** ($\Rightarrow$ part) Suppose $(E, I)$ is not a matroid. There exists an assignment of weights to the elements of $E$ such that algorithm 1 does not return a maximum-weight base.

**Proof:** If $(E, I)$ is not a matroid, it does not satisfy at least one of the two properties of the matroid. Suppose $I$ is not a downward-closed family of sets. Therefore, there exist two sets $S \subset T \in I$, but $S \not\in I$. Suppose we assign the weights as follows:

$$
\forall 1 \leq i \leq n, \quad w_i = \begin{cases} 
2 & i \in S \\
1 & i \in T \setminus S \\
0 & \text{otherwise}
\end{cases}
$$

By the weight assignment, the algorithm first considers the elements of $S$, then the elements of $T$, and then the rest of the elements. The elements in $E \setminus S$ are worth nothing, thus every optimal base must contain $T$. Suppose the algorithm selects a subset $S_1 \subset S$ after observing the elements of $S$. Since $S \not\in I$, we have $S_1 \neq S$. Out of the remaining elements, the algorithm can get value at most $|T \setminus S|$. If $S_2$ is the final set chosen by the algorithm, we have

$$w(S_2) = 2|S_1| + w(S_2 \setminus S) < 2|S| + |T \setminus S| = w(T).$$

Now suppose $(E, I)$ is not a matroid because the extension axiom is violated (assume the downward closed property). In particular, let $S, T \in I$ be two independent sets such that $|S| < |T|$, and for all $i \in T \setminus S, S + i \not\in I$. We use the following weights:

$$
\forall 1 \leq i \leq n, \quad w_i = \begin{cases} 
1 + \frac{1}{2|S|} & i \in S \\
1 & i \in T \setminus S \\
0 & \text{otherwise}
\end{cases}
$$

Note that $S$ is not necessarily a subset of $T$ here. This time, because of the downward closedness property the algorithm would select all of the elements of $S$. But this means that it can not add any element in $T \setminus S$, as this would violate independence. Further elements do not bring any value anymore, so if $S_2$ is the solution returned by the algorithm,

$$w(S_2) = w(S) = |S| \left(1 + \frac{1}{2|S|}\right) = |S| + \frac{1}{2},$$

while the value of $T$ is

$$w(T) \geq |T| \geq |S| + 1.$$

The following properties can be shown using the above theorem.
1. Let $S_i$ be the set of elements chosen by the algorithm after observing the first $i$ elements. Then $S_i$ is always a base of those $i$ elements. (By considering $w_1 = \ldots = w_i = 1$ and $w_{i+1} = \ldots = w_n = 0$.)

2. Finding the maximum-weight base in a matroid is in fact equivalent to finding the minimum-weight base. Let $w_{\text{max}} = \max_{1 \leq i \leq n} w_i$ be the maximum weight assigned to the elements; to find the minimum-weight base it is sufficient to consider $w'_i = w_{\text{max}} - w_i$, for all $i \in E$.

3. Also, it is straightforward that if the weights are non-negative, then the maximum-weight independent set is the same as the maximum-weight base. In general, we can say that the maximum-weight independent set is the maximum-weight base of the elements with non-negative weights.

2 The span function in matroids

The following definition of a “span” of a set of elements in a matroid is a generalization of the notion of span in vector spaces.

**Definition 4** Let $\mathcal{M} = (E, \mathcal{I})$ be a matroid. For any set $S \subseteq E$ define
\[
\text{span}(S) := \{i \in E : \text{rank}(S + i) = \text{rank}(S)\}.
\]

**Lemma 5** Let $S \subseteq E$. Then, any base of $S$ is also a base of $\text{span}(S)$.

**Proof:** By contradiction: Let $B$ be a base of $S$ which is not a base of $\text{span}(S)$. Since $B$ is a base of $S$, we have $|B| = \text{rank}(S)$. Also, since $B$ is not a base of $\text{span}(S)$, it means that there is some element $i \in \text{span}(S) \setminus B$, such that $B + i \in \mathcal{I}$. Therefore,
\[
\text{rank}(S + i) \geq \text{rank}(B + i) > \text{rank}(B) = \text{rank}(S).
\]
This contradicts the definition of $i \in \text{span}(S)$.

**Lemma 6** Let $S \subseteq E$. For any base $B$ of $S$ and any element $i \in E \setminus S$, $i \in \text{span}(S)$ if and only if $B + i \not\in \mathcal{I}$.

**Proof:** If $i \in \text{span}(S)$ and $B$ is a base of $S$, then by Lemma 5, $B$ is also a base of $\text{span}(S)$, and thus a base of $S + i$. Therefore, $B + i \not\in \mathcal{I}$.

Conversely, suppose $i \not\in \text{span}(S)$. Therefore, $\text{rank}(S+i) > \text{rank}(S)$, and there is an independent set $B' + i$, where $B' \subseteq S$ and $|B' + i| = \text{rank}(S) + 1$. In other words, $B'$ is a base of $S$. Now consider any base $B$ of $S$. This is also an independent subset of $S + i$. Since $|B| < |B' + i|$, by the extension axiom, it can be extended by adding an element from $B' + i$. But that element must be $i$ (otherwise, $B$ was not a base of $S$), and thus $B + i$ is independent.

Next, we prove that span preserves the ordering by inclusion.

**Lemma 7** For any $S \subseteq T \subseteq E$, $\text{span}(S) \subseteq \text{span}(T)$. 
**Proof:** Let \( B_S \) be a base of \( S \), and \( B_T \) a base of \( T \). By the extension axiom, \( B_S \) can be extended to a base \( B' \) of \( T \) from the elements of \( B_T \) (note that \( B' \setminus B_S \subseteq T \setminus S \)).

Consider \( i \in \text{span}(S) \). Since \( \text{rank}(S + i) = \text{rank}(S) \), we have \( B_S + i \notin I \). Therefore, since \( B_S \subseteq B' \), by the downward closedness axiom, \( B' + i \notin I \) either. By Lemma 6, \( i \in \text{span}(T) \). \( \square \)

Suppose we assign distinct weight to the elements of the matroid (i.e. \( w_i \neq w_j \) for all \( i, j \in E \)), then the maximum weight base is unique. Using the facts above, we can describe the maximum-weight base as follows:

**Lemma 8** Let \( M = (E, I) \) be a matroid, \( E = \{1, 2, \ldots, n\} \) and assume \( w_1 > w_2 > \ldots > w_n \). Then, the maximum-weight base is

\[
B_{\text{opt}} = \{i \in E : i \notin \text{span}(\{1, \ldots, i-1\})\}.
\]

**Proof:** Consider Algorithm 1. Let \( E_i = \{1, 2, \ldots, i\} \) be the set of the first \( i \) elements observed by the algorithm, and similar to the proof of Theorem 1, let \( S_{i-1} \) be the independent set chosen by the algorithm after observing the elements of \( E_{i-1} \). Recall that \( S_{i-1} \) is a base of \( E_{i-1} \). Therefore, by Lemma 6, \( S_{i-1} + i \in I \) if and only if \( i \notin \text{span}(S_{i-1}) = \text{span}(E_{i-1}) \). So the algorithm produces exactly the set \( B_{\text{opt}} \). From the analysis of Algorithm 1, it is also clear that in this case the maximum-weight base in unique. \( \square \)

### 3 Characterization of rank functions

In this section we prove some of the basic properties of rank functions.

**Lemma 9** The rank function of a matroid satisfies the following:

1. For any \( S \subseteq T \subseteq E \) of elements, we have \( r(S) \leq r(T) \) (monotonicity)

2. For any \( S \subseteq T \subseteq E \), \( i \in E \setminus T \), we have \( r(T + i) - r(T) \leq r(S + i) - r(S) \) (non-increasing marginal values)

**Proof:** The first property is trivial (since any base of \( S \) is also an independent set of \( T \)). To prove the second property we use Lemma 7. Observe that \( r(S + i) - r(S) = 0 \) if \( i \in \text{span}(S) \) and \( 1 \) if \( i \notin \text{span}(S) \). Lemma 7 implies that if \( i \in \text{span}(S) \) then \( i \in \text{span}(T) \). Therefore,

\[
 r(T + i) - r(T) \leq r(S + i) - r(S).
\]

For a set \( S \) and an element \( i \notin S \), we call \( r(S + i) - r(S) \) the marginal value of \( i \) with respect to \( S \). In the next lemma we show that non-increasing marginal values are equivalent to submodularity: a function \( f \) is submodular if \( f(A \cup B) + f(A \cap B) \leq f(A) + f(B) \) for every pair of sets \( A, B \).

**Lemma 10** Let \( f : 2^E \to \mathbb{R} \) be a set function on a ground set \( E \). Then \( f \) is submodular \( \forall A, B \subseteq E, f(A \cap B) + f(A \cup B) \leq f(A) + f(B) \) if and only if for all \( S \subset T \subset E \) and \( i \in E \setminus T \):

\[
f(T + i) - f(T) \leq f(S + i) - f(S).
\]
Proof: Assume for all \( S \subseteq T \) and \( i \notin T \), we have \( f(T+i) - f(T) \leq f(S+i) - f(S) \). Let \( A, B \subseteq E \) be two subsets of \( E \). If \( B \subseteq A \), the claim is trivial. Let \( B \setminus A = \{b_1, b_2, \ldots, b_k\} \). We have:

\[
\begin{align*}
 f(A \cup B) - f(A) &= \sum_{i=1}^{k} (f(A + b_1 + \ldots + b_i) - f(A + b_1 + \ldots + b_{i-1})) \\
 &\leq \sum_{i=1}^{k} (f(A \cap B + b_1 + \ldots + b_i) - f(A \cap B + b_1 + \ldots + b_{i-1})) \\
 &= f(B) - f(A \cap B).
\end{align*}
\]

Here the inequality follows from the assumption once we set \( S := A \cap B \) and \( T := A \) (note that this implies \( S \subseteq T \)).

Conversely, suppose for any two sets \( A, B \) we have \( f(A \cup B) + f(A \cap B) \leq f(A) + f(B) \). Let \( S \subseteq T \) and \( i \notin T \). Now set \( A := S + i \) and \( B := T \). By the submodularity condition, we get:

\[
 f(T+i) + f(S) = f(A \cup B) + f(A \cap B) \leq f(A) + f(B) = f(S+i) + f(T),
\]

which completes the proof.

It turns out that submodularity, together with the fact that marginal values are 0 or 1, characterizes exactly the rank functions of matroids.

Lemma 11 A function \( r : 2^E \mapsto \mathbb{R} \) is a rank function of a matroid if and only if

1. \( r(\emptyset) = 0 \) and \( r(S+i) - r(S) \in \{0, 1\} \) for all \( S \subseteq E, i \notin S \);

2. \( r \) is submodular, i.e. \( r(S \cup T) + r(S \cap T) \leq r(S) + r(T) \) for all \( S, T \subseteq E \).

Proof: We already know that any matroid rank function must satisfy these conditions. So let us assume that \( r \) satisfies the conditions and define \( I = \{ A : r(A) = |A| \} \). We claim that \( (E, I) \) is a matroid and \( r \) is its rank function.

By the first condition on \( r \), it is clear that \( I \) is closed under taking subsets. We claim that for any set \( S \), all maximal subsets of \( S \) which are in \( I \) (bases of \( S \)) have the same size. Consider any \( A \subseteq S \) such that \( r(A) = |A| \). We have \( r(A+i) - r(A) \in \{0, 1\} \). As long as \( r(A) < r(S) \), by submodularity there is \( i \in S \setminus A \) such that \( r(A+i) = r(A) + 1 \). Hence, if \( A \) is maximal such that \( r(A) = |A| \), we have \( r(A) = r(S) \). i.e., all bases have the same size, and \( r(S) \) is equal to that size.

\( \square \)