Ramsey numbers of stars versus wheels of similar sizes

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Abstract

We study the Ramsey number $R(W_m, S_n)$ for a star $S_n$ on $n$ vertices and a wheel $W_m$ on $m + 1$ vertices. We show that the Ramsey number $R(W_m, S_n) = 3n - 2$ for $n = m, m + 1,$ and $m + 2,$ where $m \geq 7$ and odd. In addition, we give the following lower bound for $R(W_m, S_n)$ where $m$ is even: $R(W_m, S_n) \geq 2n + 1$ for all $n \geq m \geq 6.$

Keywords: Ramsey number; Star; Wheel

1. Introduction

For two graphs $G$ and $H$, the Ramsey number $R(G, H)$ is the smallest positive integer $r$ such that for every graph $F$ on $r$ vertices, $F$ contains $G$ as a subgraph or the complement of $F$ contains $H$ as a subgraph.

In this paper, we study the Ramsey number $R(W_m, S_n)$ of wheels versus stars. A wheel $W_m$ is the graph on $m + 1$ vertices obtained from a cycle $C_m$ on $m$ vertices by adding one vertex $o$, called the hub of the wheel, and making $o$ adjacent to all vertices of $C_m$, called the rim of the wheel. A star $S_n$ is the graph on $n$ vertices with one vertex of degree $n - 1$, called the center, and $n - 1$ vertices of degree 1.

It was shown in [5] by Surahmat et al. that $R(W_m, S_n) = 3n - 2$ for $n \geq 2m - 4$, where $m \geq 5$ and odd. It was also shown in [4] that $R(W_4, S_n) = 2n - 1$ if $n \geq 3$ and odd, $R(W_4, S_n) = 2n + 1$ if $n \geq 4$ and even, and $R(W_5, S_n) = 3n - 2$ for each $n \geq 3.$ Baskoro et al. have also shown

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in [1] that $R(W_4, T_n) = 2n - 1$ for $n \geq 4$ and $R(W_5, T_n) = 3n - 2$ for $n \geq 3$ for any tree $T_n$ on $n$ vertices that is not a star.

In this paper we prove that $R(W_m, S_n) = 3n - 2$ for $m = n, m + 1,$ and $m + 2,$ where $m \geq 7$ and odd. In particular, this completes the calculation that $R(W_7, S_n) = 3n - 2$ for each $n \geq 7.$ In addition, we give the following lower bound: $R(W_m, S_n) \geq 2n + 1$ for all $n \geq m \geq 6$ and $m$ even.

2. Background

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G).$ For $v \in V(G)$ and $B \subset V(G),$ define $N_B(v) = \{ y \in B : vy \in E(G) \}.$ Define the degree of $v$ with respect to $B$ to be $|N_B(v)|$ and denote it by $D_B(v).$ If $B$ consists of the entire vertex set of the graph $G$ (i.e. $B = V(G)$), we use the conventional $d_G(v)$ instead of $D_{V(G)}(v)$.

Let $\overline{G}$ denote the complement of $G,$ i.e. the graph obtained from the complete graph on the vertices of $G$ by deleting the edges of $G.$

Chvátal and Harary [2] established the following lower bound for Ramsey numbers:

$$R(G, H) \geq (\chi(G) - 1) \cdot (c(H) - 1) + 1,$$

where $\chi(G)$ is the chromatic number of $G$ and $c(H)$ is the number of vertices in the largest connected component of $H.$

Corollary 1. $R(W_{2k+1}, S_n) \geq 3n - 2$ for $n \geq 2k + 1.$

The inequality follows directly from the Chvátal and Harary bound and the facts that $\chi(W_{2k+1}) = 4$ and $c(S_n) = n.$

Corollary 2. $R(W_{2k}, S_n) \geq 2n - 1$ for $n \geq 2k.$

The inequality here follows directly from the Chvátal and Harary bound and the facts that $\chi(W_{2k}) = 3$ and $c(S_n) = n.$

The following well-known theorem [3] is useful throughout the paper:

**Dirac’s Theorem.** Every graph with $n \geq 3$ vertices and minimum degree at least $n/2$ has a Hamiltonian cycle.

3. $R(W_n, S_n) = 3n - 2$ when $n$ is odd

**Theorem 3.** $R(W_{2k+1}, S_{2k+1}) = 6k + 1$ for $k \geq 3.$

**Proof.** Corollary 1 yields $R(W_{2k+1}, S_{2k+1}) \geq 3 \cdot (2k + 1) - 2 = 6k + 1.$ Therefore, it suffices to prove that $R(W_{2k+1}, S_{2k+1}) \leq 6k + 1.$
Let $F$ be an arbitrary graph on $6k + 1$ vertices. Assume that $W_{2k+1} \not\subseteq F$ and $S_{2k+1} \not\subseteq F$. Then $d_F(v) \geq 6k - (2k - 1) = 4k + 1$ for all $v \in V(F)$. Since the number of vertices in $F$ is odd, not all vertices of $F$ can have odd degree; thus, there exists a vertex $o$ such that $d_F(o) \geq 4k + 2$. Let $A$ be a subset of $N_{V(F)}(o)$ such that $|A| = 4k + 2$. Let $P = V(F) - o - A$, so that $|P| = 2k - 2$.

Observe that for all $a \in A$ we have $\delta_A(a) \geq d_F(a) - (|P| + 1)$. Therefore,

$$\delta_A(a) \geq 4k + 1 - (2k - 2 + 1) = 2k + 2.$$  \hfill (1)

By Dirac’s Theorem the latter inequality implies that $A$ contains a Hamiltonian cycle $\mathcal{H}$ on $4k + 2$ vertices.

Let $x$ be an arbitrary vertex in $A$. Number the other vertices in $A$ depending on their position with respect to $x$ on the cycle $\mathcal{H}$. More precisely, let $x, c_{2k+1}, c_{2k}, c_{2k-1}, \ldots, c_2, c_1, c_0$ be the vertices encountered in order when $\mathcal{H}$ is traversed starting at $x$ and going in one direction around the cycle. Let $x, b_0, b_1, b_2, \ldots, b_{2k-1}, b_{2k}, b_{2k+1}$ be the vertices encountered when the Hamiltonian cycle $\mathcal{H}$ is traversed starting at $x$ and going in the other direction around the cycle $\mathcal{H}$. See Fig. 1. Note that $c_0 = b_{2k-1}, c_1 = b_{2k},$ and $c_2 = b_{2k+1}$.

We now interrupt the proof of Theorem 3 in order to prove several needed observations.

**Observation 4.** For all $i$ with $1 \leq i \leq 2k$, the vertex $x$ is connected to at most one of $b_i$ and $c_i$.

**Proof.** If, for some $i$, the vertex $x$ is connected to both $b_i$ and $c_i$, then $x, b_i, b_{i+1}, b_{i+2}, \ldots, b_{2k-1}, c_i, c_{i+1}, \ldots, c_i$.
form a cycle $C$. Moreover, it is a cycle on $2k + 1$ vertices. Together $C$ and vertex $o$ form a copy of $W_{2k+1}$ contained in $F$. The inclusion $W_{2k+1} \subseteq F$ contradicts our assumption that $W_{2k+1} \not\subseteq F$. □

**Observation 5.** The vertex $x$ is not connected to vertex $c_0 = b_{2k-1}$.

**Proof.** Otherwise $x$, $b_0$, $b_1$, ..., $b_{2k-1}$ together with $o$ form $W_{2k+1} \subseteq F$. □

**Observation 6.** The vertex $x$ is not connected to vertex $b_{2k+1} = c_2$.

**Proof.** Otherwise $x$, $c_2$, $c_3$, ..., $c_{2k}$, $c_{2k+1}$ together with $o$ form $W_{2k+1} \subseteq F$. □

**Observation 7.** The vertex $x$ is connected to both $b_2$ and $c_{2k-1}$.

**Proof.** If $x$ is connected to at most one vertex among $b_2$ and $c_{2k-1}$, then by Observations 4–6 we have that

$$D_A(x) = \left(\sum_{i=1}^{2k-1} D_{b_i, c_i}(x)\right) + D_{b_0, c_{2k}, c_{2k+1}}(x)$$

$$\leq ((2k - 3) + D_{b_2, c_2}(x) + D_{b_{2k-1}, c_{2k-1}}(x)) + 3$$

$$\leq 2k + (D_{b_2}(x) + D_{c_{2k-1}}(x))$$

$$\leq 2k + 1.$$  

This inequality contradicts (1), and therefore $x$ must be connected to both $b_2$ and $c_{2k-1}$. □

**Observation 8.** $x$ is connected to both $b_1$ and $c_{2k}$.

**Proof.** Assume $x$ is connected to at most one vertex among $b_1$, $c_1$, and $c_{2k}$. Then, by Observation 4,

$$D_A(x) = \left(\sum_{i=1}^{2k-1} D_{b_i, c_i}(x)\right) + D_{b_0, c_{2k}, c_{2k+1}}(x)$$

$$= \left(\sum_{i=2}^{2k-1} D_{b_i, c_i}(x)\right) + D_{b_1, c_{2k}}(x) + D_{b_0, c_{2k+1}}(x)$$

$$\leq (2k - 2) + 1 + 2 = 2k + 1.$$  

This inequality contradicts (1). Thus $x$ is connected to at least two vertices among $b_1$, $c_1 = b_{2k}$, and $c_{2k}$. Observation 4 implies this arrangement is possible if and only if $x$ is connected to both $b_1$ and $c_{2k}$ but not to $c_1$. □

**Proof of Theorem 3 (continued).** Let $a_1$, $a_2$, ..., $a_{4k+2}$ be the vertices of $A$ numbered in the order that the Hamiltonian cycle $H$ traverses them. Let $a_i$ and $a_j$ stand for the same
vertex if \( i \equiv j \pmod{4k + 2} \). It follows from Observations 7 and 8 that, for all \( i \), the vertex \( a_i \) is connected to vertices \( a_{i+3}, a_{i-1}, a_{i+2}, \) and \( a_{i-2} \). Therefore, we can conclude that

\[
a_1, a_3, a_6, a_9, \ldots, a_{3i}, \ldots, a_{3(k+1)}, a_{3k+4}, a_{3k+5}, a_{3k+6}, \ldots, a_{4k+1}, a_{4k+2}
\]

is a cycle on \( 2k + 1 \) vertices of \( A \). This cycle, together with \( o \), forms \( W_{2k+1} \subset F \). The latter contradicts our assumption that \( W_{2k+1} \notin F \). Thus \( R(W_{2k+1}, S_{2k+1}) \leq 6k + 1 \), as desired. \( \square \)

4. \( R(W_{n-2}, S_n) = 3n - 2 \) when \( n \) is odd

**Theorem 9.** \( R(W_{2k+1}, S_{2k+3}) = 6k + 7 \) for \( k \geq 3 \).

**Proof.** Corollary 1 yields \( R(W_{2k+1}, S_{2k+3}) \geq 3 \cdot (2k + 3) - 2 = 6k + 7 \). Therefore, it suffices to prove that \( R(W_{2k+1}, S_{2k+3}) \leq 6k + 7 \).

Let \( F \) be an arbitrary graph on \( 6k + 7 \) vertices. Assume that \( W_{2k+1} \notin F \) and \( S_{2k+3} \notin F \). Then

\[
d_F(v) \geq (6k + 6) - (2k + 1) = 4k + 5
\]

for all \( v \in V(F) \).

It follows from Theorem 3 that \( R(W_{2k+3}, S_{2k+3}) = 6k + 7 \). By assumption, \( S_{2k+3} \notin \overline{F} \), and therefore \( W_{2k+3} \subset F \). Let \( o \) be the vertex of \( F \) that corresponds to the hub of \( W_{2k+3} \), and let \( Z = \{z_1, z_2, \ldots, z_{2k+3}\} \) be the set of vertices of \( F \) forming the rim of \( W_{2k+3} \) in order. Let \( z_i \) and \( z_j \) stand for the same vertex if \( i \equiv j \pmod{2k + 3} \).

Since \( d_F(o) \geq 4k + 5 \) according to (2), therefore there exists a subset \( A = \{a_1, \ldots, a_{2k+1}\} \) of \( V(F) \) such that \( a_i \neq z_j \) for all \( i \) and \( j \), and \( a_i \) is connected to \( o \) for all \( i \). Observe that we have chosen \( A \) so that \(|A| = 2k + 1\), even though it is also possible to choose \( A \) so that \(|A| = 2k + 2\).

We now interrupt the proof of Theorem 9 in order to prove several needed observations.

**Observation 10.** For all \( i \), if \( 1 \leq x \leq 2k + 1 \), then the vertex \( a_x \) is connected to at most one of \( z_i \) and \( z_{i+4} \).

**Proof.** If the vertex \( a_x \) is connected to both \( z_i \) and \( z_{i+4} \) for some \( x \) and for some \( i \), then \( z_1, z_2, \ldots, z_i, a_x, z_{i+4}, z_{i+5}, \ldots, z_{2k+3} \) form a cycle on \( 2k + 1 \) vertices. This cycle, together with \( o \), forms \( W_{2k+1} \subset F \). The latter contradicts our assumption that \( W_{2k+1} \notin F \). \( \square \)

**Observation 11.** \( \text{d}_Z(v) \leq k + 1 \) for all \( v \in A \).

**Proof.** Let \( v \in A \) be arbitrary. Consider two cases.

**Case 1:** Assume that \( k \) is even. Write \( k = 2a \), so that \(|Z| = 2k + 3 = 4a + 3\). Consider the sequence \( \{s_i\} \) defined inductively by \( s_1 = 1 \) and \( s_{i+1} \equiv s_i + 4 \pmod{4a + 3} \). Clearly, this sequence is periodic with a period of \( 4a + 3 \); indeed, the sequence is

\[
1, 5, 9, \ldots, 4a + 1, 2, 6, \ldots, 4a + 2, 3, 7, \ldots, 4a + 3, 4, 8, \ldots, 4a, 1, \ldots
\]
By Observation 10, \( v \) is connected to at most one of \( z_{s_t} \) and \( z_{s_t+1} \) for all \( t \). Therefore, \( v \) is connected to at most \( \left\lfloor \frac{4a+5}{2} \right\rfloor \) vertices in \( Z \). Since \( \left\lfloor \frac{4a+5}{2} \right\rfloor = 2a + 1 = k + 1 \), we have \( \mathcal{D}_Z(v) \leq k + 1 \) for all \( v \in A \).

Case 2: Assume that \( k \) is odd. Write \( k = 2a + 1 \), so that \( |Z| = 2k + 3 = 4a + 5 \). Consider the sequence \( \{s_t\} \) defined inductively by \( s_1 = 1 \) and \( s_{t+1} = s_t + 4 \mod (4a + 5) \). Clearly, this sequence is periodic with a period of \( 4a + 5 \); indeed, the sequence is

\[
1, 5, 9, \ldots, 4a + 5, 4, 8, \ldots, 4a + 4, 3, 7, \ldots, 4a + 3, 2, 6, \ldots, 4a + 2, 1, \ldots
\]

By Observation 10, \( v \) is connected to at most one of \( z_{s_t} \) and \( z_{s_t+1} \) for all \( t \). Therefore, \( v \) is connected to at most \( \left\lfloor \frac{4a+5}{2} \right\rfloor \) vertices in \( Z \). Since \( \left\lfloor \frac{4a+5}{2} \right\rfloor = 2a + 2 = k + 1 \), we have \( \mathcal{D}_Z(v) \leq k + 1 \) for all \( v \in A \). □

**Proof of Theorem 9 (continued).** Let \( P = V(F) - o - Z - A \). By construction,

\[
|P| = (6k + 7) - 1 - (2k + 3) - (2k + 1) = 2k + 2.
\]

Observe that, for all \( a \in A \), we have \( \mathcal{D}_A(a) \geq d_F(a) - (\mathcal{D}_Z(a) + |P| + 1) \). Therefore, by Observation 11,

\[
\mathcal{D}_A(a) \geq (4k + 5) - ((k + 1) + (2k + 2) + 1) = k + 1.
\]

Dirac’s Theorem combined with the latter inequality implies that \( A \) contains a Hamiltonian cycle \( \mathcal{C} \) on \( 2k + 1 \) vertices. Together \( \mathcal{C} \) and \( o \) form \( W_{2k+1} \subset F \). The latter contradicts our assumption that \( W_{2k+1} \not\subset F \). Thus, \( R(W_{2k+1}, S_{2k+3}) \leq 6k + 7 \). The proof is now complete. □

**5.** \( R(W_n, S_n) \leq 3n - 3 \) when \( n \) is even

**Theorem 12.** \( R(W_{2k}, S_{2k}) \leq 6k - 3 \) for \( k \geq 2 \).

**Proof.** For \( k = 2 \) the theorem follows from \( R(W_4, S_4) = 9 \), proved in [4]. Consider the case \( k > 2 \).

Let \( F \) be an arbitrary graph on \( 6k - 3 \) vertices. Assume that \( W_{2k} \not\subset F \) and \( S_{2k} \not\subset \overline{F} \). Then \( d_F(v) \geq (6k - 4) - (2k - 2) = 4k - 2 \) for all \( v \in V(F) \). Let \( o \) be an arbitrary vertex of \( F \) and let \( A \) be a subset of \( N_{V(F)}(o) \) such that \( |A| = 4k - 2 \). Let \( P = V(F) - o - A \), so that \( |P| = 2k - 2 \).

Observe that, for all \( a \in A \), we have \( \mathcal{D}_A(a) \geq d_F(a) - (|P| + 1) \). Therefore,

\[
\mathcal{D}_A(a) \geq (4k - 2) - (2k - 2 + 1) = 2k - 1.
\]
By Dirac’s Theorem, the latter inequality implies that $A$ contains a Hamiltonian cycle $H$ on $4k - 2$ vertices.

Let $x \in A$ be arbitrary. Number the other vertices in $A$ depending on their position with respect to $x$ on the cycle $H$. More precisely, let $x, c_{2k-2}, c_{2k-3}, \ldots, c_1, c_0$ be the vertices encountered in order when $H$ is traversed starting at $x$ and going in one direction around the cycle. Let $x, b_0, b_1, b_2, \ldots, b_{2k-3}, b_{2k-2}$ be the vertices encountered when the Hamiltonian cycle $H$ is traversed starting at $x$ and going in the other direction around the cycle $H$. See Fig. 2. Note that $c_0 = b_{2k-2}$.

We now interrupt the proof of Theorem 12 in order to prove several needed observations. The proofs of the first two observations are similar to those of Observations 4 and 5, and so they are omitted.

**Observation 13.** For all $i$ with $1 \leq i \leq 2k - 3$, the vertex $x$ is connected to at most one of $b_i$ and $c_i$.

**Observation 14.** The vertex $x$ is not connected to $c_0 = b_{2k-2}$.

We proceed to prove the following observation:

**Observation 15.** For all $i$ with $1 \leq i \leq 2k - 3$, the vertex $x$ is connected to precisely one of $b_i$ and $c_i$. 
Theorem 16. This inequality contradicts (3).

Proof. If, for some \( i \) such that \( 1 \leq i \leq 2k - 3 \), the vertex \( x \) is not connected to \( b_i \) and also not connected to \( c_i \), then by Observations 13 and 14, we have

\[
\mathcal{D}_A(x) = \left( \sum_{j=1}^{2k-3} \mathcal{D}_{b_j,c_j}(x) \right) + \mathcal{D}_{b_0,c_{2k-2}}(x) + \mathcal{D}_{b_{2k-2}}(x) \\
\leq ((2k - 4) + 2 + 0) = 2k - 2.
\]

This inequality contradicts (3). \( \square \)

Proof of Theorem 12 (continued). Let \( a_1, a_2, \ldots, a_{4k-2} \) be the vertices of \( A \) numbered in the order that the Hamiltonian cycle \( \mathcal{H} \) traverses them. Let \( a_i \) and \( a_j \) stand for the same vertex if \( i \equiv j \pmod{4k - 2} \). It follows from Observation 15 that, for all \( j \), the vertex \( a_j \) is connected to precisely one of \( a_{j+k} \) and \( a_{j-k} \) (choose \( x = a_j \) and \( i = k - 1 \) in the Observation 15).

In particular, \( a_{k+1} \) is connected to precisely one of \( a_1 \) or \( a_{2k+1} \).

Suppose \( a_{k+1} \) is connected to \( a_1 \) and is therefore not connected to \( a_{2k+1} \). Then \( a_{2k+1} \) must be connected to \( a_{3k+1} \) and so

\[
a_1, a_{k+1}, a_{k+2}, \ldots, a_{2k+1}, a_{3k+1}, a_{3k+2}, \ldots, a_{4k-2}
\]

is a cycle on \( 2k \) vertices.

On the other hand, suppose \( a_{k+1} \) is connected to \( a_{2k+1} \) and is therefore not connected to \( a_1 \). Then \( a_1 \) must be connected to \( a_{3k-1} \) and so

\[
a_1, a_2, \ldots, a_{k+1}, a_{2k+1}, a_{2k+2}, \ldots, a_{3k-2}, a_{3k-1}
\]

is a cycle on \( 2k \) vertices.

In both cases, the obtained cycle on \( 2k \) vertices forms the rim, and \( o \) forms the hub of \( W_{2k} \subset F \). The latter contradicts our assumption that \( W_{2k} \nsubseteq F \). The proof is now complete. \( \square \)

6. \( R(W_{n-1}, S_n) = 3n - 2 \) when \( n \) is even

Theorem 16. \( R(W_{2k+1}, S_{2k+2}) = 6k + 4 \) for \( k \geq 3 \).

Proof. Corollary 1 yields \( R(W_{2k+1}, S_{2k+2}) \geq 3 \cdot (2k + 2) - 2 = 6k + 4 \). Therefore, it suffices to prove that \( R(W_{2k+1}, S_{2k+2}) \leq 6k + 4 \).

Let \( F \) be an arbitrary graph on \( 6k + 4 \) vertices. Assume that \( W_{2k+1} \nsubseteq F \) and \( S_{2k+2} \nsubseteq \overline{F} \). Then \( d_F(v) \geq (6k + 3) - 2k = 4k + 3 \) for all \( v \in V(F) \).

Theorem 12 implies that \( R(W_{2k+2}, S_{2k+2}) \leq 6k + 3 \). By assumption we have \( S_{2k+2} \nsubseteq \overline{F} \), and therefore \( W_{2k+2} \subset F \). Let \( o \) be the vertex of \( F \) that corresponds to the hub of \( W_{2k+2} \), and let \( Z = \{z_1, z_2, \ldots, z_{2k+2}\} \) be the set of vertices of \( F \) forming the rim of \( W_{2k+2} \) in order. Let \( z_i \) and \( z_j \) stand for the same vertex if \( i \equiv j \pmod{2k+2} \).
Since \( d_F(o) \geq 4k + 3 \), there exists a subset \( A = \{a_1, \ldots, a_{2k+1}\} \) of \( V(F) \) such that \( a_i \neq z_j \) for all \( i \) and \( j \), and \( a_i \) is connected to \( o \) for all \( i \). Observe that we have chosen \( A \) so that \( |A| = 2k + 1 \).

We now interrupt the proof of Theorem 16 in order to prove several needed observations. The proof of the next observation is similar to that of Observation 10, and so it is omitted.

**Observation 17.** For all \( i \), if \( 1 \leq x \leq 2k + 1 \), then the vertex \( a_i \) is connected to at most one of \( z_i \) and \( z_{i+3} \).

We proceed to prove another observation, which is similar to Observation 11.

**Observation 18.** \( D_Z(v) \leq k + 1 \) for all \( v \in A \).

**Proof.** Let \( v \in A \) be arbitrary. Consider three cases.

- **Case 1:** Assume that \( 3 \mid k \). Write \( k = 3a \), so that \( |Z| = 2k + 2 = 6a + 2 \). Now, consider the sequence \( \{s_t\} \) defined inductively by \( s_1 = 1 \) and \( s_{t+1} = s_t + 3 \mod 6a + 2 \). Clearly, this sequence is periodic with a period of \( 6a + 2 \); indeed, the sequence is 
  
  \[ 1, 4, 7, \ldots, 6a + 1, 2, 5, \ldots, 6a + 2, 3, 6, \ldots, 6a, 1 \ldots \]

  By Observation 17, \( v \) is connected to at most one of \( z_{s_t} \) and \( z_{s_{t+1}} \) for all \( t \). Therefore, \( v \) is connected to at most \( \left\lfloor \frac{6a+2}{2} \right\rfloor \) vertices in \( Z \). Since \( \left\lfloor \frac{6a+2}{2} \right\rfloor = 3a + 1 = k + 1 \), we have \( D_Z(v) \leq k + 1 \) for all \( v \in A \).

- **Case 2:** Assume that \( k \equiv 1 \mod 3 \). Write \( k = 3a + 1 \), so that \( |Z| = 2k + 2 = 6a + 4 \). Consider the sequence \( \{s_t\} \) defined inductively by \( s_1 = 1 \) and \( s_{t+1} \equiv s_t + 3 \mod 6a + 4 \). Clearly, this sequence is periodic with a period of \( 6a + 4 \); indeed, the sequence is 
  
  \[ 1, 4, 7, \ldots, 6a + 4, 3, 6, \ldots, 6a + 3, 2, 5, \ldots, 6a + 2, 1 \ldots \]

  By Observation 17, \( v \) is connected to at most one of \( z_{s_t} \) and \( z_{s_{t+1}} \) for all \( t \). Therefore, \( v \) is connected to at most \( \left\lfloor \frac{6a+4}{2} \right\rfloor \) vertices in \( Z \). Since \( \left\lfloor \frac{6a+4}{2} \right\rfloor = 3a + 2 = k + 1 \), we have \( D_Z(v) \leq k + 1 \) for all \( v \in A \).

- **Case 3:** Assume that \( k \equiv 2 \mod 3 \). Write \( k = 3a + 2 \), so that \( |Z| = 2k + 2 = 6a + 6 \). Consider the following three sequences \( \{s_{1,t}\}, \{s_{2,t}\}, \) and \( \{s_{3,t}\} \), defined inductively by \( s_{1,1} = i \) and \( s_{1,t+1} \equiv s_{1,t} + 3 \mod 6a + 6 \), for \( i = 1, 2, 3 \). Clearly, each of the sequences \( \{s_{1,t}\}, \{s_{2,t}\}, \) and \( \{s_{3,t}\} \) is periodic with a period of \( 2a + 2 \); indeed, the sequence \( \{s_{1,t}\} \) is \( 1, 4, 7, \ldots, 6a + 4, 1, \ldots \), the sequence \( \{s_{2,t}\} \) is \( 2, 5, \ldots, 6a + 5, 2, \ldots \), and the sequence \( \{s_{3,t}\} \) is \( 3, 6, \ldots, 6a + 3, 1, \ldots \). By Observation 17, for \( i = 1, 2, 3 \), the vertex \( v \) is connected to at most one of \( z_{s_{i,t}} \) and \( z_{s_{i,t+1}} \) for all \( t \). Therefore, \( v \) is connected to at most \( 3 \cdot \left\lfloor \frac{2a+2}{2} \right\rfloor \) vertices in \( Z \). Since \( 3 \cdot \left\lfloor \frac{2a+2}{2} \right\rfloor = 3 \cdot (a + 1) = 3a + 3 = k + 1 \), we have \( D_Z(v) \leq k + 1 \) for all \( v \in A \). \( \square \)

**Proof of Theorem 16 (continued).** Let \( P = V(F) - o - Z - A \). By construction, \( |P| = (6k + 4) - 1 - (2k + 2) - (2k + 1) = 2k \). Observe that, for all \( a \in A \), we have \( D_A(a) \geq d_F(a) - (D_Z(a) + |P| + 1) \). Therefore, by Observation 18,

\[
D_A(a) \geq (4k + 3) - ((k + 1) + (2k + 1)) = k + 1.
\]
By Dirac’s Theorem, the latter inequality implies that $A$ contains a Hamiltonian cycle $C$ on $2k + 1$ vertices. Together $C$ and $o$ form $W_{2k+1} \subset F$. The latter contradicts our assumption that $W_{2k+1} \not\subset F$. Thus $R(W_{2k+1}, S_{2k+2}) \leq 6k + 4$. The proof is now complete. □

7. $R(W_m, S_n) \geq 2n + 1$ when $m$ is even and $n \geq m \geq 6$

**Theorem 19.** $R(W_{2k}, S_{2k+i}) \geq 4k + 2i + 1$ for all $i \geq 0$ and $k \geq 3$.

**Proof.** Construct a graph $F$ on $4k + 2i$ vertices as follows. Divide the vertices of $F$ into two disjoint sets: $A = \{a_1, a_2, \ldots, a_{2k+i-1}\}$ and $B = \{b_1, b_2, \ldots, b_{2k+i+1}\}$. Let $F$ have the following edges: $a_t$ is connected to $b_j$ for all $t$ and $j$. In addition, $b_t$ is connected to $b_{t+1}$ for all $1 \leq t \leq 2k + i$, and $b_{2k+i+1}$ is connected to $b_1$.

Clearly, we have $d_F(a_t) = d_F(b_j) = 2k + i + 1$ for all $t$ and $j$. Since $(4k + 2i - 1) - (2k + i + 1) = 2k + i - 2$, we have $S_{2k+i} \not\subset F$. Moreover, we can verify that $W_{2k} \not\subset F$. Indeed, for all $t$, the vertex $a_t$ cannot be a hub of $W_{2k}$ since there is no cycle of length $2k$ among the vertices of $B$. In addition, for all $j$, the vertex $b_j$ cannot be a hub of $W_{2k}$ since there is no cycle of length $2k$ among the vertices $b_{j+1}, b_{j-1}$ and $a_1, \ldots, a_{2k+i-1}$, unless $2k \leq 4$.

Thus, we have constructed a graph $F$ on $4k + 2i$ vertices, such that $W_{2k} \not\subset F$ and $S_{2k+i} \not\subset F$. Therefore, $R(W_{2k}, S_{2k+i}) > 4k + 2i$. □

This improves the lower bound given in Corollary 2 by two.

8. Conclusion

In this paper we showed that $R(W_m, S_n) = 3n - 2$ for $n = m, m + 1, \text{ and } m + 2$, where $m \geq 7$ and odd. Combined with the previously known result that $R(W_m, S_n) = 3n - 2$ for $n \geq 2m - 4$, where $m \geq 5$ and odd, this leads to a conjecture that $R(W_m, S_n) = 3n - 2$ for all $n \geq m, m \geq 5$, and $m$ odd.

The exact value for $R(W_m, S_n)$ when $m \geq 6$ and even remains an open problem.

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