Approximation Algorithms - Tools and Techniques	**Spring 2002**

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The VLSI Routing Problem

Given a graph $G(V, E)$, n pairs of nodes $\{(s_1, t_1), \ldots, , (s_n, t_n)\}$, and a set of paths P_i connecting s_i and t_i, for each i; To choose a path $p_i \in P_i$ for each i such that congestion, the maximum number of paths going through any edge $e \in E$ is minimised.

IP Formulation

Let x_p be the variable denoting whether the path p is chosen or not.

$$\text{for each } p \in P_i, \text{ for some } i; x_p = \begin{cases} 1 & \text{if path } p \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}$$

Since each source-destination pair is to be connected by only one path, we have the first constraint

$$\sum_{p \in P_i} x_p = 1;\ \forall i.$$

The problem involves minimising the maximum of a functions of the variables, and hence, cannot directly be formulated as a integer programming problem. The problem is modified by letting W be the maximum number of parallel paths allowed on any edge, and considering W also as a variable. This imposes the constraint

$$\sum_{e \in p} x_p \leq W;\ \forall e \in E$$

The objective function is just to minimise W

$$\min W.$$

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Thus, the LP relaxation of this IP problem is

$$\begin{array}{rrl} & \min W & \\ s.t. & \sum_{p \in P_i} x_p = & 1;\ \forall i, \\ & \sum_{e \in p} x_p \leq & W;\ \forall e \in E, \\ & W > 1 & \\ & 0 \leq x \leq 1, & \end{array}$$

Rounding Idea

Use the weights returned by the LP solution as a probability distribution over the paths for each source-destination pair. With this distribution, independantly pick a path at random, for each such pair. Let $\tilde{W}$ be the maximum congestion in the resulting solution. In the following, we shall show that

Theorem 1 *With probability* $\left(1 - \frac{1}{n}\right)$

$$\tilde{W} \leq W \log n$$

Consider an edge e. LP returns weights x_p on the paths p through e such that

$$\sum_{e \in p} x_p \leq W.$$

Divide the set of paths passing through e into sets Q_i, each Q_i consisting only of paths from s_i to t_i, for that i. Let y_i be defined as

$$y_i(e) = \sum_{\substack{e \in p \\ p \in P_i}} x_p;\ \forall i, e$$

Thus, y_i s form a probability distribution over the P_is, denoting the probability that a path in P_i is picked which passes through the edge e.

Let $\tilde{Y}_i(e)$ be an indicator variable denoting whether a path in P_i passes through e (We use the indicator variable to apply Chernoff bound (which are valid over independant Bernoulli variables to bound the congestion after rounding).

$$\tilde{Y}_i(e) = \begin{cases} 1; & \text{if } p \in P_i \text{ passes through } e \\ 0; & \text{otherwise} \end{cases}$$

We have,

$$\begin{aligned} & Prob\left[\tilde{Y}_i(e) = 1\right] && = y_i(e) \\ \Rightarrow \quad & E\left[\tilde{Y}_i(e)\right] && = y_i(e). \end{aligned}$$

Let congestion(e) denote the number of paths passing through edge e. It is given by

$$\text{congestion}(e) = \sum_i \tilde{Y}_i(e)$$

and it's expectation, using the linearity of expectation is

$$\begin{aligned} E\left[\text{congestion}(e)\right] = & \quad \textstyle\sum_i E\left[\tilde{Y}_i(e)\right] \\ = & \quad \textstyle\sum_i y_i(e) \\ \leq & \quad W. \end{aligned} \tag{1}$$

However this is not sufficient to claim that that the maximum of the congestion over all edges is $\leq W$ (see box).

> An example to show that for random variables X_i
>
> $$E\left[X_i\right] \leq W,\ \forall i \nRightarrow E\left[\max_i X_i\right] \leq W.$$
>
> Consider the following X_is with $i \in [1, n]$.
>
> $$X_i = \begin{cases} 0 & \text{with prob. } \frac{1}{2} \\ 1 & \text{with prob. } \frac{1}{2} \end{cases}$$
>
> We have
>
> $$E\left[X_i\right] = \frac{1}{2}$$
>
> and
>
> $$\begin{aligned} Prob\left[\max_i X_i = 1\right] = & \quad 1 - Prob\left[X_i = 0;\ \forall i\right] \\ = & \quad 1 - \frac{1}{2^n} \\ \Rightarrow \quad E\left[\max_i X_i\right] = 1 - \tfrac{1}{2^n} & \end{aligned}$$
>
> where the final step follows from Markov's inequality.

We will show that

Theorem 2 *For the above rounding scheme*

$$\max_e \mathit{congestion}(e) \leq W \log(n)$$

with high probability $(1 - \frac{1}{n})$*, for* $W > 1$*.*

Chernoff Bound

For independent Bernoulli variables X_i each with probability of success p,

$$\mathit{Prob}\left[\sum X_i > (1+\beta)M\right] \leq \left(\frac{e^{\beta}}{(1+\beta)^{1+\beta}}\right)^M$$

where

$$M = \sum E\left[X_i\right].$$

To begin with, Using the Chernoff bound, we will show

Theorem 3

$$\mathit{Prob}\left[\mathit{congestion}(e) > W \log n\right] \leq \frac{1}{n^3}$$

Proof: We let $\tilde{Y}_i(e)$s be the Bernoulli variables, note that they are independant, with M given by

$$M = \sum E\left[\tilde{Y}_i(e)\right] \leq W$$

the latter inequality from Eq. 1. Applying Chernoff bound, this immediately gives

$$\mathit{Prob}\left[\text{congestion}(e) > (1+\beta)W)\right] \leq \left(\frac{e^{\beta}}{(1+\beta)^{1+\beta}}\right)^W$$

Letting $\beta = \log n$, we get

$$\begin{aligned} \mathit{Prob}\left[\text{congestion}(e) > W \log n\right] &\leq \left(\tfrac{e^{\log n}}{(1+\log n)^{1+\log n}}\right)^W \\ &\leq \left(\tfrac{n}{n^{\log\log n}}\right)^W \\ &\leq \tfrac{1}{n^3} \end{aligned}$$

for $\log\log n > 4$ and $W > 1$. ∎

Now, we have

$$\begin{aligned}
&Prob\,[\exists \text{ an edge } e \text{ with congestion}(e) > W \log(n)] \\
&\qquad = \quad Prob\,[\textstyle\bigcup_{e\in E} \text{congestion}(e) > W \log n] \\
&\qquad = \quad \textstyle\sum_e Prob\,[\text{congestion}(e) > W \log n] \\
&\qquad \leq \quad \textstyle\sum_e \frac{1}{n^3} \\
&\qquad = \quad \frac{|E|}{n^3} \leq \frac{1}{n}.
\end{aligned}$$

Thus we have the required result

$$Prob\left[\tilde{W} > W \log(n)\right] \leq \frac{1}{n}, \forall W > 1.$$