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ON FINDING ADDITIVE, SUPERADDITIVE AND SUBADDITIVE SET-FUNCTIONS SUBJECT TO LINEAR INEQUALITIES

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ABSTRACT: Complexity results are obtained with regard to problems of finding solutions to set of linear inequalities which are compatible with some set-functions and a prescribed intersection graph. It is shown that the additive case is NP-complete, the superadditive case is coNP-complete, and the subadditive case is in P.

On Finding Additive, Superadditive and Subadditive Set-Functions Subject to Linear Inequalities

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Abstract. Complexity results are obtained with regard to problems of finding solutions to set of linear inequalities which are compatible with some set-functions and a prescribed intersection graph. It is shown that the additive case is NP-complete, the superadditive case is coNP-complete, and the subadditive case is in P.

1. Introduction

In this note we describe some problems inspired by work on a recent paper [1]. Some of the problems are related to [3].

Let X denote any set and let $v : 2^X \rightarrow R$ be a nonnegative set-function. The function v is called *additive* if for every pair of disjoint subsets $S, T \subset X$, $v(S \cup T) = v(S) + v(T)$. The function is called *superadditive* if for every pair of disjoint subsets $S, T \subset X$, $v(S \cup T) \geq v(S) + v(T)$. Note that a superadditive function (and therefore also an additive one) must be monotone, i.e., $v(S) \leq v(T)$ if $S \subset T$, and satisfy $v(\emptyset) = 0$. Finally, the function is called *subadditive* if it is monotone and for every pair of disjoint subsets $S, T \subset X$, $v(S \cup T) \leq v(S) + v(T)$.

In Section 2 we show that the problem finding a solution of linear inequalities, which is compatible with some additive set-function, can be solved in polynomial time. However, if the structure is prescribed by an intersection graph then the problem is already NP-complete. In Section 3 we discuss the superadditive case. Surprisingly, replacing "additive" by "superadditive" changes the problem from NP-complete into coNP-complete. In Section 4 we show that the subadditive case (yet with the intersection graph) is solvable in polynomial time.

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2. Additive function compatibility

In this section we consider the problem of recognizing whether a system of linear inequalities has a solution which is compatible with some additive function. More precisely, consider first the following problem:

Problem 2.1. The input consists of an $(m \times n)$-matrix $A = (a_{ij})$, an m-vector b, and a scalar c. The problem is to find a vector $x \in R^n$ such that

(i) $Ax \geq b$, and
(ii) there exist a set X, an additive set-function $v : 2^X \rightarrow R$, and subsets $E_1, \cdots, E_n \subseteq X$ such that $v(\bigcup_{j=1}^{n} E_j) = c$ and $v(E_j) = x_j$ $(j = 1, \cdots, n)$.

We wish to formulate Problem 2.1 as a mathematical programming problem. Thus, we are looking for a characterization of vectors x which satisfy condition (ii) of the problem. This is given in the following lemma:

Lemma 2.2. *For every nonnegative vector $x \in R^n$ and for any scalar c, condition (ii) of Problem 2.1 is satisfied if and only if*

$$\max_{1 \leq j \leq n} x_j \ \leq \ c \ \leq \ \sum_{j=1}^{n} x_j \ .$$

Proof: Denote

$$\bar{c} = \sum_{j=1}^{n} x_j$$

and

$$\underline{c} = \max_{1 \leq j \leq n} x_j \ .$$

Assume without loss of generality that $x_1 \geq x_2 \geq \cdots \geq x_n$. The "only if" part is obvious. For the "if" part consider first two extreme cases. Suppose $c = \bar{c}$. Obviously, there exist a set $\bar{X}$, an additive set-function $\bar{v}$ on $\bar{X}$, and pairwise disjoint subsets $\bar{E}_j \subseteq \bar{X}$ $(j = 1, \cdots, n)$ such that $\bar{v}(\bar{E}_j) = x_j$. In the other extreme case, $c = \underline{c}$, there exist a set $\underline{X}$ $(\bar{X} \cap \underline{X} = \emptyset)$, an additive set-function $\underline{v}$, and subsets $\underline{E}_j \subseteq \underline{X}$ $(j = 1, \cdots, n)$,

$$\underline{E}_1 \supseteq \underline{E}_2 \supseteq \cdots \supseteq \underline{E}_n \ ,$$

such that

$$\underline{v}(\underline{E}_j) = x_j \ .$$

In general, if $\underline{c} < c < \bar{c}$, let t be the number between 0 and 1 such that

$$c = (1 - t)\underline{c} + t\bar{c} \ .$$

Let $X = \underline{X} \cup \bar{X}$, $E_j = \underline{E}_j \cup \bar{E}_j$ $(j = 1, \cdots, n)$ and for every $S \subseteq X$, let

$$v(S) = (1-t)\underline{v}(S \cap \underline{X}) + t\bar{v}(S \cap \bar{X}) \ .$$

It is easy to verify that the function v is additive, $v(E_j) = x_j$ and $v(\cup_{j=1}^n E_j) = c$. ∎

We thus have the following:

Proposition 2.3. *Problem 2.1 can be solved in polynomial time as the following system of linear inequalities:*

$$Ax \geq b \quad , \quad \sum_{j=1}^n x_j \geq c \quad , \quad 0 \leq x_j \leq c \ \ (j = 1, \cdots, n) \ .$$

Problem 2.1 turned out to be easy, probably due to the lack of structural requirements on the sets E_j. We now consider a more structured problem. Suppose we are required to have the sets $E_1, \cdots, E_n$ so that $E_i \cap E_j \neq \emptyset$ if and only if the pair (E_i, E_j) is in a certain given set $\mathcal{E}$ of pairs. In other words, denoting $\mathcal{V} = \{E_1, \cdots, E_n\}$, the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ (called the intersection graph of the sets $E_1, \cdots, E_n$) is prescribed. Of course, we could consider more complicated structural constraints but it turns out that with a prescribed intersection graph the problem is already NP-complete. Thus, consider the following problem:

Problem 2.4. The input consists of an $(m \times n)$-matrix $A = (a_{ij})$, an m-vector b, a scalar c, and a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ $(|\mathcal{V}| = n)$. The problem is to recognize whether there exist a set X, a family of n distinct subsets $E_1, \cdots, E_n \subseteq X$ consistent with the intersection graph $\mathcal{G}$, and an additive set-function $v : 2^X \rightarrow R$, such that

$$\sum_j a_{ij} v(E_j) \geq b_i \quad (i = 1, \cdots, m)$$

$$v(\cup_{j=1}^n E_j) = c \ .$$

We first prove:

Proposition 2.5. *Problem 2.4 is* NP*-hard.*

Proof: The proof follows by reduction from the maximum independent set problem [2]. Suppose a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a number k are given, and we have to recognize whether there exists in $\mathcal{G}$ an independent set of vertices (i.e., a set $U \subset \mathcal{V}$ of vertices such that for every pair $u, v \in U$, $(u, v) \notin \mathcal{E}$) whose cardinality is at least k. Consider an instance of Problem 2.4 with a single inequality:

$$\sum_{j=1}^n x_j \geq k \ ,$$

where $c = 1$, and the prescribed intersection graph is the graph $\mathcal{G}$. Let X denote a set and let v denote an additive set-function on X. Let $E_1, \cdots, E_n$ denote subsets of X consistent with the graph $\mathcal{G}$. Without loss of generality assume

$$X = \cup_{j=1}^{n} E_j \ .$$

The problem is equivalent to the system:

$$\sum_{j=1}^{n} v(E_j) \geq k$$

$$v(X) = 1 \ .$$

For every $S \subseteq \mathcal{V}$, denote

$$A_S = \bigcap_{j \in S} E_j \cap \bigcap_{j \notin S} \bar{E}_j$$

and let

$$\pi_S = v(A_S) \ .$$

Obviously,

$$E_j = \bigcup_{S \ni j} A_S$$

so we have

$$v(E_j) = \sum_{S \ni j} \pi_S$$

$$\sum_{S \subseteq \mathcal{V}} \pi_S = v(X)$$

$$\pi_S \geq 0 \ .$$

Note that

$$E_i \cap E_j = \bigcap_{S \supseteq \{i,j\}} A_S \ ,$$

so

$$v(E_i \cap E_j) = \sum_{S \supseteq \{i,j\}} \pi_S \ .$$

Thus, for every pair i, j such that $E_i \cap E_j = \emptyset$, and for every S such that $S \supseteq \{i, j\}$, we must have $\pi_S = 0$. In other words, for every non-independent set S, $\pi_S = 0$. Let $\mathcal{I}$ denote the family of independent sets in $\mathcal{G}$. There is no difficulty with the requirements that certain pairs of sets have nonempty intersection; a solution where such a pair does not intersect can be modified by adding a common element to which the function assigns value zero. Thus, Problem 2.4 is equivalent to the following system of linear inequalities in the variables π_S:

$$\sum_{S \in \mathcal{I}} |S| \cdot \pi_S \geq k$$

$$\sum_{S \in \mathcal{I}} \pi_S = 1$$

$$\pi_S \geq 0 \quad (S \in \mathcal{I}) \ .$$

It is easy to see that the latter has a solution if and only if there exists an independent set S such that

$$|S| \geq k \ .$$

∎

Proposition 2.6. *Problem 2.4 is in the class* NP.

Proof: To prove membership in NP, note that with the notation established in the proof of Proposition 2.5, Problem 2.4 is equivalent to the following system of linear inequalities:

$$\sum_{S \in \mathcal{I}} a_{iS} \pi_S \geq b_i \quad (i = 1, \cdots, m)$$
$$\sum_{S \in \mathcal{I}} \pi_S = c$$
$$\pi_S \geq 0 \quad (S \in \mathcal{I}) \ .$$

It is well-known that if this system has a solution then it must have a solution where at most $m + 1$ of the π_S's are positive. Moreover, there exists a solution whose size is bounded by a polynomial in the size of the problem. Membership in NP is evident from these observations. ∎

We have established

Proposition 2.7. *Problem 2.4 is* NP-*complete.*

3. Superadditive function compatibility

In this section we consider problems similar to the ones of Section 2, involving superadditive rather than additive functions. However, we first need to explain a certain difference between the two problems. An analogue of Problem 2.1 (which turns out to be too easy) is phrased as follows.

Problem 3.1. The input consists of an $(m \times n)$-matrix $A = (a_{ij})$, an m-vector b, and a scalar c. The problem is to find a vector $x \in R^n$ (or conclude that none exists) such that

(i) $Ax \geq b$, and
(ii) there exist a *superadditive* set-function v on a set X and subsets $E_1, \cdots, E_n \subseteq X$ such that $v(\cup_{j=1}^n E_j) = c$ and $v(E_j) = x_j$ $(j = 1, \cdots, n)$.

Note that the superadditivity condition applies to pairs of disjoint sets. Suppose $E_1, \cdots, E_n \subseteq X$ are such that every two of them intersect and none of them contains any other one. Suppose the (nonnegative) values $v(E_j)$ $(j = 1, \cdots, n)$ are given. Now, extend the function v by defining

$$v(S) = \max_{E_j \subseteq S} v(E_j) \ .$$

Obviously, the resulting function is superadditive. This implies the following proposition which makes Problem 3.1 too easy:

Proposition 3.2. *Problem 3.1 is equivalent to the following system of linear inequalities:*

$$Ax \geq b \quad , \quad 0 \leq x_j \leq c \quad (j = 1, \cdots, n) \ .$$

As in Section 2, a more interesting problem arises when some structure is imposed on the sets E_j through a prescribed intersection graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$:

Problem 3.3. The input consists of an $(m \times n)$-matrix $A = (a_{ij})$, an m-vector b, a scalar c, and a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ $(|\mathcal{V}| = n)$. The problem is to recognize whether there exist a set X, a family of n distinct subsets $E_1, \cdots, E_n \subseteq X$ consistent with the intersection graph $\mathcal{G}$, and a superadditive set-function $v : 2^X \rightarrow R$, such that

$$\sum_j a_{ij} v(E_j) \geq b_i \quad (i = 1, \cdots, m)$$

$$v(\cup_{j=1}^n E_j) = c \ .$$

Again, we assume without loss of generality that

$$X = \cup_{j=1}^n E_j \ .$$

The analysis of the computational complexity of Problem 3.3 is based on the following lemma.

Lemma 3.4. *Let $E_1, \cdots, E_n$ be subsets of a set $X = \cup_{j=1}^n E_j$ none of which contains the other, and let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote their intersection graph. Under these conditions, a partial set-function v, satisfying $v(E_j) = w_j \geq 0$ $(i = 1, \cdots, n)$ and $v(X) = c$, can be extended into a superadditive set-function on the set X if and only if the maximum weight (in terms of the numbers of the form w_k) of an independent set of vertices is not greater than c.*

Proof: The "only if" part is obvious. For the "if" part, define a set-function u by

$$u(S) = \max \left\{ \sum_{i=1}^k v(E_{j_i}) \right\} \ ,$$

where the maximum is taken over all the sums of $v(E_{j_i})$ such that $E_{j_1}, \cdots, E_{j_k}$ are pairwise disjoint subsets of S; if S does not contain any E_j, $u(S) = 0$. The function u is superadditive since, if $S_1 \cap S_2 = \emptyset$, the family of independent sets of E_i's contained in $S_1 \cup S_2$ contains all the unions of independent sets of E_i's contained in S_1 with independent sets of E_i's contained in S_2. It is easy to see that if the condition of the lemma holds then the function u is in fact an extension of v. ∎

Corollary 3.5. *A solution to Problem 3.3 exists if and only if there exist values $v(E_j) = w_j$ such that*

(i) $\sum_j a_{ij} v(E_j) \geq b_i$ *$(i = 1, \cdots, m)$,*
(ii) *the weight (in terms of the w_j's) of any independent set in $\mathcal{G}$ is not greater than c.*

For definitions of the classes NP and coNP and related material see [2].

Proposition 3.6. *Problem 3.3 is complete for the complexity class* coNP *in polynomial-time reducibilities.*

Proof: The fact that the complement of Problem 3.3 is NP-hard follows easily from Corollary 3.5 since the maximum independent set problem in a graph can be trivially reduced to Problem 3.3. Thus the interesting part of the proposition is membership in coNP. We now show how in view of Corollary 3.5 one can recognize in nondeterministic polynomial time that the problem has no solution. A solution has to satisfy the following system of linear inequalities:

$$\begin{aligned} Ax &\geq b \\ \sum_{j \in S} x_j &\leq c \quad (S \in \mathcal{I}) \\ x_j &\geq 0 \quad (j = 1, \cdots, n) \end{aligned}$$

(where $\mathcal{I}$ is the family of independent sets in $\mathcal{G}$). By linear programming duality, this problem has a solution if and only if the following system (in the variables $y = (y_1, \cdots, y_m)^T$ and $(\pi_S)_{S \in \mathcal{I}}$) does *not* have a solution:

$$\begin{aligned} b^T y - c \sum_{S \in \mathcal{I}} \pi_S &\geq 1 \\ A^T y - \sum_{\substack{S \in \mathcal{I} \\ S \ni j}} \pi_S &\leq 0 \quad (j = 1, \cdots, n) \\ y_i, \pi_S &\geq 0 \ . \end{aligned}$$

However, if the latter has a solution then it also has one whose size is polynomial in the size of the problem, where at most $n + 1$ of the variables y_j and π_S are positive. This implies that the latter problem is in NP and hence our original problem is in coNP. ∎

4. Subadditive function compatibility

In this section we consider problems similar to the ones of the previous sections, but now with (monotone) subadditive functions.

Problem 4.1. The input consists of an $(m \times n)$-matrix $A = (a_{ij})$, an m-vector b, a scalar c, and a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ ($|\mathcal{V}| = n$). The problem is to recognize whether there exist a set X, a family of n distinct subsets $E_1, \cdots, E_n \subseteq X$ consistent with the intersection graph $\mathcal{G}$, and a subadditive set-function $v : 2^X \to R$, such that

$$\sum_j a_{ij} v(E_j) \geq b_i \quad (i = 1, \cdots, m)$$
$$v(\cup_{j=1}^n E_j) = c \ .$$

Again, we assume without loss of generality that

$$X = \cup_{j=1}^n E_j \ .$$

The analysis of the computational complexity of Problem 4.1 is based on the following lemma.

Lemma 4.2. *Let $E_1, \cdots, E_n$ be subsets of a set $X = \cup_{j=1}^n E_j$ none of which contains the other, and let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote their intersection graph. Under these conditions, a partial set-function v, satisfying $v(E_j) = w_j \geq 0$ $(i = 1, \cdots, n)$ and $v(X) = c$, can be extended into a subadditive set-function on the set X if and only if*

$$\max_{1 \leq k \leq n} \{w_k\} \ \leq \ c \ \leq \ \min_{(E_i, E_j) \notin \mathcal{E}} \{w_i + w_j\} \ .$$

Proof: The "only if" part is obvious. For the "if" part, define a set-function u by

$$u(S) = \max_{E_j \subseteq S} \{v(E_j)\} \quad (S \subset X)$$

(and $u(X) = c$). Obviously, the function u is monotone and extends v. It is also easy to check that it is subadditive if our condition is satisfied. ∎

Corollary 4.3. *Problem 4.1 can be solved in polynomial time as the following system of linear inequalities:*

$$Ax \geq b$$
$$x_i + x_j \geq c \quad ((E_i, E_j) \notin \mathcal{E})$$
$$0 \leq x_j \leq c \quad (j = 1, \cdots, n) \ .$$

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