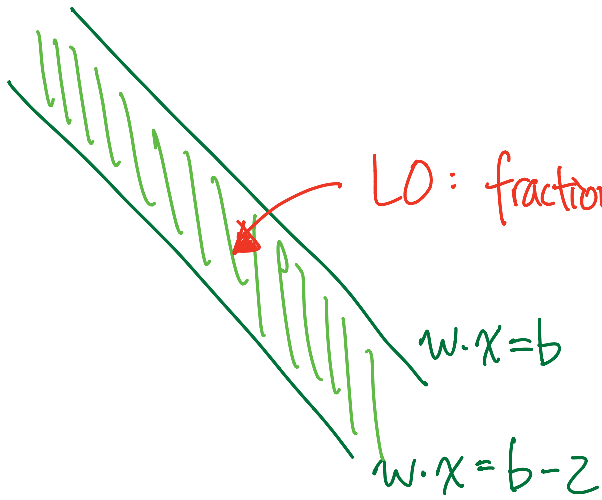
Littlewood–Offord: Suppose $w \in \mathbb{R}^n$ is such that $|w_j| \geq 1$ $\forall j \in [n]$

Then for all $b \in \mathbb{R}$, $\Pr_{x \sim \{\pm 1\}^n}\left[w \cdot x \in (b-2, b] \right] \leq O\left(\frac{1}{\sqrt{n}}\right)$

In fact, $\dfrac{\binom{n}{n/2}}{2^n}$. Extremal example: $w = 1^n$, interval $(-1, 1]$

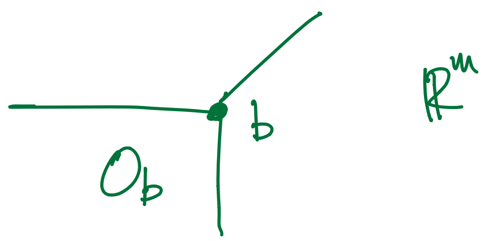
Geometrically,



LO: fraction of $\{\pm 1\}^n$ in here is $O\left(\frac{1}{\sqrt{n}}\right)$.

Notation: For $b \in \mathbb{R}^m$, let $O_b = \{ y \in \mathbb{R}^m : y_i \leq b_i \ \forall i \in [m] \}$

O_b = "translated orthant rooted at b"

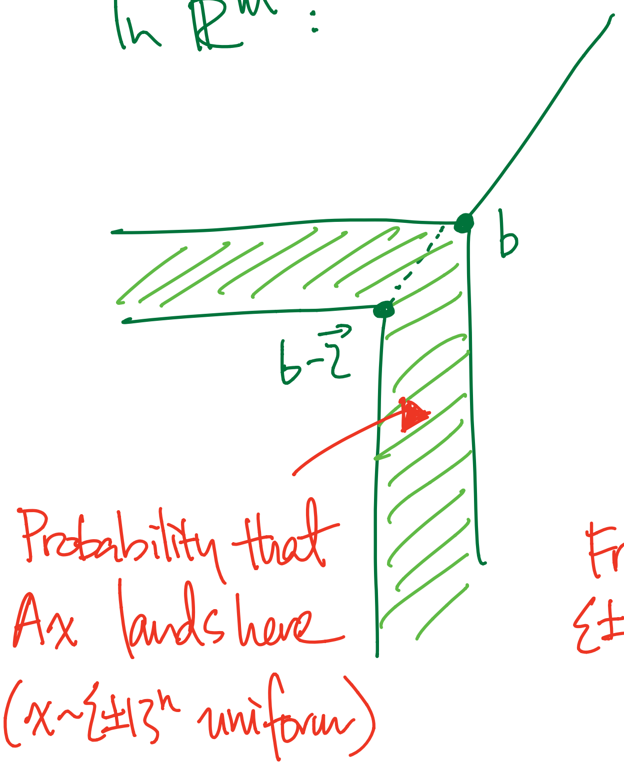


Theorem (high-dim LO): Let $A \in \mathbb{R}^{m \times n}$ be such that $|A_{ij}| \geq 1$ $\forall i \in [m], j \in [n]$. Then for all $b \in \mathbb{R}^m$,

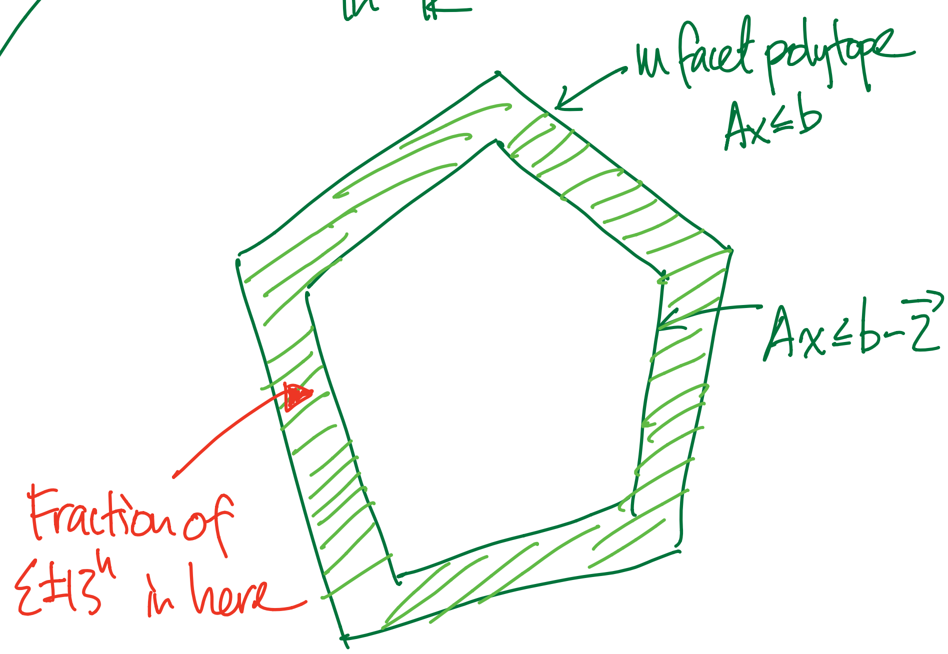
$$\Pr_{x \sim \{\pm 1\}^n}\left[Ax \in O_b \setminus O_{b - \vec{2}} \right] \leq O\left(\frac{\sqrt{\ln(m)}}{\sqrt{n}} \right)$$

2 ways to visualize this geometrically:

In $\mathbb{R}^m$:



In $\mathbb{R}^n$:



Remarks: 1. Our bound is tight

2. Gaussian case ($x \sim N(0,1)^n$) proved by Nazarov 2003

(Gaussian case is special case of Boolean)

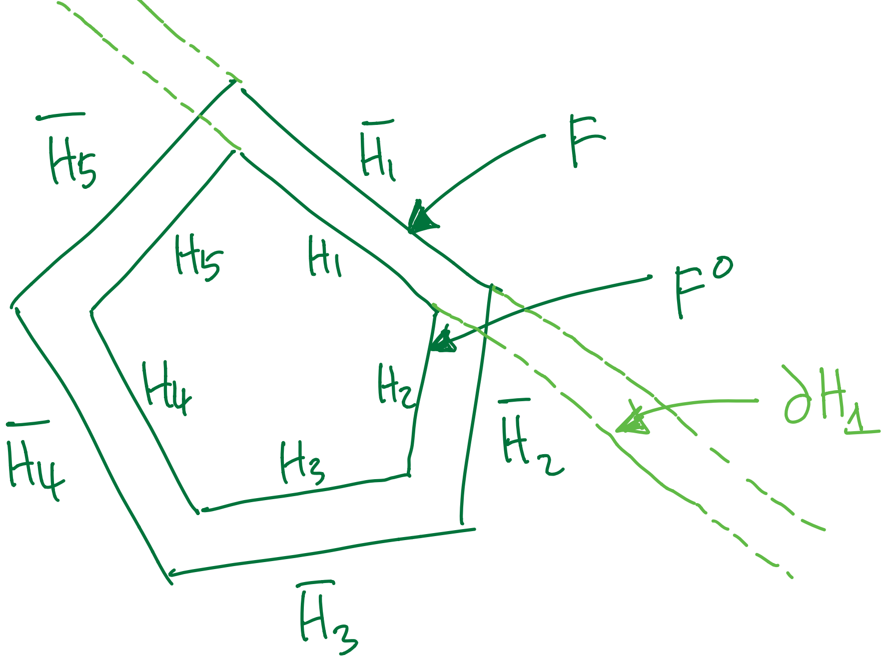
Notation: For each $i \in [m]$, let:

$$\overline{H_i} = \{x \in \{\pm1\}^n : A_i \cdot x \le b_i\} \qquad F = \overline{H_1} \cap \cdots \cap \overline{H_m}$$

$$H_i = \{x \in \{\pm1\}^n : A_i \cdot x \le b_i - 2\} \qquad F^\circ = H_1 \cap \cdots \cap H_m$$

$$\partial H_i = \overline{H_i} \setminus H_i \qquad \partial F = F \setminus F^\circ$$

In $\mathbb{R}^n$:

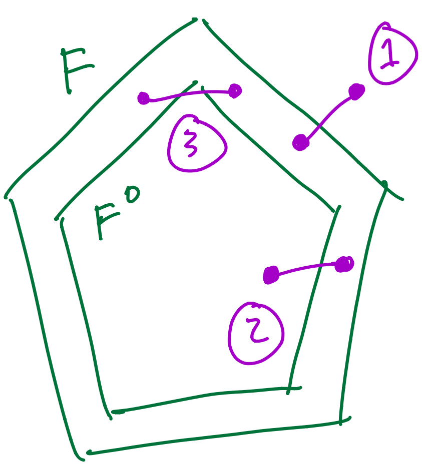


Our goal is to bound: $\Pr_{x \sim \{\pm 1\}^n} [x \in \partial F] \leq O\left(\frac{\sqrt{\ln(m)}}{\sqrt{n}}\right)$

Note that

$$\Pr_{x \sim \{\pm 1\}^n} [x \in \partial F] \leq \underbrace{\Pr_{\text{edge }(x,y)} [\overset{x \in \partial F \text{ or } y \in \partial F}{(x,y) \text{ touch } \partial F}]}_{\text{we'll bound this instead}}$$

3 types of edges that contribute to this probability:



1. Boundary-to-exterior edges
2. Interior-to-boundary edges.
3. Boundary-to-boundary edges.

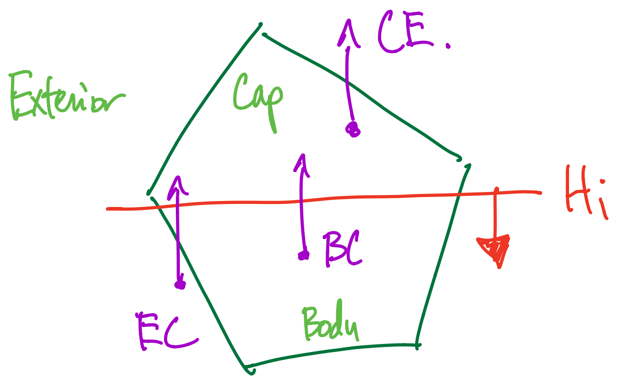
Fraction of type 1 edges $= \mathcal{E}(F) = O\left(\frac{\sqrt{\ln(m)}}{\sqrt{n}}\right)$.

" " " 2 " $= \mathcal{E}(F^\circ) = O\left(\frac{\sqrt{\ln(m)}}{\sqrt{n}}\right)$.

Remains to bound fraction of type 3 edges, boundary-to-boundary. Call this fraction μ_{BB}.

Recall Kane's proof bounding $\mathcal{E}(F^\circ)$, $F^\circ = H_1 \cap \text{—} \cap H_m$.

Consider net change in $\mathcal{E}(F^\circ)$ as each H_i added:



$$\mathcal{E}(H_1 \cap \text{—} \cap H_{i-1}) - \mathcal{E}(H_1 \cap \text{—} \cap H_i)$$
$$= BC_i - EC_i - CE_i$$
$$\leq BC_i + EC_i - CE_i \quad (*)$$

$$\mathcal{E}(F^\circ) = \sum_{i=1}^{m} BC_i + EC_i - CE_i \leq O\left(\frac{\sqrt{\ln(m)}}{\sqrt{n}}\right).$$

<u>Claim 1</u>: $\sum_{i=1}^{m} EC_i \leq \frac{1}{2} \sum_{i=1}^{m} BC_i + EC_i - CE_i$

<u>Claim 2</u>: $\mu_{BB} \leq \sum_{i=1}^{m} EC_i$

Theorem follows from Claims 1 + 2.

Proof of Claim 1: Being more careful with (*),

$$\mathcal{E}(H_1 \cap \dots \cap H_{i-1}) - \mathcal{E}(H_1 \cap \dots \cap H_i)$$
$$= BC_i - EC_i - CE_i$$
$$= (BC_i + EC_i - CE_i) - 2EC_i$$

Summing over $i \in [m]$,

$$\mathcal{E}(F^\circ) = \sum_{i=1}^{m} (BC_i + EC_i - CE_i) - 2\sum_{i=1}^{m} EC_i$$

Claim follows since $\mathcal{E}(F^\circ) \geq 0$. ■

Proof of Claim 2:
Consider boundary-to-boundary edge (x,y), $x, y \in \partial F$.
Want to "charge" this edge to some EC_i

Points in ∂F belong to at least one ∂H_i, possibly multiple.
So:

Let $i^*(x) = \min\{i \in [m] : x \in \partial H_i\}$. Similarly $i^*(y)$.

"First time that x falls into the boundary"

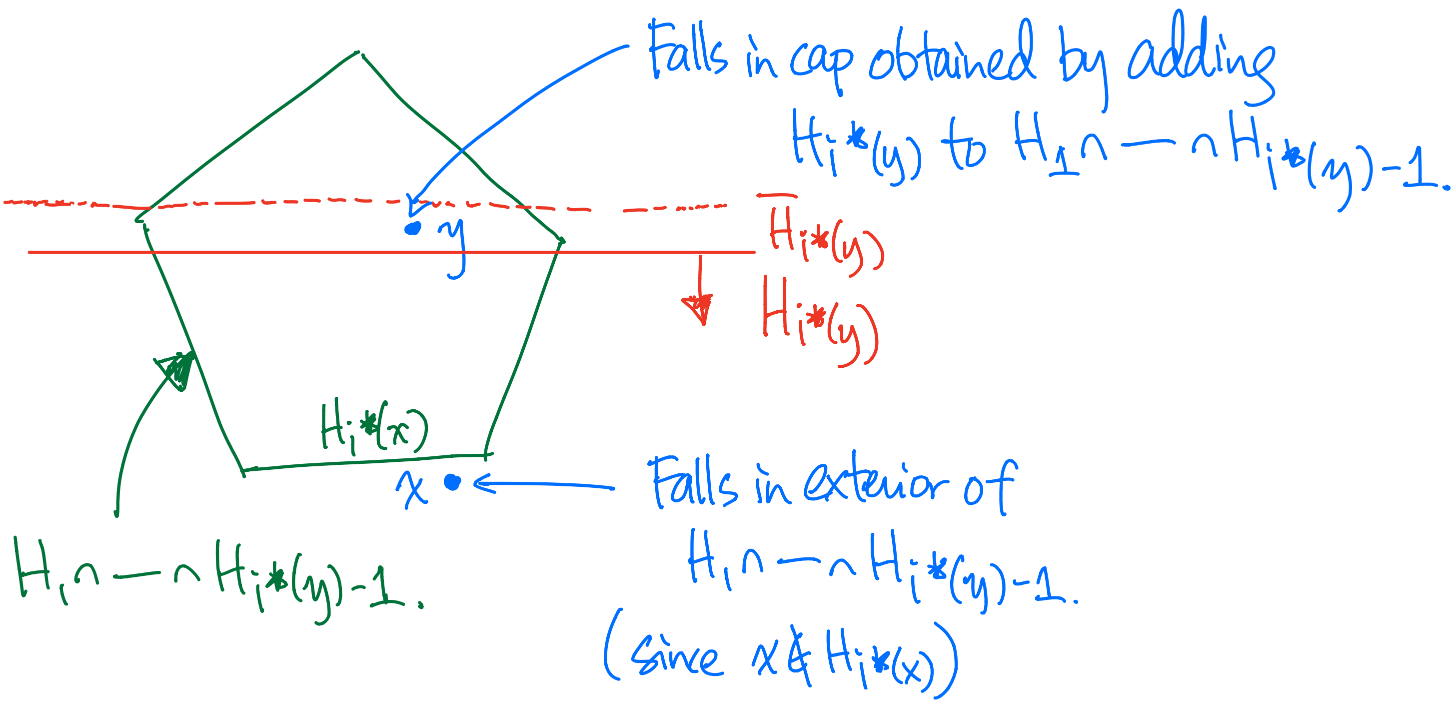
Observation: $i^*(x) \neq i^*(y)$.

Proof: If $\underbrace{x \in \partial H_i}_{A_i \cdot x \in (b_i - 2,\, b_i]}$, then $y \notin \partial H_i$.

$y = x^{\oplus j}$ and $|A_{ij}| \geq 1$.

(So flipping jth coordinate has to "swing x out of interval")

Assume WLOG that $i^*(x) < i^*(y)$. We now show that (x, y) contributes to $EC_{i^*(y)}$.



So indeed, (x, y) contributes to $EC_{i^*(y)}$. ∎