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Research Report

FINDING A LINE OF SIGHT THRU BOXES IN d-SPACE IN LINEAR TIME

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Abstract:

The following problem is addressed. Given a set of rectangular boxes with edges parallel to the axes in a Euclidean space, find a straight line that intersects all the boxes, or conclude that no such line exists. An algorithm is presented which solves the problem in linear time for any fixed dimension of the space.

Finding a Line of Sight Thru Boxes in d-Space in Linear Time

Nimrod Megiddo*

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In this note we address a problem discussed in a paper by Amenta [1], which can be stated as follows. First, given two vectors $\boldsymbol{a}, \boldsymbol{b} \in R^d$, define

$$B(\boldsymbol{a}, \boldsymbol{b}) = \{\boldsymbol{x} \in R^d \ : \ \boldsymbol{a} \le \boldsymbol{x} \le \boldsymbol{b}\} \ .$$

Problem. Given are n rectangular boxes $B(\boldsymbol{a}^1, \boldsymbol{b}^1), \ldots, B(\boldsymbol{a}^n, \boldsymbol{b}^n)$ in R^d. Find a straight line $\ell \subseteq R^d$ such that for every k $(k = 1, \ldots, n)$, $\ell \cap B^k \ne \emptyset$, or prove that no such line exists. Previous work on the subject can also be found in [2, 3, 4, 5].

Suppose a line ℓ is represented as

$$\ell = \{\boldsymbol{u} + t\boldsymbol{v} \ : \ t \in R\}$$

where $\boldsymbol{u} = (u_1, \ldots, u_d)$ and $\boldsymbol{v} = (v_1, \ldots, v_d) \ne \boldsymbol{0}$, and $B = B(\boldsymbol{a}, \boldsymbol{b})$ is a box where $\boldsymbol{a} = (a_1, \ldots, a_d)$, $\boldsymbol{b} = (b_1, \ldots, b_d)$. Obviously, $B \cap \ell \ne \emptyset$ if and only if there exists a real number t such that

$$a_i \le u_i + tv_i \le b_i \quad (i = 1, \ldots, d) \ .$$

Denote

$$I_-(\boldsymbol{v}) = \{i : v_i < 0\} \ , \ I_0(\boldsymbol{v}) = \{i : v_i = 0\} \ , \ I_+(\boldsymbol{v}) = \{i : v_i > 0\} \ .$$

Obviously, $B \cap \ell \ne \emptyset$ if and only if

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(i) $a_i \leq u_i \leq b_i$ for $i \in I_0(\boldsymbol{v})$, and
(ii) there exists a t such that

$$\begin{aligned} (a_i - u_i)/v_i \leq t \leq (b_i - u_i)/v_i \quad &\text{for } i \in I_+(\boldsymbol{v}) \quad \text{and} \\ (a_i - u_i)/v_i \geq t \geq (b_i - u_i)/v_i \quad &\text{for } i \in I_-(\boldsymbol{v}) \ . \end{aligned}$$

It follows that yet another characterization of $B \cap \ell \neq \emptyset$ is:

(i) For all $i \in I_0(\boldsymbol{v})$, $a_i \leq u_i \leq b_i$.
(ii) For all $i, j \in I_+(\boldsymbol{v})$,

$$(a_i - u_i)/v_i \leq (b_j - u_j)/v_j \ .$$

(iii) For all $i, j \in I_-(\boldsymbol{v})$,

$$(a_i - u_i)/v_i \geq (b_j - u_j)/v_j \ .$$

(iv) For all $i \in I_+(\boldsymbol{v})$ and $j \in I_-(\boldsymbol{v})$,

$$\begin{aligned} (b_i - u_i)/v_i &\geq (b_j - u_j)/v_j \text{ and} \\ (a_j - u_j)/v_j &\geq (a_i - u_i)/v_i \ . \end{aligned}$$

For every i such that $v_i \neq 0$, let

$$x_i = 1/v_i \quad \text{and} \quad y_i = u_i/v_i \ .$$

Thus,

$$v_i = 1/x_i \quad \text{and} \quad u_i = y_i/x_i \ .$$

We can now state the following lemma:

Lemma. *If $B^k = B(\boldsymbol{a}^k, \boldsymbol{b}^k)$ $(k = 1, \ldots, n)$ are nonempty boxes, then there exists a straight line ℓ such $B^k \cap \ell \neq \emptyset$ $(k = 1, \ldots, n)$ if and only if there exists a partition $\{1, \ldots, d\} = I_- \cup I_0 \cup I_+$ and there exist vectors $\boldsymbol{x}, \boldsymbol{y} \in R^d$ such that*

(i) *For every $i \in I_0$, $\max_k a_i^k \leq \min_k b_i^k$,*
(ii) *For $i \in I_+$, $x_i > 0$, and for $i \in I_-$, $x_i < 0$,*

(iii) *For all $i, j \in I_+$, and for $k = 1, \ldots, n$,*

$$a_i^k x_i - y_i \le b_j^k x_j - y_j \ .$$

(iv) *For all $i, j \in I_-$, and for $k = 1, \ldots, n$,*

$$a_i^k x_i - y_i \ge b_j^k x_j - y_j \ .$$

(v) *For every $i \in I_+$ and $j \in I_-$, and for $k = 1, \ldots, n$,*

$$b_i^k x_i - y_i \ge b_j^k x_j - y_j \quad \text{and} \quad a_j^k x_j - y_j \ge a_i^k x_i - y_i \ .$$

If such I_-, I_0, I_+, $\boldsymbol{x}$, and $\boldsymbol{y}$ exist, then a line $\ell = \{\boldsymbol{u} + t\boldsymbol{v}\}$ can be constructed as follows. For $i \in I_0$, let u_i be any number such that

$$\max_k a_i^k \le u_i \le \min_k b_i^k \ ,$$

and let $v_i = 0$. For any other i, let

$$v_i = 1/x_i \quad \text{and} \quad u_i = y_i/x_i \ .$$

For every partition, the problem of finding suitable $\boldsymbol{x}$ and $\boldsymbol{y}$ amounts to solving a system of not more than $2d^2n + d$ linear inequalities in at most $2d$ variables. Note that some of these inequalities are strict. We can, however, solve our problem by solving the linear programming problem of maximizing ξ subject to: (i) for every $i \in I_+$, $\xi \le x_i$, (ii) for every $i \in I_-$, $\xi \le -x_i$, and subject to all the weak inequalities stated in the lemma. This can thus be done in $O(n)$ time for any fixed d. Since the number of partitions is independent of n, we have the following theorem:

Theorem. *For any fixed dimension d, the problem of finding a traversal line ℓ for a collection of n rectangular boxes parallel to the coordinate axes can be solved in $O(n)$ time.*

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