

# LTL Fonts Proof Sheet

An LTL formula  $\varphi$  is defined by the following grammar:

$$\begin{aligned} \varphi ::= & p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \varphi \leftrightarrow \varphi \mid \forall x \varphi \mid \exists x \varphi \\ & \mid \Box\varphi \mid \Diamond\varphi \mid \bigcirc\varphi \mid \varphi \mathcal{U} \varphi \mid \varphi \mathcal{W} \varphi \mid \Box\varphi \mid \Diamond\varphi \mid \ominus\varphi \mid \odot\varphi \mid \varphi \mathcal{S} \varphi \mid \varphi \mathcal{B} \varphi, \end{aligned}$$

where  $p$  is an assertion and  $x$  is a variable.

The truth of an LTL formula  $\varphi$  at position  $n$  of an infinite sequence  $\sigma$  of states is denoted  $\sigma, n \models \varphi$  and defined, by induction on the structure of  $\varphi$ , as follows:

- $\sigma, n \models p$ , for  $p$  an assertion, if  $p$  holds at state  $\sigma[n]$ ;
- $\sigma, n \models \neg\varphi$  if  $\sigma, n \not\models \varphi$ ;
- $\sigma, n \models \varphi \wedge \psi$  if both  $\sigma, n \models \varphi$  and  $\sigma, n \models \psi$ ;
- $\sigma, n \models \varphi \vee \psi$  if either  $\sigma, n \models \varphi$  or  $\sigma, n \models \psi$ , or both;
- $\vdots$
- $\sigma, n \models \Box\varphi$  if  $\sigma, i \models \varphi$  for all  $i \geq n$ ;
- $\sigma, n \models \Diamond\varphi$  if there exists  $i \geq n$  such that  $\sigma, i \models \varphi$ ;
- $\sigma, n \models \bigcirc\varphi$  if  $\sigma, (i+1) \models \varphi$ ;
- $\sigma, n \models \varphi \mathcal{U} \psi$  if there exists  $i \geq n$  such that  $\sigma, i \models \psi$  and  $\sigma, j \models \varphi$  for all  $j \in [n, i]$ ;
- $\sigma, n \models \varphi \mathcal{W} \psi$  if either  $\sigma, n \models \varphi \mathcal{U} \psi$  or  $\sigma, n \models \Box\varphi$ ;
- $\sigma, n \models \Box\varphi$  if  $\sigma, i \models \varphi$  for all  $i \in [0, n]$ ;
- $\sigma, n \models \Diamond\varphi$  if there exists  $i \in [0, n]$  such that  $\sigma, i \models \varphi$ ;
- $\sigma, n \models \ominus\varphi$  if  $n > 0$  and  $\sigma, (i-1) \models \varphi$ ;
- $\sigma, n \models \odot\varphi$  if either  $n = 0$  or  $\sigma, (i-1) \models \varphi$ ;
- $\sigma, n \models \varphi \mathcal{S} \psi$  if there exists  $i \in [0, n]$  such that  $\sigma, i \models \psi$  and  $\sigma, j \models \varphi$  for all  $j \in (i, n]$ ;
- $\sigma, n \models \varphi \mathcal{B} \psi$  if either  $\sigma, n \models \varphi \mathcal{S} \psi$  or  $\sigma, n \models \Box\varphi$ .

The strict versions of the operators are defined as follows:

$$\begin{aligned} \hat{\Box}\varphi &\equiv \bigcirc\Box\varphi & \hat{\ominus}\varphi &\equiv \ominus\Box\varphi \\ \hat{\Diamond}\varphi &\equiv \bigcirc\Diamond\varphi & \hat{\odot}\varphi &\equiv \odot\Diamond\varphi \\ \varphi \hat{\mathcal{U}} \psi &\equiv \bigcirc(\varphi \mathcal{U} \psi) & \varphi \hat{\mathcal{S}} \psi &\equiv \ominus(\varphi \mathcal{S} \psi) \\ \varphi \hat{\mathcal{W}} \psi &\equiv \bigcirc(\varphi \mathcal{W} \psi) & \varphi \hat{\mathcal{B}} \psi &\equiv \ominus(\varphi \mathcal{B} \psi) \end{aligned}$$

Some examples:

$p \rightarrow \Diamond q$	$\Box(p \rightarrow \Diamond q)$	$\Box \Diamond q$
$\Diamond \Box q$	$(\neg q) \mathcal{W} p$	$\Box(p \rightarrow q \widehat{\mathcal{W}} r)$
$\Box \exists u (x = u \wedge \bigcirc(x = u + 1))$	$\forall u \Box(x = u \rightarrow \widehat{\Diamond}(y = u))$	$\exists b. b \wedge \Box(b \leftrightarrow \neg \bigcirc b) \wedge \Box(p \rightarrow b)$
$\Box(q \rightarrow \Diamond p)$	$p \Rightarrow q_m \mathcal{W} q_{m-1} \dots q_1 \mathcal{W} q_0$	$p \mathcal{U} q \sim \Diamond(q \wedge \widehat{\Box} p)$

Operator pairs:

$\Box \Box p$	$\Box \Diamond p$	$\Box \bigcirc p$	$\Box(p \mathcal{U} q)$	$\Box(p \mathcal{W} q)$	
$\Box \Box p$	$\Box \Diamond p$	$\Box \ominus p$	$\Box \ominus p$	$\Box(p \mathcal{S} q)$	$\Box(p \mathcal{B} q)$
$\Diamond \Box p$	$\Diamond \Diamond p$	$\Diamond \bigcirc p$	$\Diamond(p \mathcal{U} q)$	$\Diamond(p \mathcal{W} q)$	
$\Diamond \Box p$	$\Diamond \Diamond p$	$\Diamond \ominus p$	$\Diamond \ominus p$	$\Diamond(p \mathcal{S} q)$	$\Diamond(p \mathcal{B} q)$
$\bigcirc \Box p$	$\bigcirc \Diamond p$	$\bigcirc \bigcirc p$	$\bigcirc(p \mathcal{U} q)$	$\bigcirc(p \mathcal{W} q)$	
$\bigcirc \Box p$	$\bigcirc \Diamond p$	$\bigcirc \ominus p$	$\bigcirc \ominus p$	$\bigcirc(p \mathcal{S} q)$	$\bigcirc(p \mathcal{B} q)$
$\Box p \mathcal{U} \Box q$	$\Diamond p \mathcal{U} \Diamond q$	$\bigcirc p \mathcal{U} \bigcirc q$	$(p \mathcal{U} q) \mathcal{U} (r \mathcal{U} s)$	$(p \mathcal{W} q) \mathcal{U} (r \mathcal{W} s)$	
$\Box p \mathcal{U} \Box q$	$\Diamond p \mathcal{U} \Diamond q$	$\ominus p \mathcal{U} \ominus q$	$\ominus p \mathcal{U} \ominus q$	$(p \mathcal{S} q) \mathcal{U} (r \mathcal{S} s)$	$(p \mathcal{B} q) \mathcal{U} (r \mathcal{B} s)$
$\Box p \mathcal{W} \Box q$	$\Diamond p \mathcal{W} \Diamond q$	$\bigcirc p \mathcal{W} \bigcirc q$	$(p \mathcal{U} q) \mathcal{W} (r \mathcal{U} s)$	$(p \mathcal{W} q) \mathcal{W} (r \mathcal{W} s)$	
$\Box p \mathcal{W} \Box q$	$\Diamond p \mathcal{W} \Diamond q$	$\ominus p \mathcal{W} \ominus q$	$\ominus p \mathcal{W} \ominus q$	$(p \mathcal{S} q) \mathcal{W} (r \mathcal{S} s)$	$(p \mathcal{B} q) \mathcal{W} (r \mathcal{B} s)$
$\Box \Box p$	$\Box \Diamond p$	$\Box \bigcirc p$	$\Box(p \mathcal{U} q)$	$\Box(p \mathcal{W} q)$	
$\Box \Box p$	$\Box \Diamond p$	$\Box \ominus p$	$\Box \ominus p$	$\Box(p \mathcal{S} q)$	$\Box(p \mathcal{B} q)$
$\Diamond \Box p$	$\Diamond \Diamond p$	$\Diamond \bigcirc p$	$\Diamond(p \mathcal{U} q)$	$\Diamond(p \mathcal{W} q)$	
$\Diamond \Box p$	$\Diamond \Diamond p$	$\Diamond \ominus p$	$\Diamond \ominus p$	$\Diamond(p \mathcal{S} q)$	$\Diamond(p \mathcal{B} q)$
$\bigcirc \Box p$	$\bigcirc \Diamond p$	$\bigcirc \bigcirc p$	$\bigcirc(p \mathcal{U} q)$	$\bigcirc(p \mathcal{W} q)$	
$\bigcirc \Box p$	$\bigcirc \Diamond p$	$\bigcirc \ominus p$	$\bigcirc \ominus p$	$\bigcirc(p \mathcal{S} q)$	$\bigcirc(p \mathcal{B} q)$
$\ominus \Box p$	$\ominus \Diamond p$	$\ominus \bigcirc p$	$\ominus(p \mathcal{U} q)$	$\ominus(p \mathcal{W} q)$	
$\ominus \Box p$	$\ominus \Diamond p$	$\ominus \ominus p$	$\ominus \ominus p$	$\ominus(p \mathcal{S} q)$	$\ominus(p \mathcal{B} q)$
$\ominus \Box p$	$\ominus \Diamond p$	$\ominus \bigcirc p$	$\ominus(p \mathcal{U} q)$	$\ominus(p \mathcal{W} q)$	
$\ominus \Box p$	$\ominus \Diamond p$	$\ominus \ominus p$	$\ominus \ominus p$	$\ominus(p \mathcal{S} q)$	$\ominus(p \mathcal{B} q)$
$\Box p \mathcal{S} \Box q$	$\Diamond p \mathcal{S} \Diamond q$	$\bigcirc p \mathcal{S} \bigcirc q$	$(p \mathcal{U} q) \mathcal{S} (r \mathcal{U} s)$	$(p \mathcal{W} q) \mathcal{S} (r \mathcal{W} s)$	
$\Box p \mathcal{S} \Box q$	$\Diamond p \mathcal{S} \Diamond q$	$\ominus p \mathcal{S} \ominus q$	$\ominus p \mathcal{S} \ominus q$	$(p \mathcal{S} q) \mathcal{S} (r \mathcal{S} s)$	$(p \mathcal{B} q) \mathcal{S} (r \mathcal{B} s)$
$\Box p \mathcal{B} \Box q$	$\Diamond p \mathcal{B} \Diamond q$	$\bigcirc p \mathcal{B} \bigcirc q$	$(p \mathcal{U} q) \mathcal{B} (r \mathcal{U} s)$	$(p \mathcal{W} q) \mathcal{B} (r \mathcal{W} s)$	
$\Box p \mathcal{B} \Box q$	$\Diamond p \mathcal{B} \Diamond q$	$\ominus p \mathcal{B} \ominus q$	$\ominus p \mathcal{B} \ominus q$	$(p \mathcal{S} q) \mathcal{B} (r \mathcal{S} s)$	$(p \mathcal{B} q) \mathcal{B} (r \mathcal{B} s)$

Upper case versions of the font (right column):

$\Box P \rightarrow \bigcirc \Diamond Q$	$\Box P \rightarrow \bigcirc \Diamond Q$
$\Box \exists u (x = u \wedge \bigcirc(x = u + 1))$	$\Box \exists u (x = u \wedge \bigcirc(x = u + 1))$