Nearly equal distances and
Szemerédi’s regularity lemma

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Abstract

A point set is separated if the minimum distance between its elements is one. Two numbers are called nearly equal if they differ by at most one. If a fixed positive percentage of all pairs of points belonging to a separated set of size \( n \) in \( \mathbb{R}^3 \) determine nearly equal distances, then the diameter of the set is at least constant times \( n \). This proves a conjecture of Erdős.

1 Introduction

In 1946, Erdős [E46] raised the following problem on repeated distances determined by a point set: Given \( n \) points in the plane (or, more generally, in \( \mathbb{R}^d \)), at most how many of the \( \binom{n}{2} \) interpoint distances can coincide? It is conjectured that in the plane this maximum is \( n^{1+\frac{\log \log n}{\log n}} \), which is asymptotically sharp, for example for a \( \sqrt{n} \times \sqrt{n} \) piece of the integer lattice. The best known upper estimate is only \( O(n^{4/3}) \) [SST84], [S97]. In 3-space, the best known upper bound is \( n^{3/2} \beta(n) \), where \( \beta(n) \) is an extremely slowly increasing function related to the inverse Ackermann function [CEG+90]. However, the truth is probably closer to \( n^{4/3} \). In higher dimensions, Lenz construction gives the asymptotically tight answer, which is quadratic in \( n \) (e.g. see [PA95], [BMP05]). These questions are intimately related to various problems concerning incidences between points and curves, surfaces, etc. (See [AS02], [PS04].)

Erdős observed that the answer to the above problem does not remain the same if one counts the number of distances that are nearly equal, where several distances

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are said to be nearly equal if they differ by at most 1, i.e. they all lie in an interval \([t, t+1]\) for some \(t > 0\). Clearly, to exclude trivial examples, one needs to consider only separated point sets, i.e., point sets where the minimum distance between two points is at least 1. Erdős et al. [EMPS91] (see also [EMP93], [MPS02]) proved that for any \(t > 0\), \(d \geq 2\), and for any separated set \(P\) of \(n\) points in \(\mathbb{R}^d\), where \(n\) is sufficiently large, the number of point pairs in \(P\) whose distance lies in the interval \([t, t+1]\) is at most \(T(d, n) = \frac{n^d}{2}(1 - \frac{t}{n} + o(1))\). Here, \(T(d, n)\) denotes the number of edges of a balanced \(d\)-partite graph on \(n\) vertices [B78], which is known to be the maximum number of edges in any graph of \(n\) vertices that does not contain a complete subgraph with \(d+1\) vertices. Moreover, this bound can be attained for every \(t \geq t(d, n)\), as shown by the following construction (described here only for \(d = 3\)). Let \(t\) be a sufficiently large number, and let \(v_1, v_2, v_3\) be the vertices of an equilateral triangle in the plane \(x_3 = 0\), with edge length \(t\). At each \(v_i\) draw a line perpendicular to the plane \(x_3 = 0\), and on each of these lines pick \([n/3]\) or \([n/3]\) distinct points whose \(x_3\)-coordinates are integers between 0 and \(n/3\), so that the total number of points is \(n\) (see Figure 1). If \(t\) is sufficiently large depending on \(n\) (roughly \(\frac{1}{18}n^2\)), the distance between any pair of points selected on different perpendicular lines belongs to the interval \([t, t+1]\).

![Figure 1](image.png)

The question arises, what is the minimal diameter of a separated set of \(n\) points in \(\mathbb{R}^d\) with \(\Omega(n^2)\) nearly equal distances? In the plane the answer is \(\Theta(n^2)\), by the Pythagorean theorem. The problem becomes more interesting in three dimensions. Notice that the diameter of the configuration depicted in Figure 1 is \(\Omega(n^2)\). However, it is easy to find a set of \(n\) points in \(\mathbb{R}^3\) with \(\frac{n^2}{2}\) nearly equal distances, whose diameter is \(O(n)\): Take two \(\sqrt{\frac{n}{2}} \times \sqrt{\frac{n}{2}}\) integer grids in two parallel planes at distance \(\frac{n}{2}\) from each other (see Figure 2). Erdős conjectured that there is no such example with diameter \(o(n)\).

The aim of this note is to prove this conjecture.

**Theorem 1.1.** Let \(\varepsilon > 0\) be fixed and let \(P\) be a separated set of \(n\) points in \(\mathbb{R}^3\) containing at least \(\varepsilon n^2\) pairs \((u, v)\), \(u, v \in P\), with \(||u - v|| \in [t, t+1]\) for some fixed real number \(t > 0\). Then the diameter of \(P\) satisfies \(\text{diam}(P) = \Omega(n)\).
Figure 2. An $n$-point separated set in $\mathbb{R}^3$ which determines $\frac{1}{7}n^2$ nearly equal distances and has diameter $O(n)$.

The proof is based on Szemerédi’s regularity lemma [KS96] and on a Ramsey-type result for dot products of vectors, derived in [APPRS05]. The geometric component of the proof (see Section 3) does not easily generalize to higher dimensions. We will return to the higher-dimensional analogue of Theorem 1.1 in a subsequent paper [PRV05].

## 2 Using Szemerédi’s regularity lemma

In this section, we prove the following theorem.

**Theorem 2.1.** Let $\varepsilon > 0$ and let $P$ be a set of $n$ points in $\mathbb{R}^3$ containing at least $\varepsilon n^2$ pairs $(u, v)$, $u, v \in P$, with $|u - v| \in [t, t + 1]$ for some fixed real number $t > 0$. Then there exists a constant $c := c(\varepsilon) > 0$ and there are two subsets $Q, R \subset P$, such that $|Q| = |R| = cn$ and $|u - v| \in [t, t + 1]$ for all $u \in Q$, $v \in R$.

**Proof.** Let $G = (V(G), E(G))$ be the graph on the vertex set $V(G) := P$ in which two vertices $u, v \in V(G)$ are connected by an edge if and only if $|u - v| \in [t, t + 1]$. By the assumptions, we have $|E(G)| \geq \varepsilon n^2$.

Before we could state the “degree form” of Szemerédi’s regularity lemma (see e.g., [KS96]), we need a definition.

**Definition 2.2.** Let $\delta > 0$. Given a graph $G$ and two disjoint vertex sets $A \subset V$, $B \subset V$, we say that the pair $(A, B)$ is $\delta$-regular if for every $X \subset A$ and $Y \subset B$ satisfying $|X| \geq \delta |A|$ and $|Y| \geq \delta |B|$, we have $|d(X, Y) - d(A, B)| < \delta$.

Here, $d$ stands for the standard density function $d(X, Y) = \frac{|E(X, Y)|}{|X||Y|}$, where $E(X, Y)$ denotes the set of edges between $X$ and $Y$, two disjoint sets of vertices.

**Lemma 2.3.** (Szemerédi’s regularity lemma) For every $\delta > 0$, there is an $M := M(\delta)$ such that if $G = (V, E)$ is any graph and $\rho \in [0, 1]$ is any real number, then there is a partition of the vertex set $V$ into $k + 1$ clusters $V_0, V_1, \ldots, V_k$, and there is a subgraph $G' \subset G$ with the following properties:
1. \( k \leq M \),
2. \(|V_0| \leq \delta |V|\),
3. all clusters \( V_i, i \geq 1 \), are of the same size \( m \leq \lceil \delta |V| \rceil \),
4. \( \text{deg}_{G'}(v) > \text{deg}_{G}(v) - (\rho + \delta) |V| \) for all \( v \in V \),
5. \( E(G'(V_i)) = \emptyset \) for all \( i \geq 1 \),
6. all pairs \( G'(V_i, V_j), 1 \leq i < j \leq k \), are \( \delta \)-regular, each with a density either 0 or greater than \( \rho \).

Consider the graph \( G \) and set parameters \( \rho = 0 \), \( \delta = \min\{\frac{1}{2\pi}, \frac{2\varepsilon}{5}\} \). Using Lemma 2.3, we obtain a partition \( V_0, V_1, \ldots, V_k \) meeting the requirements. Delete all elements in \( V_0 \) to obtain a “pure” graph \( G'' \) with \( \geq (1 - \delta) |V| \) vertices, whose vertex set \( V(G'') = V(G') \setminus V_0 \) is partitioned into \( k \) clusters \( V_1, V_2, \ldots, V_k \), where \( k \leq M(\delta) \), \( |V_i| = m \) and \( E(G''(V_i)) = \emptyset \) for all \( i \geq 1 \), and all pairs \( G''(V_i, V_j), 1 \leq i < j \leq k \), are \( \delta \)-regular. The pure graph \( G'' \) still contains most of the original edges of \( G \). Indeed, we have \( \text{deg}_{G''}(v) > \text{deg}_{G}(v) - \delta n \) for every \( v \), whence \( |E(G'')| > |E(G)| - \delta n^2 \). To obtain \( G'' \), we remove another set of at most \( |V_0| n \leq \delta n^2 \) edges, which implies

\[ |E(G'')| > |E(G)| - 2\delta n^2 \geq (\epsilon - 2\delta)n^2. \]

We claim that there is a pair \((V_i, V_j)\) of clusters in \( G'' \) with density \( d(V_i, V_j) \geq \alpha := 2(\varepsilon - 2\delta) \). Indeed, otherwise, using the fact that \( E(G''(V_i)) = \emptyset \), we conclude that \( |E(G'')| \) is too small:

\[ |E(G'')| \leq \left( \frac{k}{2} \right) \alpha m^2 < \frac{\alpha k^2 m^2}{2} \leq (\epsilon - 2\delta)n^2. \]

Let \((V_i, V_j)\) be a \( \delta \)-regular pair with density \( d(V_i, V_j) \geq 2(\varepsilon - 2\delta) \). Define maps \( \omega_1, \omega_2 : V_i \cup V_j \rightarrow \mathbb{R}^5 \) as follows:

\[ \omega_1(u) = (u_x, u_y, u_z, ||u||^2 - t^2, 1), \]

\[ \omega_2(u) = (u_x, u_y, u_z, ||u||^2 - (t + 1)^2, 1), \]

\[ \omega_1(v) = (-2v_x, -2v_y, -2v_z, 1, ||v||^2), \]

\[ \omega_2(v) = (2v_x, 2v_y, 2v_z, -1, -||v||^2), \]

for all \( u = (u_x, u_y, u_z) \in V_i \subset \mathbb{R}^3 \), \( v = (v_x, v_y, v_z) \in V_j \subset \mathbb{R}^3 \).

Then, for all \( u \in V_i, v \in V_j \), the edge \( \{u, v\} \) is in \( E(G'') \), that is, \( ||u - v|| \in [t, t+1] \), if and only if \( \omega_1(u) \cdot \omega_1(v) \geq 0 \) and \( \omega_2(u) \cdot \omega_2(v) \geq 0 \).

Recall the following lemma of Alon et al. [APRS05], which can be proved using a Borsuk-Ulam type result in range searching, due to Yao and Yao [YY85].
Lemma 2.4. [APPRS05] Let \( U \) and \( V \) be finite sets of vectors in \( \mathbb{R}^d \). Then there are subsets \( U' \subset U \) and \( V' \subset V \) such that |\( U' \)\| \( \geq \frac{1}{2d+1} |U| \), |\( V' \)\| \( \geq \frac{1}{2d+1} |V| \) and either \( u \cdot v \geq 0 \) for all \( u \in U' \), \( v \in V' \), or \( u \cdot v < 0 \) for all \( u \in U' \), \( v \in V' \).

Applying Lemma 2.4 to the sets \( U := \omega_1(V_i) \) and \( V := \omega_1(V_j) \), we obtain two subsets \( V'_i \subset V_i \) and \( V'_j \subset V_j \) such that \( |V'_i| \geq \frac{1}{2d} |V_i| \), \( |V'_j| \geq \frac{1}{2d} |V_j| \), and either \( \omega_1(u) \cdot \omega_1(v) \geq 0 \) for all \( u \in V'_i \), \( v \in V'_j \), or \( \omega_1(u) \cdot \omega_1(v) < 0 \) for all \( u \in V'_i \), \( v \in V'_j \).

We claim that \( \omega_1(u) \cdot \omega_1(v) \geq 0 \) for all \( u \in V'_i \), \( v \in V'_j \). Indeed, otherwise \( |u - v| < t \) holds for all \( u \in V'_i \), \( v \in V'_j \), which implies that \( d(V'_i, V'_j) = 0 \). However, by the \( \delta \)-regularity of the pair \( (V_i, V_j) \), we have \( d(V'_i, V'_j) = d(V_i, V_j) - \delta \geq 2(\varepsilon - 2\delta) - \delta \geq 0 \), since \( \delta = \min\{\frac{1}{2d}, \frac{\varepsilon}{2}\} \) and \( |V'_i| \geq \frac{1}{2d} |V_i| > \delta |V_i| \), \( |V'_j| \geq \frac{1}{2d} |V_j| > \delta |V_j| \).

Therefore, we have \( \omega_1(u) \cdot \omega_1(v) \geq 0 \) for all \( u \in V'_i \), \( v \in V'_j \).

Next, we apply Lemma 2.4 to the sets \( U := \omega_2(V'_i) \) and \( V := \omega_2(V'_j) \), and we obtain two subsets \( V''_i \subset V'_i \) and \( V''_j \subset V'_j \) such that \( |V''_i| \geq \frac{1}{2d} |V'_i| \), \( |V''_j| \geq \frac{1}{2d} |V'_j| \), and either \( \omega_2(u) \cdot \omega_2(v) \geq 0 \) for all \( u \in V''_i \), \( v \in V''_j \), or \( \omega_2(u) \cdot \omega_2(v) < 0 \) for all \( u \in V''_i \), \( v \in V''_j \).

We now claim that \( \omega_2(u) \cdot \omega_2(v) \geq 0 \) for all \( u \in V''_i \), \( v \in V''_j \). Indeed, otherwise \( |u - v| > t + 1 \) holds for all \( u \in V''_i \), \( v \in V''_j \), and we have \( d(V''_i, V''_j) = 0 \). However, by the \( \delta \)-regularity of the pair \( (V'_i, V'_j) \), we obtain \( d(V''_i, V''_j) = d(V'_i, V'_j) - \delta \geq 2(\varepsilon - 2\delta) - \delta \geq 0 \), since \( \delta = \min\{\frac{1}{2d}, \frac{\varepsilon}{2}\} \) and \( |V''_i| \geq \frac{1}{2d} |V'_i| \geq \frac{1}{2d} |V_i| \geq \delta |V_i| \), \( |V''_j| \geq \frac{1}{2d} |V'_j| \geq \frac{1}{2d} |V_j| \geq \delta |V_j| \).

Thus, we conclude that \( \omega_1(u) \cdot \omega_1(v) \geq 0 \) and \( \omega_2(u) \cdot \omega_2(v) \geq 0 \) for all \( u \in V''_i \), \( v \in V''_j \). Furthermore, both \( V''_i \) and \( V''_j \) are of size at least \( \delta |V_i| = \delta |V_j| \geq \frac{\delta(1-\delta)}{M(\delta)} n \), where \( \delta = \min\{\frac{1}{2d}, \frac{\varepsilon}{2}\} \). Therefore, we have found two subsets \( Q := V''_i \subset P \) and \( R := V''_j \subset P \) such that \( |Q| = c(\varepsilon)n \) and \( |u - v| \in [t, t + 1] \) for all \( u \in P \), \( v \in Q \). This completes the proof of Theorem 2.1. \( \square \)

3 Proof of Theorem 1.1

In the previous section we have established that there are two sets of points \( Q, R \) of size \( \Omega(n) \) such that all pairwise distances between points in \( Q \) and \( R \) are in the interval \( [t, t + 1] \). In the rest of the paper, we conclude that this is impossible unless \( t = \Omega(n) \). The proof proceeds in two steps.

1. We prove that \( R \) must contain a special configuration, either two points at large distance or three points forming a triangle of large area and small circumradius.

2. We show that this configuration forces \( Q \) to be contained in a region of volume \( O(t^2/n) \). Since \( Q \) requires volume \( \Omega(n) \), this yields \( t = \Omega(n) \).
Lemma 3.1. Let $t \leq \frac{n}{64}$ and let $R$ be a separated set of $n \geq 316$ points in $\mathbb{R}^3$ such that $t \leq ||x|| \leq t + 1$ for all $x \in R$. Then we have $t \geq 2$, and either there are two points $x_1, x_2 \in R$ such that $||x_1 - x_2|| > 0.24t$, or there are three points $y_1, y_2, y_3 \in R$ such that the area of triangle $\{y_1, y_2, y_3\}$ is at least $\frac{n}{6\pi}$ and the radius of its circumscribed circle is at most $\frac{1}{8}$.

Proof. Let $B$ denote the family of $n$ balls of radius $\frac{1}{2}$, each centered at a point of $R$. Note that the interiors of any two balls in $B$ are disjoint, since $R$ is separated. Let $A(t)$ denote the spherical annulus $A(t) = \{x : t - \frac{1}{4} \leq ||x|| \leq t + \frac{1}{2}\}$. Clearly, $A(t)$ contains $B$ and its volume is

\[
\text{Vol}(A(t)) = \frac{4\pi}{3} \left( \left( t + \frac{3}{2} \right)^3 - \left( t - \frac{1}{2} \right)^3 \right) = \pi \left( 8t^2 + 8t + \frac{14}{3} \right).
\]

Since the balls in $B$ are pairwise disjoint and each has volume $\pi/6$, the volume of $A(t)$ must be at least $n\pi/6 \geq 316\pi/6$, which implies $t \geq 2$.

Let $x_1, x_2$ denote two points in $R$ whose distance $h := ||x_1 - x_2||$ is maximal. If $h > 0.24t$, we are done. Assume that $h \leq 0.24t$. By a similar volume argument, we obtain that $h$ must be at least $\sqrt{n}/4$. Otherwise, all balls belonging to $B$ are contained in a sphere of radius $\frac{\sqrt{n}}{4} + \frac{1}{2}$, which intersects $A(t)$ in a region of volume less than $4(\frac{\sqrt{n}}{4} + \frac{1}{2})^2 \pi < n\pi/6$, which is a contradiction. Therefore, we obtain

\[ h = ||x_1 - x_2|| \geq \frac{1}{4} \sqrt{n} \geq 2\sqrt{t}, \]

which yields

\[ x_1 \cdot (x_1 - x_2) = \frac{1}{2} (||x_1||^2 - ||x_2||^2 + ||x_1 - x_2||^2) \geq \frac{1}{2} (t^2 - (t + 1)^2 + 4t) > 0. \]

Similarly, we have $x_2 \cdot (x_1 - x_2) < 0$. Hence, there is a point $w = \beta x_1 + (1 - \beta) x_2, \beta \in [0, 1]$ such that $w \cdot (x_1 - x_2) = 0$. Let $H$ be a plane through the origin, orthogonal to vector $w$. Project the points of $R$ onto $H$, and let $\phi(x) \in H$ denote the projection of $x \in R$. Let $p = \phi(x_1), q = \phi(x_2)$, and note that $||p - q|| = ||x_1 - x_2|| = h$. Without loss of generality, we can assume that $pq$ is a vertical line in $H$. Note that all points in $R$ are at most $0.24t$ away from $x_1$ and $x_2$, so they project to within $0.24t$ of the origin. Divide the plane between $p$ and $q$ into horizontal strips of height 1. Every point in $\phi(R)$ is contained in one of these strips (see Figure 3). In the $i$-th strip, let $l_i$ and $r_i$ be the horizontal coordinate of the leftmost and rightmost points belonging to $\phi(R)$.

The balls in $B$ project to disks of radius $\frac{1}{2}$ in the plane $H$. The area of the region $D$ covered by these disks

\[ \text{Area}(D) \leq \sum_{i=1}^{\left\lfloor \frac{h}{2} \right\rfloor} 2(r_i - l_i + 1). \]
Consider a stabbing line through a point of $D$, perpendicular to the plane $H$. In view of our choice of $H$, the distance of the origin from the line is at most $0.24t + \frac{1}{2}$. On each side of $H$, such a line intersects $A(t)$ in an interval of length at most

$$\sqrt{(t + \frac{3}{2})^2 - \left(0.24t + \frac{1}{2}\right)^2} - \sqrt{(t - \frac{1}{2})^2 - \left(0.24t + \frac{1}{2}\right)^2},$$

which is less than 2.5, provided that $t \geq 2$. Since the points in $R$ have distance at most $0.24t$ from $x_1$, they lie in the same halfspace of $H$ as $x_1$. The balls in $\mathcal{B}$ are contained in the region of $A(t)$ which projects to $D$, and whose volume is therefore at most $2.5 \cdot \text{Area}(D)$. On the other hand, each of these $n$ balls has volume $\pi/6$. Therefore,

$$\frac{\pi n}{6} \leq 2.5 \cdot \text{Area}(D) = \frac{5}{3} \sum_{i=1}^{[h]} (r_i - l_i + 1),$$

and, hence, using $h \geq \sqrt{n}/4 > 4.44$ and $[h] \leq 1.2h$, we deduce that there exists a $j$ such that

$$r_j - l_j + 1 \geq \frac{\pi n}{30[h]} \geq \frac{\pi n}{36h}.$$

Since $h \leq \frac{1}{4} \leq \frac{n}{256}$, we obtain $\pi n/36h \geq 22.2$ and

$$r_j - l_j \geq \frac{n}{12h}.$$

Let $s$ and $t$ denote the leftmost and rightmost points in the $j$-th horizontal strip, whose coordinates are $l_j$ and $r_j$, respectively. Let $\theta$ denote the angle between the line $st$ and the $x$-axis in $H$ (see Figure 3). Then, $\tan \theta \in [-\frac{1}{20}, \frac{1}{30}]$. We have

$$\text{Area}(\{p, s, t\}) + \text{Area}(\{q, s, t\}) = \frac{1}{2}h(r_j - l_j) \geq \frac{n}{24}.$$
Without loss of generality, assume that \( \text{Area}(\{p, s, t\}) \geq \frac{n}{36} \). Then the distance of \( p \) from line \( st \) is at least \( \frac{1}{2}h \cos \theta \geq \frac{1}{2}h \sqrt{\frac{100}{401}} > 0.49h \). Set \( y_1 := \phi^{-1}(p), y_2 := \phi^{-1}(s), \) and \( y_3 := \phi^{-1}(t) \). Since the area cannot increase by projection, we have \( \text{Area}(\{y_1, y_2, y_3\}) \geq \frac{n}{36} \).

Finally, we show that the radius of the circumscribed circle of the triangle \( \{y_1, y_2, y_3\} \) is not too large. Let \( h' \) denote the distance of \( y_1 \) from the line \( y_2y_3 \) and let \( \gamma \) denote the inner angle at \( y_3 \). We have \( h' > 0.49h \) and \( \sin \gamma = h' / ||y_1 - y_3|| \). The radius of the circumscribed circle is

\[
r = \frac{||y_1 - y_2||}{\sin \gamma} = \frac{||y_1 - y_2|| \cdot ||y_1 - y_3||}{h'} \leq \frac{h'^2}{0.49h} \leq \frac{t}{2}.
\]

\[
\square
\]

**Lemma 3.2.** If \( x_1, x_2 \in \mathbb{R}^3, ||x_1 - x_2|| = a \), then the ring-like region

\[
R(x_1, x_2) := \{x : \forall i = 1, 2; \ t - \frac{1}{2} \leq ||x - x_i|| \leq t + \frac{3}{2}\}
\]

has volume at most \( 4\pi \left(\frac{2t+1}{a}\right)^2 \).

![Figure 4. \( R(x_1, x_2) \) and the slab \( S_a \).](image)

**Proof.** We claim that \( R(x_1, x_2) \) is contained in the parallel slab \( S_a \) (bounded by two parallel planes) of thickness \( 2(2t + 1)a \), orthogonal to line \( x_1x_2 \) (see Figure 4). Indeed, assume \( x_1 = -x_2, ||x_1|| = ||x_2|| = \frac{a}{2} \), and \( ||x - x_1||^2, ||x - x_2||^2 \in [(t - \frac{1}{2})^2, (t + \frac{3}{2})^2] \). Then,

\[
||x - x_1||^2 - ||x - x_2||^2 = 2|x \cdot (x_2 - x_1)| \leq 4t + 2,
\]

which means that \( x \) can deviate from the plane of symmetry between \( x_1 \) and \( x_2 \) by at most \( \frac{2t+1}{a} \).
Consider the annulus \( A(t) \) between the spheres of radius \( t - \frac{1}{2} \) and \( t + \frac{1}{2} \), centered at \( x_1 \). Take the intersection of this annulus with \( S_a \). The planes of \( S_a \) are at distances \( l = \frac{a}{2} - \frac{2t+1}{a} \) and \( r = \frac{a}{2} + \frac{2t+1}{a} \) from \( x_1 \). The resulting volume is

\[
\text{Vol}(R(x_1, x_2)) \leq \text{Vol}(A(t) \cap S_a) \\
\leq \int_l^r \left( \pi \left( \left( t + \frac{3}{2} \right)^2 - x^2 \right) - \pi \left( \left( t - \frac{1}{2} \right)^2 - x^2 \right) \right) dx \\
= \int_l^r 2\pi(2t+1)dx = \frac{4\pi(2t+1)^2}{a}.
\]

\[\square\]

**Lemma 3.3.** Let \( t \geq 2 \), and let \( \{x_1, x_2, x_3\} \) be a triangle of area \( A \geq 2t+1 \) whose sides are of length at most \( \frac{a}{4} \) and whose circumscribing circle has radius at most \( \frac{a}{2} \). Then the region

\[
R(x_1, x_2, x_3) = \{ x : \forall i = 1, 2, 3; \ t - \frac{1}{2} \leq ||x - x_i|| \leq t + \frac{3}{2} \}
\]

has volume at most \( \frac{16(2t+1)^2}{A} \).

![Figure 5. \( R' \), the intersection of the infinite prism \( S_a \cap S_b \) with the plane of the triangle \( \{x_1, x_2, x_3\} \).](image)

**Proof.** Consider two points \( x_1, x_2 \) at distance \( a \) and the region

\[
R(x_1, x_2) = \{ x : \forall i = 1, 2; \ t - \frac{1}{2} \leq ||x - x_i|| \leq t + \frac{3}{2} \}.
\]

As before, \( R(x_1, x_2) \) is contained in a slab \( S_a \) of thickness \( \frac{2t+1}{a} \), orthogonal to line \( x_1x_2 \). The same holds for \( R(x_1, x_3) \), that is, if \( ||x_1 - x_3|| = b \), then \( R(x_1, x_3) \) is
contained in a slab $S_b$ of thickness $\frac{2(2t+1)}{b}$. Let $\gamma$ denote the angle between $x_1 x_2$ and $x_1 x_3$. Then the angle between the normal vectors to $S_a$ and $S_b$ is also $\gamma$. The area $A$ of the triangle $\{x_1, x_2, x_3\}$ is $A = \frac{1}{2}ab\sin\gamma$.

The intersection $S_a \cap S_b$ is an infinite prism which intersects the plane of the triangle in a parallelogram $R'$ (see Figure 5). The sides of $R'$ are of length $\frac{2(2t+1)}{a\sin\gamma}$ and $\frac{2(2t+1)}{b\sin\gamma}$, and the angle between them is $\gamma$, which implies

$$\text{Area}(R') = \frac{4(2t+1)^2}{ab\sin\gamma} = \frac{2(2t+1)^2}{A}.$$

The center of $R'$ is the center of the circle circumscribed around $\{x_1, x_2, x_3\}$. Since the radius of the circumscribing circle is at most $\frac{1}{2}$, the distance between any point in $R'$ and $x_1$ is at most

$$\frac{t}{2} + \frac{2t+1}{a\sin\gamma} + \frac{2t+1}{b\sin\gamma} = \frac{t}{2} + \frac{2t+1}{2A} (b+ a) \leq \frac{t}{2} + \frac{a+b}{2} \leq \frac{3t}{4},$$

where the first inequality follows from the assumption that $A \geq 2t+1$. This implies that the prism $S_a \cap S_b$ intersects the annulus $A(t)$ centered at $x_1$ within distance $\frac{3t}{4}$ from $x_1$. Any line within distance $\frac{3t}{4}$ from the center intersects $A(t)$ in two intervals of length at most

$$\sqrt{\left(\frac{t+3}{2}\right)^2 - \left(\frac{3t}{4}\right)^2} - \sqrt{\left(\frac{t-1}{2}\right)^2 - \left(\frac{3t}{4}\right)^2},$$

which is smaller than 4 for $t \geq 2$. Therefore, we have

$$\text{Vol}(R(x_1, x_2, x_3)) \leq \text{Vol}(A(t) \cap S_a \cap S_b) \leq 8 \text{ Area}(R') = \frac{16(2t+1)^2}{A}.$$

\[\square\]

**Theorem 3.4.** Let $Q$ and $R$ be two separated sets of points in $\mathbb{R}^3$, each of size $n \geq 316$, such that $t \leq ||x - y|| \leq t + 1$ for all $x \in Q, y \in R$. Then, $t \geq \frac{n}{800}$.

*Proof.* Suppose that $n \geq 316$ and $t < \frac{n}{800}$. Assume that one of the points in $Q$ is the origin. Then Lemma 3.1 implies that $t \geq 2$, and either there exist two points $x_1, x_2 \in R$ at distance at least $0.24t$, or there exist three points $y_1, y_2, y_3 \in R$ such that the triangle $\{y_1, y_2, y_3\}$ has area at least $\frac{n}{38}$, its edges are of length at most $\frac{t}{4}$, and its circumradius is at most $\frac{t}{4}$. In the first case, Lemma 3.2 implies that the volume of $R(x_1, x_2)$ is at most $\frac{60(2t+1)^2}{t}$. In the second case, Lemma 3.3 implies that the volume of $R(y_1, y_2, y_3)$ is at most $\frac{268(2t+1)^2}{t} \leq \frac{32(2t+1)^2}{t}$. In either case, the region must contain a ball of radius $\frac{1}{2}$ around each point of $Q$, and these balls are pairwise disjoint. Therefore, the volume of the region must be at least $\frac{n\pi}{6}$. For $2 \leq t < \frac{n}{800}$, this number is at most $\frac{60(2t+1)^2}{t} \leq 375t < \frac{n\pi}{6}$, which is a contradiction. \[\square\]
Now we are in a position to prove Theorem 1.1. Let $P$ be a separated $n$-point set in $\mathbb{R}^3$ such that there exist $\varepsilon n^2$ pairs $(u, v)$, $u, v \in P$, $\varepsilon > 0$, with $||u - v|| \in [t, t + 1]$ for some fixed real number $t > 0$. Then, by Theorem 2.1, there exist a constant $c := c(\varepsilon) > 0$ and two subsets $Q, R \subset P$ such that $|Q| = |R| = cn$ and $||u - v|| \in [t, t + 1]$ for all $u \in Q$, $v \in R$. Then, by Theorem 3.4, we obtain $t \geq \frac{cn}{800}$. Hence, we have $\text{diam}(P) = \Omega(n)$, as required.

References


