Fooling Gaussian PTFs via Local Hyperconcentration

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Joint work with
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Polynomial threshold functions (PTFs) over $\mathbb{R}^n$

$F : \mathbb{R}^n \rightarrow \{\pm 1\}$

$F(x) = \text{sign}(p(x))$

polynomial $p : \mathbb{R}^n \rightarrow \mathbb{R}$

- High-dimensional ($n \rightarrow \infty$) geometric objects
- Intensively studied within TCS and beyond TCS
Main complexity measure for this talk: **degree of $p$**

$$F(x) = \text{sign}(p(x)), \quad \text{deg}(p) \leq d$$

Think of $d$ as growing with $n = \text{dimension of } \mathbb{R}^n$

For concreteness, can think of $d = \log n$
Discrepancy Sets for PTFs over Gaussian space

Consider $\mathbb{R}^n$ endowed with $\mathcal{N}(0,1)^n$. Want **small** set of points such that:

For all degree-$d$ PTFs $F$,
if $F$’s Gaussian volume is $\Delta$
... then $F$ accepts $(\Delta \pm 0.01)$ fraction of points

Random set of points works great. Want **explicit** set.
Pseudorandom generators over Gaussian space

**Definition:** An $\epsilon$-PRG for a class $C$ is an explicit function $G : \{0,1\}^r \rightarrow \mathbb{R}^n$ such that: for every function $F$ in $C$,

$$
\left| \mathbb{E}_{x \sim N(0,1)^n} [F(x)] - \mathbb{E}_{s \sim \{0,1\}^r} [F(G(s))] \right| \leq \epsilon
$$

**This work:**

$C =$ { degree-$d$ PTFs }  

$r =$ “seed length”  

= log(size of discrepancy set)
Our main result: PRG for PTFs over Gaussian space

An $\varepsilon$-PRG for degree-$d$ PTFs over $\mathbb{R}^n$ with seed length:

$$\left(\frac{\varepsilon}{d}\right)^{O(\log d)} \cdot \log n$$

- Previous best seed lengths: $2^{O(d)}$ dependence
- Probabilistic method: $O(d \log n + \log(1/\varepsilon))$
- Gaussian space special case of Boolean space
  - Current best Boolean seed length: still $2^{O(d)}$
# Comparison with prior results

<table>
<thead>
<tr>
<th>Reference</th>
<th>Seed length</th>
<th>Allowable / nontrivial range of $d$’s</th>
</tr>
</thead>
<tbody>
<tr>
<td>[DKN10]</td>
<td>$\tilde{O}(1/\varepsilon^9) \cdot \log n$</td>
<td>$d \leq 2$</td>
</tr>
<tr>
<td>[MZ13, MZ10]</td>
<td>$(d/\varepsilon)^{O(d)} \cdot \log n$</td>
<td>$d \leq O(\log n / \log \log n)$</td>
</tr>
<tr>
<td>[Kan11a]</td>
<td>$O_d(\varepsilon^{-2^{O(d)}}) \cdot \log n$</td>
<td>$d \leq \text{slightly superconstant}$</td>
</tr>
<tr>
<td>[Kan11b]</td>
<td>$2^{O(d)} \cdot \text{poly}(1/\varepsilon) \cdot \log n$</td>
<td>$d \leq O(\log n)$</td>
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<tr>
<td>[Kan12]</td>
<td>$A(d, 1/\varepsilon) \cdot (1/\varepsilon)^{2+c} \cdot \log n$ for any $c &gt; 0$</td>
<td>$d \leq \text{slightly superconstant}$</td>
</tr>
<tr>
<td>[Kan14]</td>
<td>$A(d, 1/\varepsilon)^c \cdot \log n$ for any $c &gt; 0$</td>
<td>$d \leq \text{slightly superconstant}$</td>
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<tr>
<td>[Kan15]</td>
<td>$O(\log^6(1/\varepsilon) \log \log(n/\varepsilon) \log n)$</td>
<td>$d \leq 2$</td>
</tr>
<tr>
<td>[KM15]</td>
<td>$O(\log(1/\varepsilon) \log \log(1/\varepsilon) + \log n)$</td>
<td>$d = 1$</td>
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<tr>
<td><strong>This work</strong></td>
<td>$(d/\varepsilon)^{O(\log d)} \cdot \log n$</td>
<td>$d \leq 2^{O(\sqrt{\log n})}$</td>
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</table>
The rest of this talk

- **Key technical ingredient: New structural lemma about polynomials**
  
  “An arbitrary polynomial, when hit with a mild random restriction, collapses to a simple polynomial with many enjoyable properties”

- **Sketch of the proof of this lemma**

With this lemma, still lots of technical work to get PRG!

Bulk of project and paper
A polynomial $p$ is **concentrated** if

$$\text{Var}[p] \ll \mathbb{E}[p]^2$$

- “Morally constant”, i.e. “morally degree 0”
- $\text{sign}(p(x)) = \text{sign}(\mathbb{E}[p])$ for most $x$’s
Our new structural lemma for polynomials

**Theorem:**
An arbitrary degree-$d$ polynomial, when hit with a mild random restriction, becomes **concentrated** with high probability.

In fact, we show that they become **hyper**concentrated:

\[ \text{HyperVar}[p] \ll \mathbb{E}[p]^2 \]
Random restrictions over Gaussian space

**Definition:** Let \( p \) be a polynomial. A \( \lambda \)-random restriction of \( p \) is the polynomial

\[
q(x) = p(\sqrt{\lambda}x + \sqrt{1 - \lambda}y), \quad y \sim N(0, 1)^n.
\]

- The smaller \( \lambda \) is, the “harsher” the random restriction
- \( q = p \)'s behavior in a **local neighborhood** around \( y \), where \( \lambda = \text{radius of neighborhood} \)
Our new structural lemma, restated

Local Hyperconcentration Theorem

Hyperconcentrated degree-$d$ polynomials

$\lambda = d^{-\log d}$

Arbitrary degree-$d$ polynomial $p$

“For most points $y$, the polynomial $p$ behaves like a constant function in a fairly large neighborhood around $y$.”

$\lambda = d^{-\log d}$
The proof of our Local Hyperconcentration Theorem

Degree-d polynomial

Hyperconcentrated ("morally degree 0")

Degree-d polynomial

"morally" degree $\frac{d}{2}$

"morally" degree $\frac{d}{4}$

log $d$ iterations

Technical ingredients:

- Hermite analysis
- Carbery-Wright anticoncentration inequality

Thanks Avi!
Open Problems

- Conjecture: $\lambda = d^{-O(1)}$ suffices
- Random restriction lemma for polynomials over Boolean space?

Local Hyperconcentration Theorem

Hyperconcentrated degree-$d$ polynomials

$\lambda = d^{-\log d}$

Arbitrary degree-$d$ polynomial $p$
Thank you!