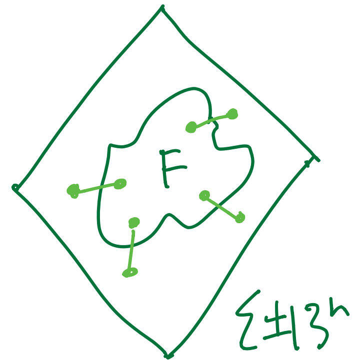
Def$^{\underline{n}}$: Let $F : \{\pm 1\}^n \to \{0,1\}$. The fractional edge boundary of F is

$$\mathcal{E}(F) = \mathop{\mathbb{E}}_{\text{edge }(x,y)}\left[F(x) \neq F(y)\right]$$



Equivalent to "average sensitivity", "total influence".
A basic complexity measure of Boolean functions.

Theorem (Kane 14): Let $F = H_1 \wedge \cdots \wedge H_m$ be the intersection of m halfspaces. Then

$$\mathcal{E}(F) = O\left(\frac{\sqrt{\ln(m)}}{\sqrt{n}}\right).$$

Tight: $\mathcal{E}(\mathrm{MAJ}_n) = \Theta\left(\frac{1}{\sqrt{n}}\right)$

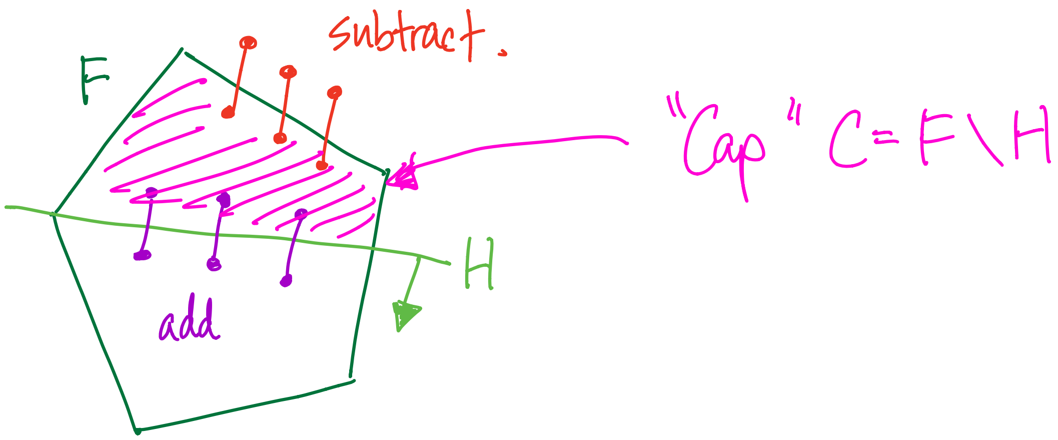
Well-known (and easy to show): $\mathcal{E}(\text{single halfspace}) = O\left(\frac{1}{\sqrt{n}}\right)$

$\Longrightarrow \mathcal{E}(H_1 \wedge \cdots \wedge H_m) = O\left(\frac{m}{\sqrt{n}}\right)$ by union bound argument.

Kane achieves exponential improvement over this naive bound.

Proof strategy: Induction of m (!!)

Understand the net change in $\mathcal{E}(F = H_1 \wedge \cdots \wedge H_k)$ when a new halfspace H is added to the intersection.



Key lemma: $\mathcal{E}(F \cap H) \le \mathcal{E}(F) + \dfrac{\mathcal{U}(\mathrm{vol}(C))}{\sqrt{n}}$,

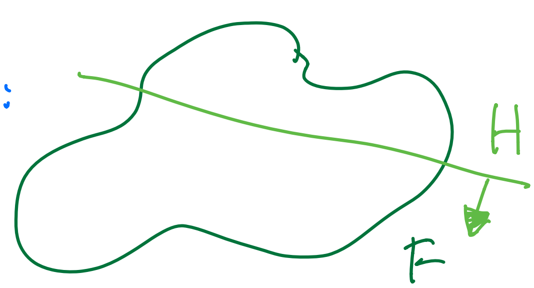
where $\mathrm{vol}(C) = |C^{-1}(1)|$

$\mathcal{U}(p) = O(p\sqrt{\ln(\frac{1}{p})})$

1. Proof of Kane's theorem is an easy consequence

2. $\mathcal{U}$ = the "Gaussian isoperimetric function".
   $\mathcal{U}(p)$ = Gaussian surface area of halfspace with volume p

3. Actually F can be arbitrary set:

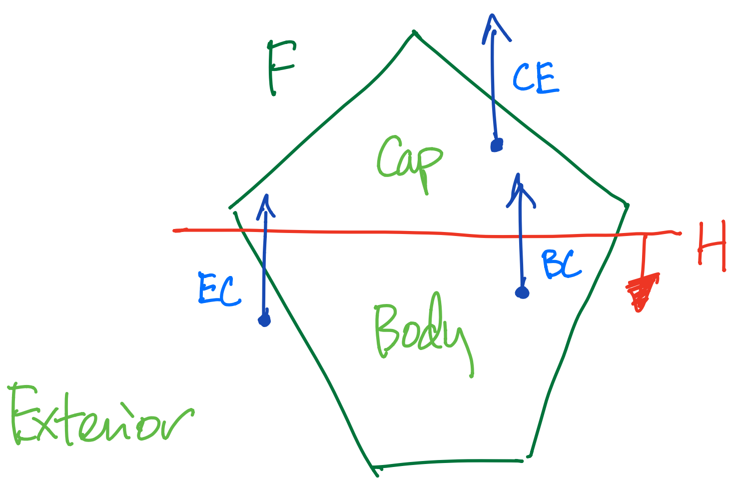


Proof of key lemma

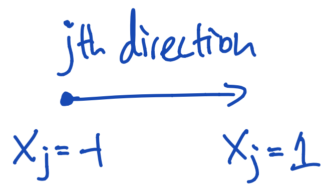
For notational simplicity, we'll assume that H is antimonotone, meaning $H(x^{j=1}) \le H(x^{j=-1})$ $\forall x \in \{\pm 1\}^n$ and $j \in [n]$.

$H = \mathrm{sign}(w \cdot x - \theta)$ antimonotone if $w_i \le 0$ $\forall i \in [n]$.

New picture, directed version of previous:



$F \cap H$ = "Body"
$F \setminus H$ = "Cap"
$\overline{F}$ = "Exterior"



3 types of edges: Body $\to$ Cap (BC)
Exterior $\to$ Cap (EC)
Cap $\to$ Exterior (CE)

Note: No Cap $\to$ Body edges since H antimonotone.
Indeed, we've drawn edges so they are "oriented away" from H

$$\text{Net change} = \mathcal{E}(F \cap H) - \mathcal{E}(F)$$
$$= BC - EC - CE \quad \leftarrow \text{fraction of such edges.}$$
$$\leq \underbrace{BC + EC - CE}$$

We'll be more careful with this inequality in 2nd part of talk

Fraction of edges going into C
$-$ Fraction of edges going out of C.

Well-studied quantity in AOBF.

Writing $C : \{\pm1\}^n \to \{0,1\}$ for indicator of C, this quantity is

$$2 \mathop{\mathbb{E}}_{\substack{x \sim \{\pm1\}^n \\ j \sim [n]}} [C(x) x_j] \quad \left(= \frac{2}{n} \sum_{j=1}^{n} \hat{C}(\{j\}), \text{ normalized} \right)$$

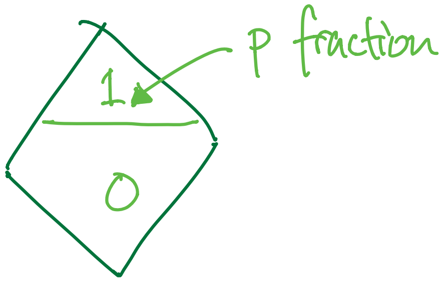
sum of degree-1 Fourier coefficients

Let's bound $\mathop{\mathbb{E}}_{\substack{x \sim \{\pm1\}^n \\ j \sim [n]}} [C(x) x_j]$ (*). Let $p = \text{vol}(C) = \Pr_{x \sim \{\pm1\}^n} [C(x) = 1]$.

$$= \frac{1}{n} \sum_{j=1}^{n} \mathbb{E}[C(x)(x_1 + \cdots + x_n)]$$

What $C : \{\pm1\}^n \to \{0,1\}^n$ maximizes this, subject to $\Pr[C(x)=1] = p$?

A:



For this C, (*) $= \Theta\left(\frac{p\sqrt{\ln(\frac{1}{p})}}{\sqrt{n}} \right)$

$\blacksquare$

We just proved the key lemma:

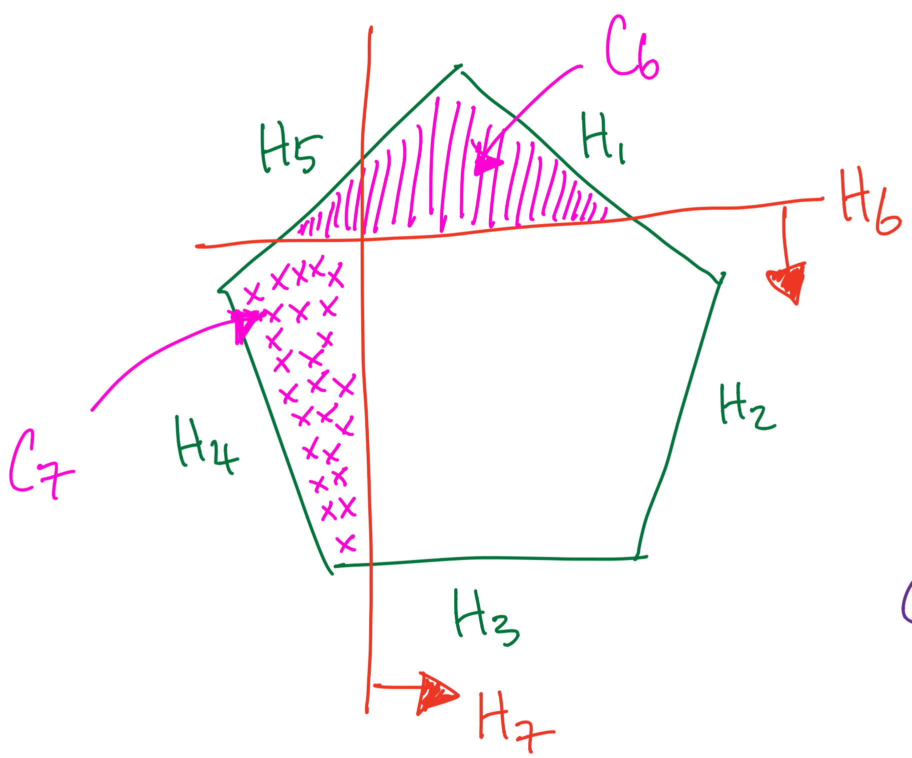
$$\mathcal{E}(F \cap H) \leq \mathcal{E}(F) + \frac{\mathcal{U}(\mathrm{vol}(C))}{\sqrt{n}},$$

where $\mathcal{U}(p) = O\left(p\sqrt{\ln\left(\frac{1}{p}\right)}\right)$.

Kane's theorem as easy corollary of key lemma:

Let $F = H_1 \cap \cdots \cap H_m$ be the intersection of m halfspaces.
Think of F as being formed by successively adding each H_i one by one.
Each H_i can be associated with a cap

$$C_i = (H_1 \cap \cdots \cap H_{i-1}) \setminus H_i$$



Observation:
C_i's are disjoint

$$\mathcal{E}(F) = \sum_{i=1}^{m} \mathcal{E}(H_1 \cap \cdots \cap H_i) - \mathcal{E}(H_1 \cap \cdots \cap H_{i-1})$$

Net change of each stage, summed across all stages

Key lemma $\leq \sum_{i=1}^{m} \dfrac{\mathcal{U}(\mathrm{vol}(C_i))}{\sqrt{n}}$

Concavity of $\mathcal{U}$ $\leq \dfrac{1}{\sqrt{n}} \, m \cdot \mathcal{U}\left(\dfrac{\sum_{i=1}^{m} \mathrm{vol}(C_i)}{m} \right)$

Disjointness of the C_i's. $\leq \dfrac{1}{\sqrt{n}} \cdot m \cdot \mathcal{U}\left(\dfrac{1}{m}\right) = O\left(\dfrac{\sqrt{\ln(m)}}{\sqrt{n}}\right)$

$\blacksquare$