Littlewood-Offord: Suppose $w \in \mathbb{R}^n$ is such that $|w_j| \geq 1 \forall j \in [n]$.

Then for all $b \in \mathbb{R}$, 
$$\Pr_{x \sim \pm \mathbb{I}^n} \left[ w \cdot x \in (b-2, b] \right] = O\left( \frac{1}{\sqrt{n}} \right)$$

In fact, $\frac{\binom{n}{\frac{n}{2}}}{2^n}$. Extremal example: $w = 1^n$, interval $(-1, 1]$.

Geometrically,

$LO$: fraction of $\pm \mathbb{I}^n$ in here is $O\left( \frac{1}{\sqrt{n}} \right)$.

Notation: For $b \in \mathbb{R}^m$, let $O_b = \{ y \in \mathbb{R}^m : y_i \leq b_i \ \forall i \in [m] \}$.

$O_b$ = “translated orthant rooted at $b$”

Theorem (high-dim LO): Let $A \in \mathbb{R}^{mn \times n}$ be such that $|A_{ij}| \geq 1 \forall i \in [m], j \in [n]$. Then for all $b \in \mathbb{R}^m$,

$$\Pr_{x \sim \pm \mathbb{I}^n} \left[ Ax \in O_b \setminus O_{b-2} \right] \leq O\left( \frac{\sqrt{m}}{\sqrt{n}} \right)$$
2 ways to visualize this geometrically:

In $\mathbb{R}^m$:

Probability that $Ax \land \text{here}$
($x \sim \pm \mathcal{N}$ uniform)

In $\mathbb{R}^n$:

Fraction of $\pm \mathcal{N}$ in here

Remarks:
1. Our bound is tight
2. Gaussian case ($x \sim \mathcal{N}(0,1)^n$) proved by Nazarov 2003
   (Gaussian case is special case of Boolean)

Notation:
For each $i \in [m]$, let:

$H_i = \exists x \in \pm \mathcal{N}^n : A_i \cdot x \leq b_i$ \quad \quad $H = H_1 \land \cdots \land H_m$

$\overline{H_i} = \exists x \in \pm \mathcal{N}^n : A_i \cdot x \geq b_i$ \quad \quad $F = \overline{H_1} \land \cdots \land \overline{H_m}$

$H_i = \exists x \in \pm \mathcal{N}^n : A_i \cdot x \leq b_i - 2$ \quad \quad $H = H_1 \land \cdots \land H_m$

$\delta H_i = \overline{H_i} \setminus H_i$ \quad \quad $F^0 = H_1 \land \cdots \land H_m$

$\delta F = F \setminus F^0$
Our goal is to bound: \( \Pr_{x \sim \mathcal{D}^n} [x \in \partial \mathcal{F}] \leq O\left(\frac{\sqrt{\ln(m)}}{\sqrt{n}}\right) \)

Note that \( x \in \partial \mathcal{F} \) or \( y \in \partial \mathcal{F} \)

\[ \Pr_{x \sim \mathcal{D}^n} [x \in \partial \mathcal{F}] \leq \Pr_{x \sim \mathcal{D}^n} [x \in \mathcal{F}] \quad \text{edge } (xy) \]

we'll bound this instead

3 types of edges that contribute to this probability:

1. Boundary-to-exterior edges
2. Interior-to-boundary edges
3. Boundary-to-boundary edges
Fraction of type 1 edges = $\mathcal{E}(F) = O\left(\frac{\sqrt{\ln(m)}}{\sqrt{n}}\right)$.

" " " Z " = $\mathcal{E}(F^0) = O\left(\frac{\sqrt{\ln(m)}}{\sqrt{n}}\right)$.

Remains to bound fraction of type 3 edges, boundary-to-boundary. Call this fraction $\mu_{BB}$.

Recall Kane's proof bounding $\mathcal{E}(F^0)$, $F^0 = H_1 \cap \cdots \cap H_m$.

Consider net change in $\mathcal{E}(F^0)$ as each $H_i$ added:

\[ \mathcal{E}(H_1 \cap \cdots \cap H_{i-1}) - \mathcal{E}(H_1 \cap \cdots \cap H_i) = BC_i - EC_i - CE_i \leq BC_i + EC_i - CE_i \]  

\[ \mathcal{E}(F^0) = \sum_{i=1}^{m} BC_i + EC_i - CE_i \leq O\left(\frac{\sqrt{\ln(m)}}{\sqrt{n}}\right). \]

Claim 1: \( \sum_{i=1}^{m} EC_i \leq \frac{1}{2} \sum_{i=1}^{m} BC_i + EC_i - CE_i \)

Claim 2: \( \mu_{BB} \leq \sum_{i=1}^{m} EC_i \)  \quad \text{Theorem follows from Claims 1 + 2.}
Proof of Claim 1: Being more careful with $(\ast)$,

\[
\begin{align*}
\mathcal{E}(H_i \cap \neg H_{i+1}) - \mathcal{E}(H_i \cap \neg H_i) \\
= BC_i - EC_i - CE_i \\
= (BC_i + EC_i - CE_i) - 2 EC_i
\end{align*}
\]

Summing over $i \in [m]$,

\[
\mathcal{E}(F^0) = \sum_{i=1}^{m} (BC_i + EC_i - CE_i) - 2 \sum_{i=1}^{m} EC_i
\]

Claim follows since $\mathcal{E}(F^0) \geq 0$. 

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Proof of Claim 2:
Consider boundary-to-boundary edge $(xy)$, $x, y \in \partial F$.
Want to "charge" this edge to some $EC_i$

Points in $\partial F$ belong to at least one $\partial Hi$, possibly multiple.
So:

let $i^*(x) = \min \{ i \in [m] : x \in \partial Hi \}$. Similarly $i^*(y)$.

"First time that $x$ falls into the boundary"
Observation: \( i^*(x) = i^*(y) \).

**Proof:** If \( x \notin \delta H_i \), then \( y \notin \delta H_i \).

\[ \forall i \cdot x \in (b_i-2, b_i] \quad x = x^{\oplus_j} \quad \text{and} \quad |A_{ij}| \geq 1. \]

(So flipping \( j \)th coordinate has to "swing \( x \) out of interval")

Assume WLOG that \( i^*(x) < i^*(y) \). We now show that \((x, y)\) contributes to \( EC_{i^*(y)} \).

So indeed, \((x, y)\) contributes to \( EC_{i^*(y)} \).