# Research Problem
## The Maximum Acyclic Subgraph Problem

**Problem 4-1.** Consider the maximum acyclic subgraph problem discussed in Homework 2-1. Refer to that homework for several simple 0.5 approximations.

We can formulate linear ordering as a set of integer constraints. Let $x_{ij}$ be set to 1 if $i$ appears before $j$ in the ordering and 0 otherwise. Show that the following set of constraints encode a valid linear ordering:

$$\begin{array}{rcll} x_{ij} + x_{ji} & = & 1 & \forall i, j \\ x_{ij} + x_{jk} + x_{ki} & \leq & 2 & \forall i, j, k \\ x_{ij} & \in & \{0, 1\} & \forall i, j \end{array}$$

Let $c_{ij} = 1$ if there is an edge from $i$ to $j$ in $G(V, E)$, and 0 otherwise. The objective function is therefore to maximize $\sum_{i,j} c_{ij} x_{ij}$. The linear relaxation of the above integer program has an integrality gap of 0.5. For a proof, refer [1].

We can consider semidefinite relaxations for this problem. Let $\mathbf{e}$ denote an arbitrary unit vector. Let $\mathbf{x}_{ij}$ be a vector for every ordered pair $(i, j)$ in the graph. The goal of the SDP is to set $\mathbf{x}_{ij} = \mathbf{e}$ if $i$ comes before $j$ in the ordering and to $-\mathbf{e}$ otherwise. Show that the following is a valid SDP relaxation for this problem (note that $\mathbf{e}$ is a variable and *not* a constant):

$$\text{Maximize} \sum_{i,j} c_{ij} \frac{\mathbf{x}_{ij} \cdot \mathbf{e} + 1}{2}$$

$$\begin{array}{rcll} \| \mathbf{x}_{ij} + \mathbf{x}_{ji} \| & = & 0 & \forall i, j \\ \| \mathbf{x}_{ij} + \mathbf{x}_{jk} + \mathbf{x}_{ki} \| & = & 1 & \forall i, j, k \\ \| \mathbf{x}_{ij} \| & = & 1 & \forall i, j \\ \| \mathbf{e} \| & = & 1 & \end{array}$$

What is the integrality gap of this SDP? Can you round it to a ratio of $0.5 + \epsilon$ for some $\epsilon > 0$? For related work on semidefinite programming, please refer [2, 3, 4].

## References

[1] A. Newman and S. Vempala. Fences are futile: On relaxations for the linear ordering problem. Integer Programming and Combinatorial Optimization, 2001 **URL:** http://theory.lcs.mit.edu/~alantha/ipco01.ps

[2] M. Goemans and J. Kleinberg. The Lovasz theta function and a semi-definite programming relaxation of vertex cover. *SIAM J. Discrete Math*, 1998.

[3] M. Goemans and D. Williamson. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *JACM*, 1995.

[4] , D. Karger, R. Motwani and M. Sudan. Approximate graph coloring by semidefinite programming. *IEEE Symposium on Foundations of Computer Science*, 1994.