ON THE COMPLEXITY OF LOCATING LINEAR FACILITIES IN THE PLANE *

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We consider the computational complexity of linear facility location problems in the plane, i.e., given n demand points, one wishes to find r lines so as to minimize a certain objective-function reflecting the need of the points to be close to the lines. It is shown that it is NP-hard to find r lines so as to minimize any isotone function of the distances between given points and their respective nearest lines. The proofs establish NP-hardness in the strong sense. The results also apply to the situation where the demand is represented by r lines and the facilities by n single points.

p-line center, p-line median, planar location, NP-complete, strongly NP-complete

Most of the literature on location theory focuses on models where both the facilities and the users are represented by points in the appropriate space, e.g., points in the Euclidean plane, vertices of a given graph, etc. In this paper we consider a planar case where the facilities are modeled as straight lines. For example, we may view the problems faced by a planner who has to locate r (linear) segments of a new railroad system so as to minimize the average cost to the users who have to reach the tracks from a number of different small communities. Thus, a straight line or a line segment is of natural importance in this context. Sometimes such problems are easier than those with point facilities. For example, it is much easier to find a line, so as to minimize the sum of distances to it from a set of given points, than to find a single point with the same objective (see [7, 8]). Thus, it is interesting to find out that our problems are NP-hard. Our proof establishes NP-hardness of several different problems (e.g., replace ‘sum of distances’ by ‘maximum distance’ and also by the point-line duality we obtain more

problems: point facilities and linear demand sets). We establish a non-trivial reduction from the 3-satisfiability problem to prove strong NP-hardness (see [3]). The complexity of the problems with point facilities and demands was investigated in [2,5,6,9].

Formally, we consider the following problem of locating linear facilities. Given p points \((x_1, y_1), \ldots, (x_p, y_p)\) in the plane, find a set of r straight lines \(L_1, \ldots, L_r\) so as to minimize \(\sum \min_{1 \leq j \leq r} d(x_i, y_i; L_j)\), where \(d(x_i, y_i; L_j)\) denotes the distance between the point \((x_i, y_i)\) and the line \(L_j\), relative to a certain metric on the plane.

A weighted version of the problem when \(r = 1\) was solved in [7] relative to the Euclidean metric in \(O(n^2 \log n)\) time and relative to the rectilinear metric in \(O(n \log^2 n)\) time.

A graphic version of the problem is obtained by replacing the points by the vertices of a given graph and the r lines by edges or paths in the graph. The graphic version is easily verified to be NP-hard. However, testing whether some set of r edges ensures a zero value for the objective-function (i.e., testing whether r edges suffice to cover all vertices) can be accomplished in polynomial-time (via matching). In this paper we prove that the planar analogue (i.e., the question whether r straight lines suffice for covering p given points) is strongly NP-hard. In particular, this proves that
the planar problem formulated above is NP-hard
not only for that specific objective-function but
also for any isotope function of the distances
between points and lines, relative to any non-trivial
metric. Using the duality between points and
lines, our proof also yields the NP-hardness of
location problems with point facilities and linear
demand sets.

Henceforth, we will be dealing with the follow-
ing problems:

1. **Point Covering** (PC). A set of points \((x_1,
y_1), \ldots, (x_p, y_p)\) \((x_i, y_i)\) rationals, \(i = 1, \ldots, p\)
is given. Find a collection of straight lines \(l_1, \ldots, l_s\)
of minimum cardinality, such that \((x_i, y_i)\) lies on
at least one \(l_j\).

2. **Line Covering** (LC). A set of straight lines
\(L_1, \ldots, L_r\) is given. Find a set of points \((x_1,
y_1), \ldots, (x_p, y_p)\) of minimum cardinality such that
each \(L_j\) contains at least one \((x_i, y_i)\).

In view of the renowned duality between lines
and points (see [4], for example) which is discussed
later, the two problems are obviously very close in
nature.

We first mention briefly the trivial cases. First,
PC is trivial when no three points are colinear, in
which case \(\lceil p/2 \rceil\) lines are necessary to cover all
points. Analogously, LC is trivial when every sub-
set of three lines has an empty intersection and
there are no parallel lines.

If there may be parallel lines, but still no inter-
section points of three lines or more, then an
optimal solution for LC can be easily computed as
follows. Partition the set of lines into classes
\(R_1, \ldots, R_s\) such that two lines are parallel if and
only if they belong to the same class \(R_i\). Let
\(r_i = |R_i|, i = 1, \ldots, s\), and assume \(r_1 \geq r_2 \geq \ldots, r_s \geq 1\).
Now, select an arbitrary line from \(R_1\) and an
arbitrary line from \(R_2\). The point of intersection
of two selected lines will belong to the final solution.
Next, drop the two selected lines from \(R_1\) and \(R_2\),
rename the classes so as to conform with the
requirement \(r_i \geq r_{i+1}\) for \(i < j\) and continue in
the same manner. We observe that this is in fact a
particular case of a well-known scheduling prob-
lem, namely minimal-length scheduling of unit-ex-
ecution-time tasks with tree-structured precedence
constraints (see [1, p. 54]), which is solvable by the
'strategy'. The embedding of our problem in
the scheduling problem is by viewing each line as a
task where members of the same \(R_i\) form a chain
and the different chains are disjoint. The number
of machines is two and the interpretation is that at
each time unit at most two lines can be processed
and this is feasible if they are not parallel. Fur-
thermore, the value of the optimal solution is simply
\(\max(r_i, \lceil r/2 \rceil)\). This is easily proved by induction,
distinguishing between the case \(r_1 > r_2\) (where both
\(r_1\) and \(\lceil r/2 \rceil\) decrease by one after the first time
unit) and the case \(r_1 = r_2\) (where only \(\lceil r/2 \rceil\)
decreases but \(\lceil r/2 \rceil > r_1\), so that \(\max(r_1, \lceil r/2 \rceil)\)
decreases in any case).

We now turn to the NP-hardness of the prob-
lems in the general case. First it is easily verified
that both problems are in NP.

We now reduce 3-satisfiability to PC. Let \(E_1 \land \ldots \land E_m\)
be an instance of 3-satisfiability, where
\(E_j = x_j \lor \bar{y}_j \lor \bar{z}_j, \langle x_j, y_j, z_j \rangle \subseteq \langle v_1, \bar{v}_1, \ldots, v_m, \bar{v}_m \rangle, j = 1, \ldots, m\). Assume \(\langle v_i, \bar{v}_i \rangle \cap \langle x_j, y_j, z_j \rangle \subseteq 1\). The
general idea of the reduction is as follows. We
shall construct a set of \(m + m^2\) points, \(m\) cor-
responding to the clauses \(E_1, \ldots, E_m\) and \(m^2\) ones
corresponding to each pair of variables \((v_i, \bar{v}_i)\).
Also, a set of \(2nm\) lines will be constructed with
the following properties (for an example, see Fig. 1):

1. Each clause \(E_j\) is represented by a point \(P_j\).
2. Each pair of variables \((v_i, \bar{v}_i)\) is presented by a
grid of \(m^2\) points \(P_{kl}^i (1 \leq k, l \leq m)\).
3. For each \(i (i = 1, \ldots, n)\) and \(j (j = 1, \ldots, m)\),
the points \(P_{ij}^1, \ldots, P_{ij}^m\) lie on a straight line
denoted by \(L_{ij}\) and the points \(P_{ij}^1, \ldots, P_{ij}^m\) be on
a straight line denoted by \(L_{ij}\).
4. Except for the lines \(L_{ij}, L_{ij} (i = 1, \ldots, m; j = 1, \ldots, m)\),
o other straight line of the plane contains more than two points of the set
\(\langle P_{kl}, i = 1, \ldots, n; k = 1, \ldots, m; l = 1, \ldots, m \rangle \cup (P_1, \ldots, P_m)\).
5. For every \(j (j = 1, \ldots, m)\) the point \(P_j\) lies on
the line \(L_{i_j}\) if and only if \(j = k\) and \(v_i \in \langle x_j, y_j, z_j \rangle\) and \(P_j\) lies on \(L_{i_j}\) if and only if \(j = k\) and
\(\bar{v}_i \in \langle x_j, y_j, z_j \rangle\).

The above five properties establish the reduc-
tion by the following argument. The points of the
form \(P_{kl}^i\) cannot be covered by less than \(nm\) lines,
since no straight line contains more than \(m\) of
them and altogether they number \(nm^2\). Moreover,
to achieve that number, for every \( i (i = 1, \ldots, n) \), the points \( P_{k,l} \) \((1 \leq k, l \leq m)\) must be covered either by the lines \( L_{i,j} \) \((j = 1, \ldots, m)\) or by the lines \( \bar{L}_{i,j} \) \((j = 1, \ldots, m)\). No other collection of \( m \) lines can cover the collection of \( m^2 \) points \( P_{k,l} \) \((1 \leq k, l \leq m)\) (assuming \( m > 2 \)). We claim that \( E_1 \wedge \cdots \wedge E_m \) is satisfiable if and only if the entire collection of points \( \{P_1, \ldots, P_m\} \cup \{\bar{P}_k \mid i = 1, \ldots, n; k = 1, \ldots, m\} \) can be covered by \( nm \) lines. For, the choice between \( \{L_{i,j}\}_{i=1}^m \) and \( \{\bar{L}_{i,j}\}_{i=1}^m \) for a given \( i \) simply corresponds to the assignment of a truth-value to \( (v_i, \bar{v}_i) \). Specifically, for each \( i \), \( v_i \) is true if and only if \( \{L_{i,j}\} \) is chosen to cover the \( m^2 \) points \( P_{k,l} \).
Finally, we have to discuss the actual construction of the points $P_j$ and $P_{ij}$. We will construct points with rational coordinates, maintaining the numerators and the denominators separately. The numerical values of all the numerators and denominators will be bounded by a polynomial in $m$ and $n$. First, let $P_j = (j, j^2)$, $j = 1, \ldots, m$. Thus, no three of the points $P_1, \ldots, P_m$ are collinear. The construction of the points $P_{ij}$ will be carried out with the aid of the lines $L_{ij}$, $\bar{L}_{ij}$ as follows. For each $i$ ($i = 1, \ldots, n$), $P_{ij}$ is the point of intersection of $L_{ik}$ with $L_{ij}$. The lines $L_{ij}$, $\bar{L}_{ij}$ are successively constructed in the order $L_{11}, L_{12}, L_{13}, \ldots, L_{1n}$, $L_{21}, \ldots, L_{2m}$, $\bar{L}_{31}, \ldots$ so as to satisfy properties 3, 4, 5. When a specific line $L_{ij}$ has to be constructed the following conditions should be satisfied:

(i) $L_{ij}$ should contain $P_j$ if and only if $v_i \in (x_j, y_j, z_j)$;

(ii) $L_{ij}$ should not contain any previously constructed point of the form $P_k$ (except possibly for $P_j$ as explained before) or $P_{ij}$;

(iii) $L_{ij}$ should not coincide with any previously constructed line.

When a specific line $\bar{L}_{ij}$ has to be constructed the following conditions should be satisfied:

(i) $\bar{L}_{ij}$ should contain $P_j$ if and only if $\bar{v}_i \in (x_j, y_j, z_j)$;

(ii) $\bar{L}_{ij}$ should not contain any previously constructed point (except possibly for $P_j$);

(iii) $\bar{L}_{ij}$ should not contain a point of intersection of two lines of the form $L_{ik}, L_{ij}$ (in order for the two points $P_{ik}, P_{ij}$ to be distinct);

(iv) $\bar{L}_{ij}$ should not contain a point of intersection between some $L_{ik}$ and another line which contains at least two previously constructed points (in order to satisfy condition 4); the intersection $\bar{L}_{ij}$ with $L_{ik}$ becomes the point $P_{ik}$;

(v) $\bar{L}_{ij}$ should not be parallel to any $L_{ik}$, in order to ensure the existence of the point $P_{ik}$.

Thus, a typical step is that a line has to be constructed so as to (possibly) contain one specified point of the $P_j$'s and not any other point from a finite collection of 'forbidden' points, and also so as not to parallel any one of a finitely numbered lines. Suppose that we always construct the line whose slope is the integer closest to zero among the feasible slopes. The number of 'forbidden' slopes is obviously bounded by some polynomial in $m$ and $n$ and hence the slope of every constructed line is an integer whose absolute value is bounded by that polynomial. If the constructed line also crosses through one of the $P_j$'s (whose coordinates are of the form $(j, j^2)$) then the coefficients of its equation will be polynomially bounded integers. Similarly, if the line should not cross through any $P_j$, then we may construct it so as to cross through an integer point, which is not forbidden, whose distance from the origin is minimal. It follows that the coordinates of such a point are polynomially bounded and hence all our constructed lines will have polynomially bounded integers as their coefficients. This implies that all the points $P_{ij}$ will have coordinates which are rationals with polynomially bounded numerators and denominators. This establishes that PC is strongly NP-hard (see [3]).

To establish that LC is strongly NP-hard we reduce PC to LC by using the point-line duality argument. Specifically, given the points $(a_1, b_1), \ldots, (a_n, b_n)$, we first find a translation $(a_j, b_j) = (a_j + a, b_j + b)$ that will assure that no two points are collinear with the origin. Next, we represent the point $(a_j, b_j)$ by a line $a_j x + b_j y + l = 0$. Thus, two lines corresponding to two distinct points are not parallel and the main property is that points are collinear if and only if their corresponding lines all intersect at a single point.

References