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Ramsey numbers of stars versus wheels of similar sizes

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Abstract

We study the Ramsey number $R(W_m, S_n)$ for a star S_n on n vertices and a wheel W_m on $m+1$ vertices. We show that the Ramsey number $R(W_m, S_n) = 3n-2$ for $n = m, m+1$, and $m+2$, where $m \geqslant 7$ and odd. In addition, we give the following lower bound for $R(W_m, S_n)$ where m is even: $R(W_m, S_n) \geqslant 2n+1$ for all $n \geqslant m \geqslant 6$.
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1. Introduction

For two graphs G and H, the *Ramsey number* $R(G, H)$ is the smallest positive integer r such that for every graph F on r vertices, F contains G as a subgraph or the complement of F contains H as a subgraph.

In this paper, we study the Ramsey number $R(W_m, S_n)$ of wheels versus stars. A *wheel* W_m is the graph on $m+1$ vertices obtained from a cycle C_m on m vertices by adding one vertex o, called the *hub* of the wheel, and making o adjacent to all vertices of C_m, called the *rim* of the wheel. A *star* S_n is the graph on n vertices with one vertex of degree $n-1$, called the *center*, and $n-1$ vertices of degree 1.

It was shown in [5] by Surahmat et al. that $R(W_m, S_n) = 3n-2$ for $n \geqslant 2m-4$, where $m \geqslant 5$ and odd. It was also shown in [4] that $R(W_4, S_n) = 2n-1$ if $n \geqslant 3$ and odd, $R(W_4, S_n) = 2n+1$ if $n \geqslant 4$ and even, and $R(W_5, S_n) = 3n-2$ for each $n \geqslant 3$. Baskoro et al. have also shown

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in [1] that $R(W_4, T_n) = 2n - 1$ for $n \geqslant 4$ and $R(W_5, T_n) = 3n - 2$ for $n \geqslant 3$ for any tree T_n on n vertices that is not a star.

In this paper we prove that $R(W_m, S_n) = 3n - 2$ for $n = m, m + 1$, and $m + 2$, where $m \geqslant 7$ and odd. In particular, this completes the calculation that $R(W_7, S_n) = 3n - 2$ for each $n \geqslant 7$. In addition, we give the following lower bound: $R(W_m, S_n) \geqslant 2n + 1$ for all $n \geqslant m \geqslant 6$ and m even.

2. Background

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$ and $B \subset V(G)$, define $N_B(v) = \{y \in B : vy \in E(G)\}$. Define the *degree of v with respect to B* to be $|N_B(v)|$ and denote it by $\mathscr{D}_B(v)$. If B consists of the entire vertex set of the graph G (i.e. $B = V(G)$), we use the conventional $d_G(v)$ instead of $\mathscr{D}_{V(G)}(v)$.

Let $\overline{G}$ denote the complement of G, i.e. the graph obtained from the complete graph on the vertices of G by deleting the edges of G.

Chvátal and Harary [2] established the following lower bound for Ramsey numbers:

$$R(G, H) \geqslant (\mathscr{X}(G) - 1) \cdot (c(H) - 1) + 1,$$

where $\mathscr{X}(G)$ is the chromatic number of G and $c(H)$ is the number of vertices in the largest connected component of H.

Corollary 1. *$R(W_{2k+1}, S_n) \geqslant 3n - 2$ for $n \geqslant 2k + 1$.*

The inequality follows directly from the Chvátal and Harary bound and the facts that $\mathscr{X}(W_{2k+1}) = 4$ and $c(S_n) = n$.

Corollary 2. *$R(W_{2k}, S_n) \geqslant 2n - 1$ for $n \geqslant 2k$.*

The inequality here follows directly from the Chvátal and Harary bound and the facts that $\mathscr{X}(W_{2k}) = 3$ and $c(S_n) = n$.

The following well-known theorem [3] is useful throughout the paper:

Dirac's Theorem. *Every graph with $n \geqslant 3$ vertices and minimum degree at least $n/2$ has a Hamiltonian cycle.*

3. $R(W_n, S_n) = 3n - 2$ when n is odd

Theorem 3. *$R(W_{2k+1}, S_{2k+1}) = 6k + 1$ for $k \geqslant 3$.*

Proof. Corollary 1 yields $R(W_{2k+1}, S_{2k+1}) \geqslant 3 \cdot (2k+1) - 2 = 6k + 1$. Therefore, it suffices to prove that $R(W_{2k+1}, S_{2k+1}) \leqslant 6k + 1$.

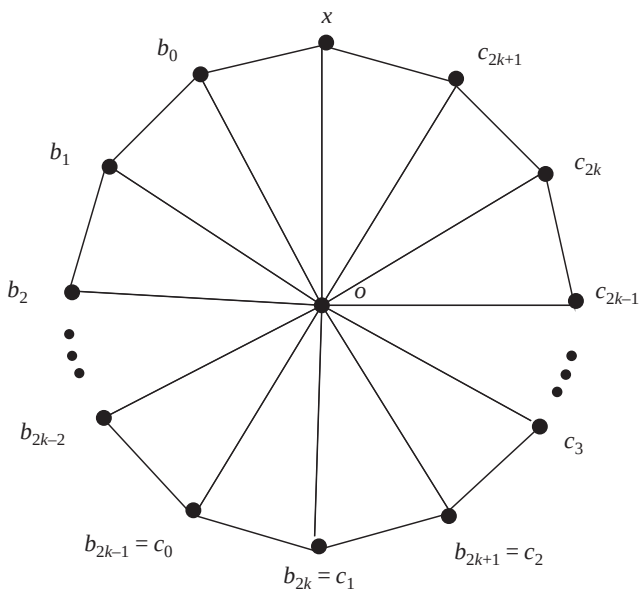

Fig. 1. Numbering of the $4k+2$ vertices of A depending on their position with respect to x on the cycle $\mathscr{H}$.

Let F be an arbitrary graph on $6k+1$ vertices. Assume that $W_{2k+1} \not\subset F$ and $S_{2k+1} \not\subset \overline{F}$. Then $d_F(v) \geqslant 6k-(2k-1) = 4k+1$ for all $v \in V(F)$. Since the number of vertices in F is odd, not all vertices of F can have odd degree; thus, there exists a vertex o such that $d_F(o) \geqslant 4k+2$. Let A be a subset of $N_{V(F)}(o)$ such that $|A| = 4k+2$. Let $P = V(F) - o - A$, so that $|P| = 2k-2$.

Observe that for all $a \in A$ we have $\mathscr{D}_A(a) \geqslant d_F(a) - (|P|+1)$. Therefore,

$$\mathscr{D}_A(a) \geqslant (4k+1) - (2k-2+1) = 2k+2. \tag{1}$$

By Dirac's Theorem the latter inequality implies that A contains a Hamiltonian cycle $\mathscr{H}$ on $4k+2$ vertices.

Let x be an arbitrary vertex in A. Number the other vertices in A depending on their position with respect to x on the cycle $\mathscr{H}$. More precisely, let $x, c_{2k+1}, c_{2k}, c_{2k-1}, \ldots, c_1, c_0$ be the vertices encountered in order when $\mathscr{H}$ is traversed starting at x and going in one direction around the cycle. Let $x, b_0, b_1, b_2, \ldots, b_{2k-1}, b_{2k}, b_{2k+1}$ be the vertices encountered when the Hamiltonian cycle $\mathscr{H}$ is traversed starting at x and going in the other direction around the cycle $\mathscr{H}$. See Fig. 1. Note that $c_0 = b_{2k-1}$, $c_1 = b_{2k}$, and $c_2 = b_{2k+1}$.

We now interrupt the proof of Theorem 3 in order to prove several needed observations.

Observation 4. *For all i with $1 \leqslant i \leqslant 2k$, the vertex x is connected to at most one of b_i and c_i.*

Proof. If, for some i, the vertex x is connected to both b_i and c_i, then

$$x, b_i, b_{i+1}, b_{i+2}, \ldots, b_{2k-1}, c_1, c_2, \ldots, c_i$$

form a cycle $\mathscr{C}$. Moreover, it is a cycle on $2k+1$ vertices. Together $\mathscr{C}$ and vertex o form a copy of W_{2k+1} contained in F. The inclusion $W_{2k+1} \subset F$ contradicts our assumption that $W_{2k+1} \not\subset F$. $\square$

Observation 5. *The vertex x is not connected to vertex $c_0 = b_{2k-1}$.*

Proof. Otherwise $x, b_0, b_1, \ldots, b_{2k-1}$ together with o form $W_{2k+1} \subset F$. $\square$

Observation 6. *The vertex x is not connected to vertex $b_{2k+1} = c_2$.*

Proof. Otherwise $x, c_2, c_3, \ldots, c_{2k}, c_{2k+1}$ together with o form $W_{2k+1} \subset F$. $\square$

Observation 7. *The vertex x is connected to both b_2 and c_{2k-1}.*

Proof. If x is connected to at most one vertex among b_2 and c_{2k-1}, then by Observations 4–6 we have that

$$\begin{aligned}\mathscr{D}_A(x) &= \left(\sum_{i=1}^{2k-1} \mathscr{D}_{b_i,c_i}(x)\right) + \mathscr{D}_{b_0,c_{2k},c_{2k+1}}(x)\\ &\leqslant ((2k-3) + \mathscr{D}_{b_2,c_2}(x) + \mathscr{D}_{b_{2k-1},c_{2k-1}}(x)) + 3\\ &\leqslant 2k + \left(\mathscr{D}_{b_2}(x) + \mathscr{D}_{c_{2k-1}}(x)\right)\\ &\leqslant 2k+1.\end{aligned}$$

This inequality contradicts (1), and therefore x must be connected to both b_2 and c_{2k-1}. $\square$

Observation 8. *x is connected to both b_1 and c_{2k}.*

Proof. Assume x is connected to at most one vertex among b_1, c_1, and c_{2k}. Then, by Observation 4,

$$\begin{aligned}\mathscr{D}_A(x) &= \left(\sum_{i=1}^{2k-1} \mathscr{D}_{b_i,c_i}(x)\right) + \mathscr{D}_{b_0,c_{2k},c_{2k+1}}(x)\\ &= \left(\sum_{i=2}^{2k-1} \mathscr{D}_{b_i,c_i}(x)\right) + \mathscr{D}_{b_1,c_1,c_{2k}}(x) + \mathscr{D}_{b_0,c_{2k+1}}(x)\\ &\leqslant (2k-2) + 1 + 2 = 2k+1.\end{aligned}$$

This inequality contradicts (1). Thus x is connected to at least two vertices among $b_1, c_1 = b_{2k}$, and c_{2k}. Observation 4 implies this arrangement is possible if and only if x is connected to both b_1 and c_{2k} but not to c_1. $\square$

Proof of Theorem 3 (*continued*). Let $a_1, a_2, \ldots, a_{4k+2}$ be the vertices of A numbered in the order that the Hamiltonian cycle $\mathscr{H}$ traverses them. Let a_i and a_j stand for the same

vertex if $i \equiv j \pmod{4k+2}$. It follows from Observations 7 and 8 that, for all i, the vertex a_i is connected to vertices $a_{i+3}, a_{i-3}, a_{i+2}$, and a_{i-2}. Therefore, we can conclude that

$$a_1, a_3, a_6, a_9, \ldots, a_{3i}, \ldots, a_{3(k+1)}, a_{3k+4}, a_{3k+5}, a_{3k+6}, \ldots, a_{4k+1}, a_{4k+2}$$

is a cycle on $2k+1$ vertices of A. This cycle, together with o, forms $W_{2k+1} \subset F$. The latter contradicts our assumption that $W_{2k+1} \not\subset F$. Thus $R(W_{2k+1}, S_{2k+1}) \leqslant 6k+1$, as desired. □

4. $R(W_{n-2}, S_n) = 3n-2$ when n is odd

Theorem 9. *$R(W_{2k+1}, S_{2k+3}) = 6k+7$ for $k \geqslant 3$.*

Proof. Corollary 1 yields $R(W_{2k+1}, S_{2k+3}) \geqslant 3\cdot(2k+3)-2 = 6k+7$. Therefore, it suffices to prove that $R(W_{2k+1}, S_{2k+3}) \leqslant 6k+7$.

Let F be an arbitrary graph on $6k+7$ vertices. Assume that $W_{2k+1} \not\subset F$ and $S_{2k+3} \not\subset \overline{F}$. Then

$$d_F(v) \geqslant (6k+6) - (2k+1) = 4k+5 \tag{2}$$

for all $v \in V(F)$.

It follows from Theorem 3 that $R(W_{2k+3}, S_{2k+3}) = 6k+7$. By assumption, $S_{2k+3} \not\subset \overline{F}$, and therefore $W_{2k+3} \subset F$. Let o be the vertex of F that corresponds to the hub of W_{2k+3}, and let $Z = \{z_1, z_2, \ldots, z_{2k+3}\}$ be the set of vertices of F forming the rim of W_{2k+3} in order. Let z_i and z_j stand for the same vertex if $i \equiv j \pmod{2k+3}$.

Since $d_F(o) \geqslant 4k+5$ according to (2), therefore there exists a subset $A = \{a_1, \ldots, a_{2k+1}\}$ of $V(F)$ such that $a_i \neq z_j$ for all i and j, and a_i is connected to o for all i. Observe that we have chosen A so that $|A| = 2k+1$, even though it is also possible to choose A so that $|A| = 2k+2$.

We now interrupt the proof of Theorem 9 in order to prove several needed observations.

Observation 10. *For all i, if $1 \leqslant x \leqslant 2k+1$, then the vertex a_x is connected to at most one of z_i and z_{i+4}.*

Proof. If the vertex a_x is connected to both z_i and z_{i+4} for some x and for some i, then $z_1, z_2, \ldots, z_i, a_x, z_{i+4}, z_{i+5}, \ldots, z_{2k+3}$ form a cycle on $2k+1$ vertices. This cycle, together with o, forms $W_{2k+1} \subset F$. The latter contradicts our assumption that $W_{2k+1} \not\subset F$. □

Observation 11. *$\mathscr{D}_Z(v) \leqslant k+1$ for all $v \in A$.*

Proof. Let $v \in A$ be arbitrary. Consider two cases.

Case 1: Assume that k is even. Write $k = 2a$, so that $|Z| = 2k+3 = 4a+3$. Consider the sequence $\{s_t\}$ defined inductively by $s_1 = 1$ and $s_{t+1} \equiv s_t + 4 \pmod{4a+3}$. Clearly, this sequence is periodic with a period of $4a+3$; indeed, the sequence is

$$1, 5, 9, \ldots, 4a+1, 2, 6, \ldots, 4a+2, 3, 7, \ldots, 4a+3, 4, 8, \ldots, 4a, 1, \ldots .$$

By Observation 10, v is connected to at most one of z_{s_t} and $z_{s_{t+1}}$ for all t. Therefore, v is connected to at most $\lfloor \frac{4a+3}{2} \rfloor$ vertices in Z. Since $\lfloor \frac{4a+3}{2} \rfloor = 2a+1 = k+1$, we have $\mathscr{D}_Z(v) \leqslant k+1$ for all $v \in A$.

Case 2: Assume that k is odd. Write $k = 2a+1$, so that $|Z| = 2k+3 = 4a+5$. Consider the sequence $\{s_t\}$ defined inductively by $s_1 = 1$ and $s_{t+1} \equiv s_t + 4 \pmod{4a+5}$. Clearly, this sequence is periodic with a period of $4a+5$; indeed, the sequence is

$$1, 5, 9, \ldots, 4a+5, 4, 8, \ldots, 4a+4, 3, 7, \ldots, 4a+3, 2, 6, \ldots, 4a+2, 1, \ldots .$$

By Observation 10, v is connected to at most one of z_{s_t} and $z_{s_{t+1}}$ for all t. Therefore, v is connected to at most $\lfloor \frac{4a+5}{2} \rfloor$ vertices in Z. Since $\lfloor \frac{4a+5}{2} \rfloor = 2a+2 = k+1$, we have $\mathscr{D}_Z(v) \leqslant k+1$ for all $v \in A$. $\square$

Proof of Theorem 9 (*continued*). Let $P = V(F) - o - Z - A$. By construction,

$$|P| = (6k+7) - 1 - (2k+3) - (2k+1) = 2k+2.$$

Observe that, for all $a \in A$, we have $\mathscr{D}_A(a) \geqslant d_F(a) - (\mathscr{D}_Z(a) + |P| + 1)$. Therefore, by Observation 11,

$$\mathscr{D}_A(a) \geqslant (4k+5) - ((k+1) + (2k+2) + 1) = k+1.$$

Dirac's Theorem combined with the latter inequality implies that A contains a Hamiltonian cycle $\mathscr{C}$ on $2k+1$ vertices. Together $\mathscr{C}$ and o form $W_{2k+1} \subset F$. The latter contradicts our assumption that $W_{2k+1} \not\subset F$. Thus, $R(W_{2k+1}, S_{2k+3}) \leqslant 6k+7$. The proof is now complete. $\square$

5. $R(W_n, S_n) \leqslant 3n-3$ when n is even

Theorem 12. $R(W_{2k}, S_{2k}) \leqslant 6k-3$ *for* $k \geqslant 2$.

Proof. For $k=2$ the theorem follows from $R(W_4, S_4) = 9$, proved in [4]. Consider the case $k > 2$.

Let F be an arbitrary graph on $6k-3$ vertices. Assume that $W_{2k} \not\subset F$ and $S_{2k} \not\subset \overline{F}$. Then $d_F(v) \geqslant (6k-4) - (2k-2) = 4k-2$ for all $v \in V(F)$. Let o be an arbitrary vertex of F and let A be a subset of $N_{V(F)}(o)$ such that $|A| = 4k-2$. Let $P = V(F) - o - A$, so that $|P| = 2k-2$.

Observe that, for all $a \in A$, we have $\mathscr{D}_A(a) \geqslant d_F(a) - (|P|+1)$. Therefore,

$$\mathscr{D}_A(a) \geqslant (4k-2) - (2k-2+1) = 2k-1. \tag{3}$$

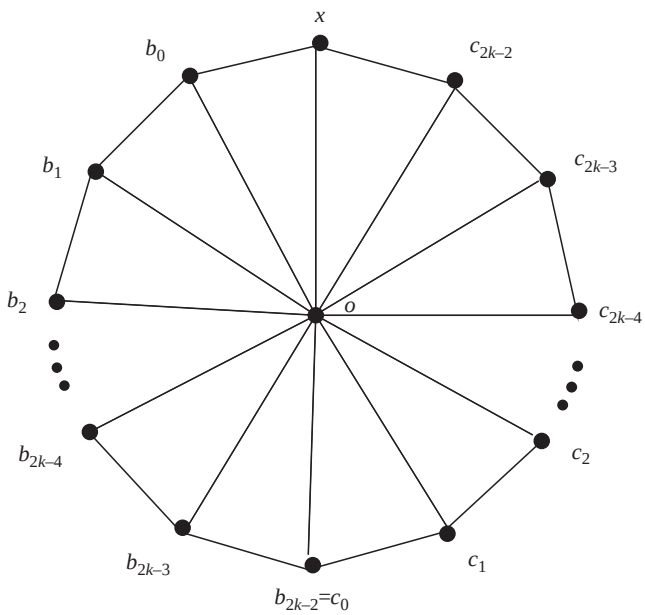

Fig. 2. Numbering of the $4k-2$ vertices of A depending on their position with respect to x on the cycle $\mathscr{H}$.

By Dirac's Theorem, the latter inequality implies that A contains a Hamiltonian cycle $\mathscr{H}$ on $4k-2$ vertices.

Let $x \in A$ be arbitrary. Number the other vertices in A depending on their position with respect to x on the cycle $\mathscr{H}$. More precisely, let $x, c_{2k-2}, c_{2k-3}, c_{2k-4}, \ldots, c_1, c_0$ be the vertices encountered in order when $\mathscr{H}$ is traversed starting at x and going in one direction around the cycle. Let $x, b_0, b_1, b_2, \ldots, b_{2k-3}, b_{2k-2}$ be the vertices encountered when the Hamiltonian cycle $\mathscr{H}$ is traversed starting at x and going in the other direction around the cycle $\mathscr{H}$. See Fig. 2. Note that $c_0 = b_{2k-2}$.

We now interrupt the proof of Theorem 12 in order to prove several needed observations. The proofs of the first two observations are similar to those of Observations 4 and 5, and so they are omitted.

Observation 13. *For all i with $1 \leqslant i \leqslant 2k-3$, the vertex x is connected to at most one of b_i and c_i.*

Observation 14. *The vertex x is not connected to $c_0 = b_{2k-2}$.*

We proceed to prove the following observation:

Observation 15. *For all i with $1 \leqslant i \leqslant 2k-3$, the vertex x is connected to precisely one of b_i and c_i.*

Proof. If, for some i such that $1 \leqslant i \leqslant 2k-3$, the vertex x is not connected to b_i and also not connected to c_i, then by Observations 13 and 14, we have

$$\begin{aligned}\mathscr{D}_A(x) &= \left(\sum_{j=1}^{2k-3} \mathscr{D}_{b_j,c_j}(x)\right) + \mathscr{D}_{b_0,c_{2k-2}}(x) + \mathscr{D}_{b_{2k-2}}(x)\\ &\leqslant ((2k-4)+2+0) = 2k-2.\end{aligned}$$

This inequality contradicts (3). □

Proof of Theorem 12 (*continued*). Let $a_1, a_2, \ldots, a_{4k-2}$ be the vertices of A numbered in the order that the Hamiltonian cycle $\mathscr{H}$ traverses them. Let a_i and a_j stand for the same vertex if $i \equiv j \pmod{4k-2}$. It follows from Observation 15 that, for all j, the vertex a_j is connected to precisely one of a_{j+k} and a_{j-k} (choose $x = a_j$ and $i = k-1$ in the Observation 15).

In particular, a_{k+1} is connected to precisely one of a_1 or a_{2k+1}.

Suppose a_{k+1} is connected to a_1 and is therefore not connected to a_{2k+1}. Then a_{2k+1} must be connected to a_{3k+1} and so

$$a_1, a_{k+1}, a_{k+2}, \ldots, a_{2k+1}, a_{3k+1}, a_{3k+2}, \ldots, a_{4k-2}$$

is a cycle on $2k$ vertices.

On the other hand, suppose a_{k+1} is connected to a_{2k+1} and is therefore not connected to a_1. Then a_1 must be connected to a_{3k-1} and so

$$a_1, a_2, \ldots, a_{k+1}, a_{2k+1}, a_{2k+2}, \ldots, a_{3k-2}, a_{3k-1}$$

is a cycle on $2k$ vertices.

In both cases, the obtained cycle on $2k$ vertices forms the rim, and o forms the hub of $W_{2k} \subset F$. The latter contradicts our assumption that $W_{2k} \not\subset F$. The proof is now complete. □

6. $R(W_{n-1}, S_n) = 3n-2$ when n is even

Theorem 16. $R(W_{2k+1}, S_{2k+2}) = 6k+4$ *for* $k \geqslant 3$.

Proof. Corollary 1 yields $R(W_{2k+1}, S_{2k+2}) \geqslant 3\cdot(2k+2)-2 = 6k+4$. Therefore, it suffices to prove that $R(W_{2k+1}, S_{2k+2}) \leqslant 6k+4$.

Let F be an arbitrary graph on $6k+4$ vertices. Assume that $W_{2k+1} \not\subset F$ and $S_{2k+2} \not\subset \overline{F}$. Then $d_F(v) \geqslant (6k+3) - 2k = 4k+3$ for all $v \in V(F)$.

Theorem 12 implies that $R(W_{2k+2}, S_{2k+2}) \leqslant 6k+3$. By assumption we have $S_{2k+2} \not\subset \overline{F}$, and therefore $W_{2k+2} \subset F$. Let o be the vertex of F that corresponds to the hub of W_{2k+2}, and let $Z = \{z_1, z_2, \ldots, z_{2k+2}\}$ be the set of vertices of F forming the rim of W_{2k+2} in order. Let z_i and z_j stand for the same vertex if $i \equiv j \pmod{2k+2}$.

Since $d_F(o) \geqslant 4k+3$, there exists a subset $A=\{a_1, \ldots, a_{2k+1}\}$ of $V(F)$ such that $a_i \neq z_j$ for all i and j, and a_i is connected to o for all i. Observe that we have chosen A so that $|A| = 2k+1$.

We now interrupt the proof of Theorem 16 in order to prove several needed observations. The proof of the next observation is similar to that of Observation 10, and so it is omitted.

Observation 17. *For all i, if $1 \leqslant x \leqslant 2k+1$, then the vertex a_x is connected to at most one of z_i and z_{i+3}.*

We proceed to prove another observation, which is similar to Observation 11.

Observation 18. $\mathscr{D}_Z(v) \leqslant k+1$ *for all* $v \in A$.

Proof. Let $v \in A$ be arbitrary. Consider three cases.

Case 1: Assume that $3 \mid k$. Write $k = 3a$, so that $|Z| = 2k+2 = 6a+2$. Now, consider the sequence $\{s_t\}$ defined inductively by $s_1 = 1$ and $s_{t+1} \equiv s_t + 3 \pmod{6a+2}$. Clearly, this sequence is periodic with a period of $6a+2$; indeed, the sequence is

$$1, 4, 7, \ldots, 6a+1, 2, 5, \ldots, 6a+2, 3, 6, \ldots, 6a, 1, \ldots.$$

By Observation 17, v is connected to at most one of z_{s_t} and $z_{s_{t+1}}$ for all t. Therefore, v is connected to at most $\lfloor \frac{6a+2}{2} \rfloor$ vertices in Z. Since $\lfloor \frac{6a+2}{2} \rfloor = 3a+1 = k+1$, we have $\mathscr{D}_Z(v) \leqslant k+1$ for all $v \in A$.

Case 2: Assume that $k \equiv 1 \pmod 3$. Write $k = 3a+1$, so that $|Z| = 2k+2 = 6a+4$. Consider the sequence $\{s_t\}$ defined inductively by $s_1 = 1$ and $s_{t+1} \equiv s_t + 3 \pmod{6a+4}$. Clearly, this sequence is periodic with a period of $6a+4$; indeed, the sequence is

$$1, 4, 7, \ldots, 6a+4, 3, 6, \ldots, 6a+3, 2, 5, \ldots, 6a+2, 1, \ldots.$$

By Observation 17, v is connected to at most one of z_{s_t} and $z_{s_{t+1}}$ for all t. Therefore, v is connected to at most $\lfloor \frac{6a+4}{2} \rfloor$ vertices in Z. Since $\lfloor \frac{6a+4}{2} \rfloor = 3a+2 = k+1$, we have $\mathscr{D}_Z(v) \leqslant k+1$ for all $v \in A$.

Case 3: Assume that $k \equiv 2 \pmod 3$. Write $k = 3a+2$, so that $|Z| = 2k+2 = 6a+6$. Consider the following three sequences $\{s_{1,t}\}$, $\{s_{2,t}\}$, and $\{s_{3,t}\}$, defined inductively by $s_{i,1} = i$ and $s_{i,t+1} \equiv s_{i,t} + 3 \pmod{6a+6}$, for $i = 1, 2, 3$. Clearly, each of the sequences $\{s_{1,t}\}$, $\{s_{2,t}\}$, and $\{s_{3,t}\}$ is periodic with a period of $2a+2$; indeed, the sequence $\{s_{1,t}\}$ is $1, 4, 7, \ldots, 6a+4, 1, \ldots$, the sequence $\{s_{2,t}\}$ is $2, 5, \ldots, 6a+5, 2, \ldots$, and the sequence $\{s_{3,t}\}$ is $3, 6, \ldots, 6a+3, 1, \ldots$. By Observation 17, for $i = 1, 2, 3$, the vertex v is connected to at most one of $z_{s_{i,t}}$ and $z_{s_{i,t+1}}$ for all t. Therefore, v is connected to at most $3 \cdot \lfloor \frac{2a+2}{2} \rfloor$ vertices in Z. Since $3 \cdot \lfloor \frac{2a+2}{2} \rfloor = 3 \cdot (a+1) = 3a+3 = k+1$, we have $\mathscr{D}_Z(v) \leqslant k+1$ for all $v \in A$. $\square$

Proof of Theorem 16 (*continued*). Let $P = V(F) - o - Z - A$. By construction, $|P| = (6k+4) - 1 - (2k+2) - (2k+1) = 2k$. Observe that, for all $a \in A$, we have $\mathscr{D}_A(a) \geqslant d_F(a) - (\mathscr{D}_Z(a) + |P| + 1)$. Therefore, by Observation 18,

$$\mathscr{D}_A(a) \geqslant (4k+3) - ((k+1) + (2k) + 1) = k+1.$$

By Dirac's Theorem, the latter inequality implies that A contains a Hamiltonian cycle $\mathscr{C}$ on $2k+1$ vertices. Together $\mathscr{C}$ and o form $W_{2k+1} \subset F$. The latter contradicts our assumption that $W_{2k+1} \not\subset F$. Thus $R(W_{2k+1}, S_{2k+2}) \leqslant 6k+4$. The proof is now complete. $\square$

7. $R(W_m, S_n) \geqslant 2n+1$ when m is even and $n \geqslant m \geqslant 6$

Theorem 19. *$R(W_{2k}, S_{2k+i}) \geqslant 4k+2i+1$ for all $i \geqslant 0$ and $k \geqslant 3$.*

Proof. Construct a graph F on $4k+2i$ vertices as follows. Divide the vertices of F into two disjoint sets: $A = \{a_1, a_2, \ldots, a_{2k+i-1}\}$ and $B = \{b_1, b_2, \ldots, b_{2k+i+1}\}$. Let F have the following edges: a_t is connected to b_j for all t and j. In addition, b_t is connected to b_{t+1} for all $1 \leqslant t \leqslant 2k+i$, and b_{2k+i+1} is connected to b_1.

Clearly, we have $d_F(a_t) = d_F(b_j) = 2k+i+1$ for all t and j. Since $(4k+2i-1) - (2k+i+1) = 2k+i-2$, we have $S_{2k+i} \not\subset \overline{F}$. Moreover, we can verify that $W_{2k} \not\subset F$. Indeed, for all t, the vertex a_t cannot be a hub of W_{2k} since there is no cycle of length $2k$ among the vertices of B. In addition, for all j, the vertex b_j cannot be a hub of W_{2k} since there is no cycle of length $2k$ among the vertices b_{j+1}, b_{j-1} and $a_1, \ldots, a_{2k+i-1}$, unless $2k \leqslant 4$.

Thus, we have constructed a graph F on $4k+2i$ vertices, such that $W_{2k} \not\subset F$ and $S_{2k+i} \not\subset \overline{F}$. Therefore, $R(W_{2k}, S_{2k+i}) > 4k+2i$. $\square$

This improves the lower bound given in Corollary 2 by two.

8. Conclusion

In this paper we showed that $R(W_m, S_n) = 3n-2$ for $n = m, m+1$, and $m+2$, where $m \geqslant 7$ and odd. Combined with the previously known result that $R(W_m, S_n) = 3n-2$ for $n \geqslant 2m-4$, where $m \geqslant 5$ and odd, this leads to a conjecture that $R(W_m, S_n) = 3n-2$ for all $n \geqslant m$, $m \geqslant 5$, and m odd.

The exact value for $R(W_m, S_n)$ when $m \geqslant 6$ and even remains an open problem.

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