Fooling Gaussian PTFs via Local Hyperconcentration

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Joint work with

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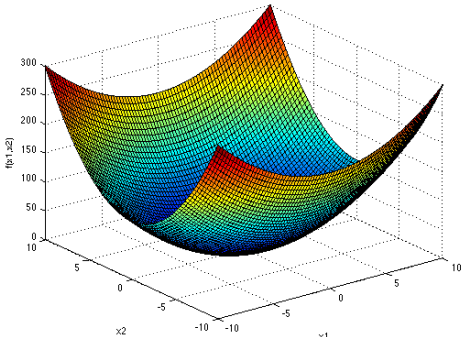
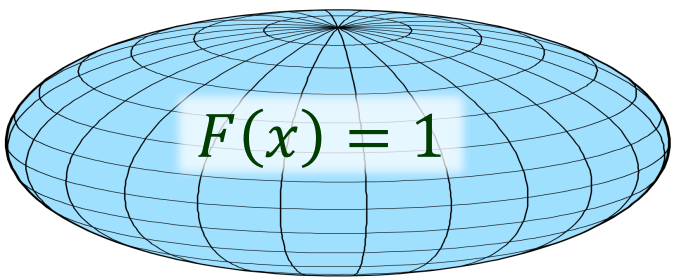


Polynomial threshold functions (PTFs) over R^n

$$F : \mathbb{R}^n \to \{\pm 1\}$$

$$F(x) = \mathrm{sign}(p(x))$$

polynomial $p : \mathbb{R}^n \to \mathbb{R}$



$F(x) = -1$

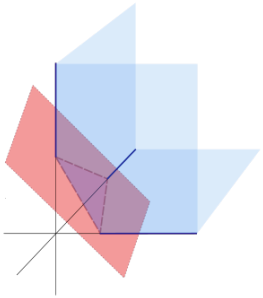
$$p(x) = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} - 1$$

living in $\mathbb{R}^3$

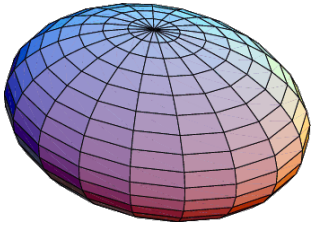
- High-dimensional ($n \to \infty$) geometric objects
- Intensively studied within TCS and beyond TCS

Main complexity measure for this talk: **degree of *p***

$$F(x) = \mathrm{sign}(p(x)), \quad \deg(p) \leq d$$

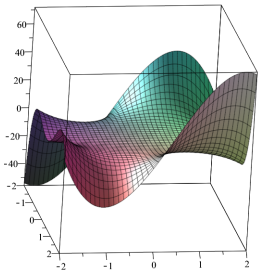


d=1

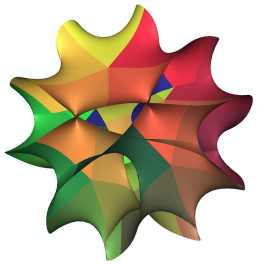


d=2

. . .



d=5



d=6

. . .

More complex

Think of *d* as growing with *n* = dimension of R^n

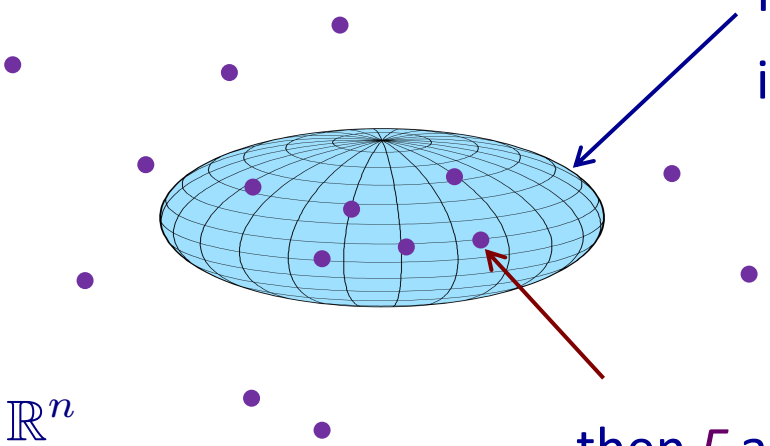
For concreteness, can think of *d* = log *n*

Discrepancy Sets for PTFs over Gaussian space

Consider R^n endowed with $N(0,1)^n$. Want **small** set of points such that:

For all degree-*d* PTFs *F*,

if *F* 's Gaussian volume is Δ



... then *F* accepts (Δ ± 0.01) fraction of points

Random set of points works great. Want **explicit** set.

Pseudorandom generators over Gaussian space

<u>Definition</u>: An ***ε*-PRG** for a class *C* is an explicit function $G : \{0,1\}^r \to \mathbb{R}^n$ such that: for every function *F* in *C*,

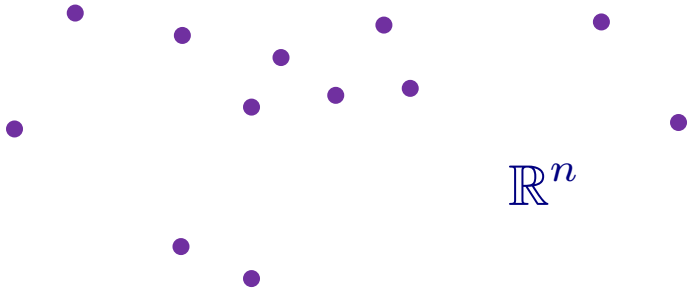
$$\left| \mathop{\mathbb{E}}_{\boldsymbol{x} \sim N(0,1)^n} [F(\boldsymbol{x})] - \mathop{\mathbb{E}}_{\boldsymbol{s} \sim \{0,1\}^r} [F(G(\boldsymbol{s}))] \right| \leq \varepsilon$$

This work:

C = { degree-*d* PTFs  }

r = "seed length"

= log(size of discrepancy set)



Our main result: PRG for PTFs over Gaussian space

An ε-PRG for degree-d PTFs over R^n with seed length:

$$\left(\frac{d}{\varepsilon}\right)^{O(\log d)} \cdot \log n$$

- Previous best seed lengths: $2^{O(d)}$ dependence
- Probabilistic method: $O(d \log n + \log(1/\varepsilon))$
- Gaussian space special case of Boolean space
  - Current best Boolean seed length: still $2^{O(d)}$

Comparison with prior results

Reference	Seed length	Allowable / nontrivial range of d's
[DKN10]	$\tilde{O}(1/\varepsilon^9) \cdot \log n$	$d \leq 2$
[MZ13, MZ10]	$(d/\varepsilon)^{O(d)} \cdot \log n$	$d \leq O(\log n / \log\log n)$
[Kan11a]	$O_d\left(\varepsilon^{-2^{O(d)}}\right) \cdot \log n$	$d \leq$ slightly superconstant
[Kan11b]	$2^{O(d)} \cdot \mathrm{poly}(1/\varepsilon) \cdot \log n$	$d \leq O(\log n)$
[Kan12]	$A(d, \frac{1}{c}) \cdot (1/\varepsilon)^{2+c} \cdot \log n$ for any $c > 0$	$d \leq$ slightly superconstant
[Kan14]	$A(d, \frac{1}{c}) \cdot (1/\varepsilon)^{c} \cdot \log n$ for any $c > 0$	$d \leq$ slightly superconstant
[Kan15]	$O(\log^6(1/\varepsilon) \log\log(n/\varepsilon) \log n)$	$d \leq 2$
[KM15]	$O(\log(1/\varepsilon) \log\log(1/\varepsilon) + \log n)$	$d = 1$
This work	$(d/\varepsilon)^{O(\log d)} \cdot \log n$	$d \leq 2^{O(\sqrt{\log n})}$

The rest of this talk

- Key technical ingredient: New structural lemma about polynomials

"An arbitrary polynomial, when hit with a mild random restriction, collapses to a simple polynomial with many enjoyable properties"

- Sketch of the proof of this lemma

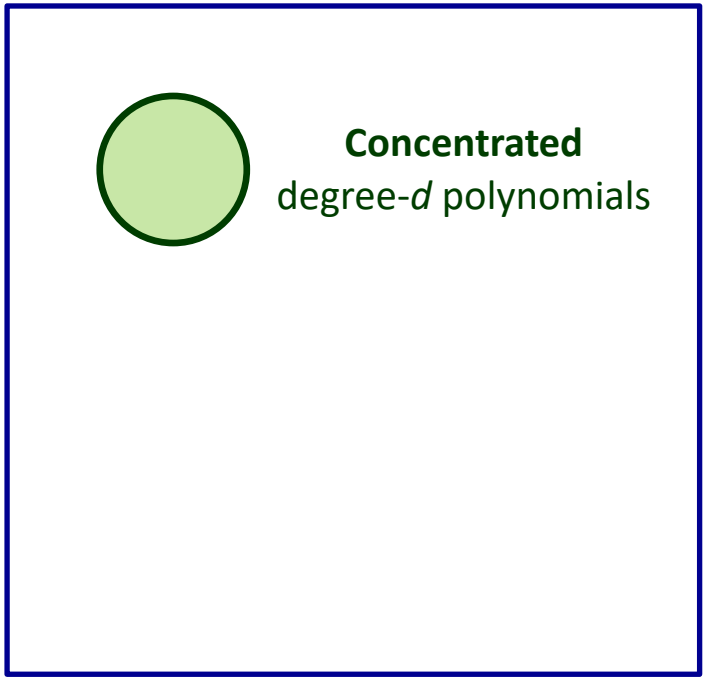
With this lemma, still lots of $\underbrace{\text{technical work to get PRG!}}_{\text{Bulk of project and paper}}$

Concentrated polynomials

A polynomial *p* is **concentrated** if

$$\mathrm{Var}[p] \ll \mathbb{E}[p]^2$$

- "Morally constant", i.e. "morally degree 0"
- sign(*p*(x)) = sign(E[*p*]) for most x's

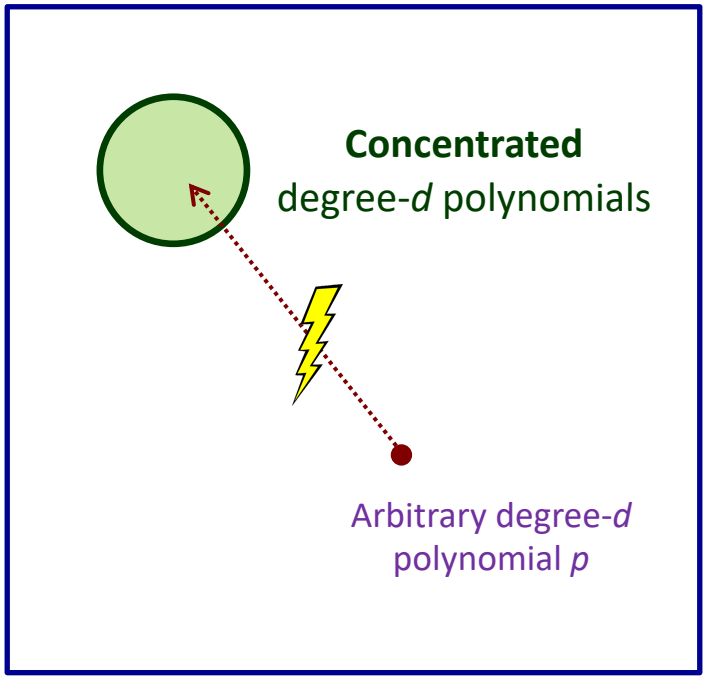


Space of all degree-*d* polynomials

Our new structural lemma for polynomials

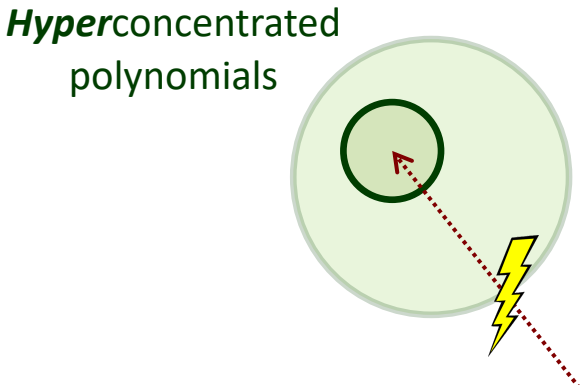
<u>Theorem</u>:

An arbitrary degree-*d* polynomial, when hit with a mild random restriction, becomes **concentrated** with high probability.



In fact, we show that they become ***hyper***concentrated:

$$\boldsymbol{Hyper}\,\mathrm{Var}[p] \ll \mathbb{E}[p]^2$$



Random restrictions over Gaussian space

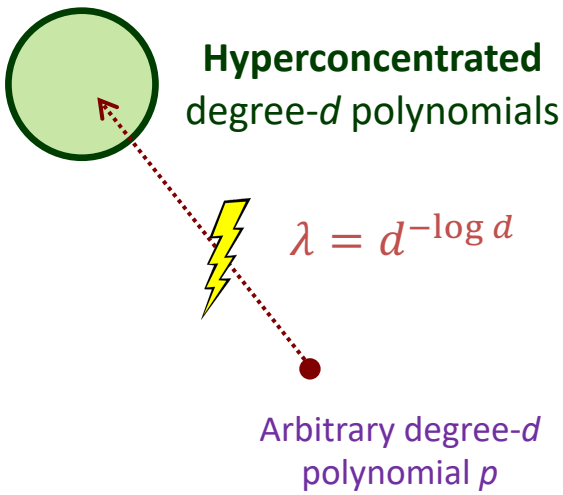
<u>Definition</u>: Let p be a polynomial. A λ-random restriction of p is the polynomial

$$\boldsymbol{q}(x) = p(\sqrt{\lambda}x + \sqrt{1-\lambda}\boldsymbol{y}), \quad \boldsymbol{y} \sim N(0,1)^n.$$

- The smaller λ is, the "harsher" the random restriction
- $\boldsymbol{q}$ = p's behavior in a local neighborhood around $\boldsymbol{y}$, where λ = radius of neighborhood

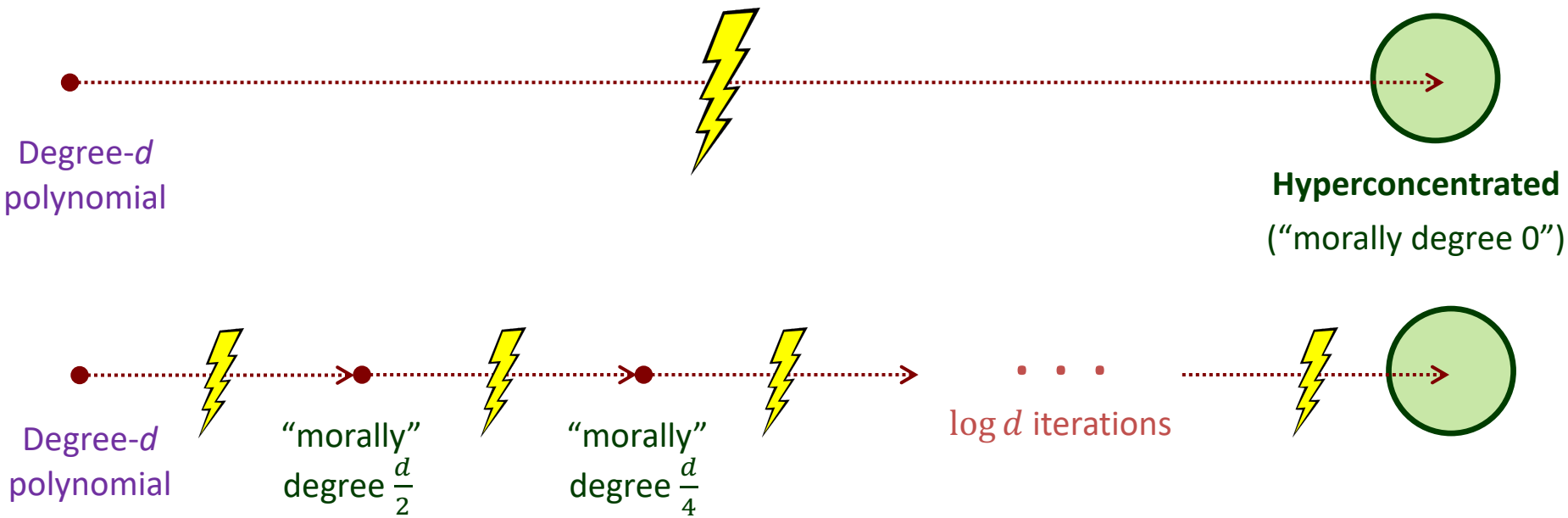
Our new structural lemma, restated

Local Hyperconcentration Theorem



"For most points ***y***, the polynomial *p* behaves like a constant function in a $\underbrace{\text{fairly large neighborhood}}_{\lambda = d^{-\log d}}$ around ***y***."

The proof of our Local Hyperconcentration Theorem



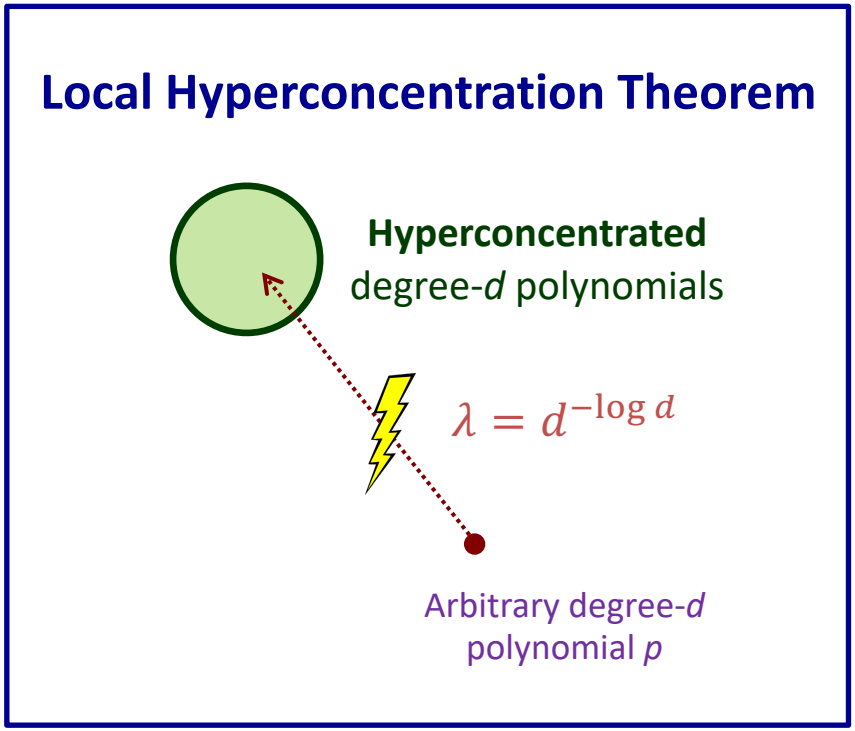
Technical ingredients:

- Hermite analysis
- Carbery-Wright anticoncentration inequality

Thanks Avi!

Open Problems

- Conjecture: $\lambda = d^{-O(1)}$ suffices
- Random restriction lemma for polynomials over Boolean space?




Thank you!