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A Linear Programming Instance with Many Crossover Events*

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Crossover events for a linear programming problem were introduced by Vavasis and Ye and provide important insight into the behavior of the path of centers. The complexity of a layered-step interior-point algorithm presented by them depends on the number of disjoint crossover events and the coefficient matrix A, but not on b and c. In this short note, we present a linear programming instance with more than $n^2/8$ disjoint crossover events.

1. INTRODUCTION

Interior-point methods for linear programming have been developed tremendously since the presentation of Karmarkar (1984). Recently Vavasis and Ye (1994) proposed a layered-step primal–dual interior-point algo-

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rithm, whose number of operations has an upper bound that depends only on the coefficient matrix A and not on b and c.

Crossover events for a linear programming problem were introduced by Vavasis and Ye (1994) and provide important insight into the behavior of the path of centers. The number of operations of the layered-step interior-point algorithm depends on the number of disjoint crossover events. Although the number depends on b and c, they prove that it is bounded by $n(n-1)/2$. The question of whether there could be more than n crossover events was left open in Vavasis and Ye (1994). If one could prove that the number is bounded by $O(n)$, the complexity of the layered-step interior-point algorithm could be reduced by a factor of n. In this short note, we present a linear programming instance with more than $n^2/8$ disjoint crossover events. We believe that the instance is of great help in understanding the behavior of the path of centers.

2. Crossover Events

Let $n \geq m > 0$ be integers. For an $m \times n$ matrix A and vectors $b \in R^m$ and $c \in R^n$, the instance $LP(A, b, c)$ denotes the primal–dual pair of linear programming problems

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, x \geq 0 \end{array}$$

and

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c, s \geq 0. \end{array}$$

We assume that the primal–dual pair has a feasible interior point (x, y, s) (i.e., $x > 0$ and $s > 0$). For each $\mu > 0$, we denote by $(x(\mu), y(\mu), s(\mu))$ the solution of the system

$$\begin{aligned} Ax = b, &\qquad x \geq 0, \\ A^T y + s = c, &\qquad s \geq 0, \\ Xs = \mu e, & \end{aligned}$$

where e is the vector of 1's and $X = \operatorname{diag}(x)$ is the diagonal matrix such that $Xe = x$. The solution is called a center and the set of centers $P = \{(x(\mu), y(\mu), s(\mu)) : \mu > 0\}$ is called the path of centers.

In this note, we define crossover events for the path of centers for simplicity and clarification of the definition, although Vavasis and Ye (1994) defined them for a neighborhood of the path. Moreover their definition depends on a point generated by an algorithm, while ours depends only on the instance $LP(A, b, c)$.

For a given $g > 10$ and $(x, y, s) \in P$, we partition the index set $\{1, 2, \ldots, n\}$ into layers $J_1, J_2, \ldots, J_p$ as follows: Let π be a permutation that sorts the components of s in nondecreasing order:

$$s_{\pi(1)} \leq s_{\pi(2)} \leq \cdots \leq s_{\pi(n)}.$$

Let $J_1 = \{\pi(1), \ldots, \pi(i)\}$ be the set of successive indices of π such that the ratio-gap $s_{\pi(j+1)}/s_{\pi(j)}$ is less than or equal to g for each $j = 1, \ldots, i - 1$ but $s_{\pi(i+1)}/s_{\pi(i)}$ is greater than g. Then put $\pi(i + 1), \pi(i + 2), \ldots$ in J_2, until another ratio-gap greater than g is encountered, and so on. Let J_p be the last set which contains $\pi(n)$.

For two indices $i, j \in \{1, 2, \ldots, n\}$, we denote

$$i \prec j \qquad \text{at } (x, y, s)$$

if there exists $k \in \{1, 2, \ldots, p - 1\}$ such that $i \in J_1 \cup \cdots \cup J_k$ and $j \in J_{k+1} \cup \cdots \cup J_p$. We also denote

$$i \preceq j \qquad \text{at } (x, y, s)$$

if there exists $k \in \{1, 2, \ldots, p\}$ such that $i \in J_1 \cup \cdots \cup J_k$ and $j \in J_k \cup \cdots \cup J_p$.

Definition 1. For an instance $LP(A, b, c)$ and a constant $g > 10$, we say that the 4-tuple (μ, μ', i, j) defines a *crossover event*, if $\{i, j\} \subseteq \{1, 2, \ldots, n\}$,

$$\begin{aligned} &i \preceq j \text{ at the point } (x(\mu), y(\mu), s(\mu)) \in P, \\ &j \prec i \text{ at any } (x(\mu''), y(\mu''), s(\mu'')) \in P \text{ where } \mu'' \in (0, \mu'], \end{aligned} \tag{1}$$

and $s(\mu'')_j \to 0$, $x(\mu'')_i \to 0$ as $\mu'' \to 0$.

Note that one must have $\mu > \mu'$. The definition above is slightly different from the one given in Vavasis and Ye (1994), but the essential meaning is the same. It is not hard to see that whenever our crossover event occurs, theirs does too. So our definition is sufficient to give a lower bound for their crossover events.

Set g to a value that depends only on A, not on b or c; this is important for our instance, where b and c depend on a positive parameter ε that must be much less than $1/g$.

We say that two crossover events $(\mu_1, \mu_1', i_1, j_1)$ and $(\mu_2, \mu_2', i_2, j_2)$ with $\mu_1 > \mu_2$ are *disjoint* if $\mu_1' \geq \mu_2$. Vavasis and Ye proved that the number of disjoint crossover events is bounded by $n(n-1)/2$.

3. The Linear Programming Instance

Let m be a positive integer and let $n = 2m$. We denote the $m \times m$ identity matrix by I. Let $\varepsilon > 0$ be a constant which is much smaller than $1/g$. Then we define an instance $LP(A, b, c)$ by

$$A = (I, I), b = (\varepsilon, \varepsilon^2, \ldots, \varepsilon^m)^T, c = (0, \ldots, 0, \varepsilon^m, \varepsilon^{2m}, \ldots, \varepsilon^{m^2})^T,$$

where the number of 0's in c is m. Then for any $\mu > 0$, the center $(x(\mu), y(\mu), s(\mu))$ is the positive solution of the system

$$\begin{aligned} x_i + x_{i+m} &= \varepsilon^i && \text{for } i = 1, 2, \ldots, m, \\ y_i + s_i &= 0 && \text{for } i = 1, 2, \ldots, m, \\ y_i + s_{i+m} &= \varepsilon^{im} && \text{for } i = 1, 2, \ldots, m, \\ x_i s_i &= \mu && \text{for } i = 1, 2, \ldots, m, \\ x_{i+m} s_{i+m} &= \mu && \text{for } i = 1, 2, \ldots, m. \end{aligned}$$

By solving the system of five variables $(x_i, x_{i+m}, y_i, s_i, s_{i+m})$ and five equations for each i, we have that

$$\begin{aligned} x_i &= \frac{\sqrt{4\mu^2 + \varepsilon^{2i(m+1)}} - (2\mu - \varepsilon^{i(m+1)})}{2\varepsilon^{im}} \\ &= \frac{2\varepsilon^i \mu}{\sqrt{4\mu^2 + \varepsilon^{2i(m+1)}} + (2\mu - \varepsilon^{i(m+1)})}, \\ x_{i+m} &= \frac{-\sqrt{4\mu^2 + \varepsilon^{2i(m+1)}} + (2\mu + \varepsilon^{i(m+1)})}{2\varepsilon^{im}} \\ &= \frac{2\varepsilon^i \mu}{\sqrt{4\mu^2 + \varepsilon^{2i(m+1)}} + (2\mu + \varepsilon^{i(m+1)})}. \end{aligned}$$

Then $s_i = \mu/x_i$, $s_{i+m} = \mu/x_{i+m}$, and $y_i = -s_i$ are easily obtained from these expressions. When the value of μ decreases from 1 to 0, approximate values of x_i, x_{i+m}, s_i and s_{i+m} at a center are shown in Table I for each $i = 1, 2, \ldots, m$, where λ_j ($j = 1, 2, 3, 4$) are real numbers between 1 and 3. When μ is much greater than $\varepsilon^{i(m+1)}$, x_i and x_{i+m} are almost constant with respect to μ, and s_i and s_{i+m} are almost linear with respect to μ. When μ is much less than $\varepsilon^{i(m+1)}$, x_i and s_{i+m} are almost constant with respect to μ, and s_i and x_{i+m} are almost linear with respect to μ.

From Table I, we can compute the layers at the centers $(x(\mu), y(\mu), s(\mu))$ as follows: If $\mu \in [\varepsilon^{m+1}, 1]$ then $p = m$ and

$$J_1 = \{1, m+1\}, J_2 = \{2, m+2\}, \ldots, J_m = \{m, 2m\},$$

if $\mu \in [10g\varepsilon^{m+2}, (10g)^{-1}\varepsilon^{m+1}]$ then $p = m + 1$ and

$$J_1 = \{1\}, J_2 = \{m+1\}, J_3 = \{2, m+2\}, \ldots, J_{m+1} = \{m, 2m\},$$

if $\mu = \varepsilon^{m+2}$ then $p = m$ and

$$J_1 = \{1\}, J_2 = \{2, m+1, m+2\}, \ldots, J_m = \{m, 2m\},$$

and so on. If $\mu = \varepsilon^{2m+1}$ then $p = m + 1$ and

$$J_1 = \{1\}, J_2 = \{2, m+2\}, \ldots, J_m = \{m, 2m\}, J_{m+1} = \{m+1\}.$$

It is easy to see that if $j \in \{1, 2, \ldots, m\}$ then

$$j < m + 1 \text{ at } (x(\mu''), y(\mu''), s(\mu'')) \in P \text{ where } \mu'' \in (0, \varepsilon^{m+j+1}]$$

and $s(\mu'')_j \to 0$, $x(\mu'')_{m+1} \to 0$ as $\mu'' \to 0$. Hence while μ decreases from ε^{m+1} to ε^{2m+1}, there are m disjoint crossover events $(\varepsilon^{m+1}, \varepsilon^{m+2}, m+1, 1)$, $(\varepsilon^{m+2}, \varepsilon^{m+3}, m+1, 2)$, $\ldots$, $(\varepsilon^{2m}, \varepsilon^{2m+1}, m+1, m)$. Similarly, while μ

TABLE I
APPROXIMATE VALUES AT A CENTER

μ	1	$\cdots$	$(2/3)\varepsilon^{i(m+1)}$	$\cdots$	0
x_i	$(1/2)\varepsilon^i$	$(1/\lambda_1)\varepsilon^i$	$(2/3)\varepsilon^i$	$(1/\lambda_2)\varepsilon^i$	ε^i
x_{i+m}	$(1/2)\varepsilon^i$	$(1/\lambda_3)\varepsilon^i$	$(1/3)\varepsilon^i$	$(1/\lambda_4)\mu\varepsilon^{-im}$	0
s_i	$2\varepsilon^{-i}$	$\lambda_1\mu\varepsilon^{-i}$	ε^{im}	$\lambda_2\mu\varepsilon^{-i}$	0
s_{i+m}	$2\varepsilon^{-i}$	$\lambda_3\mu\varepsilon^{-i}$	$2\varepsilon^{im}$	$\lambda_4\varepsilon^{im}$	ε^{im}

decreases from $\varepsilon^{2(m+1)}$ to ε^{3m+2}, there are $m - 1$ crossover events, and so on. As a total, the number of disjoint crossover events is

$$m + (m - 1) + \cdots + 1 = m(m + 1)/2 \geq n^2/8.$$

4. Concluding Remarks

We have presented a linear programming instance with more than $n^2/8$ crossover events. This result indicates that the path of centers consists of more than $n^2/8$ parts each of which defines a partition of $\{1, 2, \ldots, n\}$ as layers. As discussed in Vavasis and Ye (1994), we can also see that the path consists of almost straight parts and curved parts, and the number of such parts is bounded by twice of the number of crossover events. From Table I, we can observe that the projection $\{(x_i, x_{i+m}, s_i, s_{i+m}) : (x, y, s) \in P\}$ of the path of centers consists of two almost straight parts and one curved part for μ around $\varepsilon^{i(m+1)}$. Thus the path of centers appears to consist of $(n/2) + 1$ straight parts and $n/2$ curved parts, which is much less than the number of crossover events. So if we could trace each almost straight part of the path in a constant number of steps, it might yield a very efficient algorithm for linear programming.

Vavasis and Ye have told us that Mike Todd constructed a similar example with the order of n^2 disjoint crossover events for their earlier primal layered-step algorithm.

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