**Definition:** Let \( F: \mathbb{Z}^n \to \{0,1\} \). The fractional edge boundary of \( F \) is

\[
\mathcal{E}(F) = \mathbb{E}_{\text{edge } (x,y)} \left[ F(x) \neq F(y) \right]
\]

Equivalent to "average sensitivity", "total influence". A basic complexity measure of Boolean functions.

**Theorem (Kane 14):** Let \( F = H_1 \land \cdots \land H_m \) be the intersection of \( m \) halfspaces. Then

\[
\mathcal{E}(F) = O\left( \frac{\sqrt{\ln(m)}}{\sqrt{n}} \right).
\]

Tight: \( \mathcal{E}(\text{MAJ}_n) = \Theta\left( \frac{1}{\sqrt{n}} \right) \)

Well-known (and easy to show): \( \mathcal{E}(\text{single halfspace}) = O\left( \frac{1}{\sqrt{n}} \right) \)

\[
\Rightarrow \mathcal{E}(H_1 \land \cdots \land H_m) = O\left( \frac{m}{\sqrt{n}} \right) \text{ by union bound argument.}
\]

Kane achieves exponential improvement over this naive bound.
Proof strategy: Induction of $m$ (!!!)

Understand the net change in $E(F=H, \ldots H_k)$ when a new halfspace $H$ is added to the intersection.

Key lemma: $E(F \cap H) \leq E(F) + \frac{U(\text{vol}(C))}{\sqrt{n}}$,

where

$\text{vol}(C) = |C^{-1}(1)|$

$U(p) = O(p\sqrt{\ln(p)})$

1. Proof of Kane's theorem is an easy consequence

2. $U =$ the "Gaussian isosceles function". $U(p) =$ Gaussian surface area of halfspace with volume $p$

3. Actually $F$ can be arbitrary set.
Proof of key lemma

For notational simplicity, we'll assume that $H$ is antimonotone, meaning $H(x^{j=1}) \leq H(x^{j=+1}) \forall x \in \mathbb{R}^n$ and $j \in \mathbb{N}$. If $H = \text{sign}(w \cdot x - \theta)$ antimonotone if $w_i \leq 0 \forall i \in \mathbb{N}$.

New picture, directed version of previous:


3 types of edges: Body → Cap (BC)  
   Exterior → Cap (EC)  
   Cap → Exterior (CE)

Note: No Cap → Body edges since $H$ antimonotone. Indeed, we've drawn edges so they are "oriented away" from $H"
Net change = $E(F \cap H) - E(F)$
= $BC - EC - CE$
= $BC + EC - CE$

We'll be more careful with this inequality in 2nd part of talk.

Fraction of edges going into $C$
- Fraction of edges going out of $C$.

Well-studied quantity in AORF.

Writing $C: \pm 13^n \rightarrow 0$, for indicator of $C$, this quantity is

$$2 \sum_{x \sim \pm 13^n, j \sim [n]} E[C(x) x_j]$$

(normalized sum of degree-1 Fourier coefficients)

Let's bound $E[C(x) x_j]$. Let $p = \text{vol}(C) = \Pr_{x \sim \pm 13^n}[C(x) = 1]$. Then

$$= \frac{1}{n} \sum_{j=1}^{n} E[C(x)(x_1 + \ldots + x_n)]$$

What $C: \pm 13^n \rightarrow 0$ maximizes this, subject to $\Pr[C(x) = 1] = p$?

A:

For this $C$, $\ast = \Theta\left(\frac{\text{vol}(C)}{\sqrt{n}}\right)$
We just proved the key lemma:

\[ \varepsilon(F \cap H) \leq \varepsilon(F) + \frac{U(\text{vol}(C))}{\sqrt{n}}, \]

where \( U(p) = O(p \sqrt{\ln(p)}) \).

Kane’s theorem as easy corollary of key lemma:

Let \( F = H_1 \cap \cdots \cap H_m \) be the intersection of \( m \) halfspaces. Think of \( F \) as being formed by successively adding each \( H_i \) one by one.

Each \( H_i \) can be associated with a cap

\[ C_i = (H_1 \cap \cdots \cap H_{i-1}) \setminus H_i \]

\[ \text{Observation: } C_i's \text{ are disjoint} \]
\[
\mathcal{E}(F) = \sum_{i=1}^{m} \mathcal{E}(H_i \cap \cdots \cap H_i) - \mathcal{E}(H_i \cap \cdots \cap H_i+1)
\]

Key lemma \[
\leq \sum_{i=1}^{m} \frac{U(\text{vol}(C_i))}{\sqrt{n}}
\]

Concavity of \( U \) \[
\leq \frac{1}{\sqrt{n}} \cdot m \cdot U\left(\frac{\sum_{i=1}^{m} \text{vol}(C_i)}{m}\right)
\]

Disjointness of the \( C_i \)'s \[
\leq \frac{1}{\sqrt{n}} \cdot m \cdot U\left(\frac{1}{m}\right) = O\left(\frac{\sqrt{m}}{\sqrt{n}}\right)
\]

Net change of each stage, summed across all stages