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CYCLIC ORDERING IS NP-COMPLETE

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Abstract. The cyclic ordering problem is to recognize whether a collection of cyclically ordered triples of elements of a set T is derived from an arrangement of all the elements of T on a circle. This problem is shown to be NP-complete.

A cyclic ordering of a set $T = \{1, \ldots, t\}$ is essentially an arrangement of the elements of T on a circle. A specific definition is as follows (see [4]). Two linear orders, $(a_1, \ldots, a_t)$ and $(b_1, \ldots, b_t)$, on T are called *cyclically equivalent* if there exists a number q, $1 \leq q \leq t$, such that $\mu - 1 \equiv (\nu - 1 + q) \pmod t$ implies $a_\nu = b_\mu$. A *cyclic ordering* of T is an equivalence class of linear orders on T modulo cyclic equivalence; the equivalence class containing $(a_1, \ldots, a_t)$ will be denoted by $a_1 a_2 \cdots a_t$.

Cyclic ordering is the following recognition problem. The input is a set Δ of cyclically ordered triples (abbreviated COT's) out of T. The property to be recognized is: There is a cyclic ordering of T from which all the COT's in Δ are derived; Δ is called *consistent* if it has this property.

Evidence for the hardness of *cyclic ordering* was given in [4]. On the other hand, the linear analogue of this problem is known to be easy. Specifically, the property that a set of ordered pairs out of T is derived from a linear order on T, is recognizable in linear time (see [3, Section 2.2.3]).

Our goal here is to prove that *cyclic ordering* is NP-complete[1]. Our problem is obviously in NP since it requires not more than polynomial time to verify that a set of COT's is derived from a certain cyclic ordering. In the remainder of the paper we shall show that *satisfiability with at most* 3 *literals per clause* (abbreviated ST3) is

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[1] The reader is assumed to be familiar with NP-completeness and related topics (see [1, 2]); the notation for *satisfiability* is taken from [2].

reducible to *cyclic ordering*. This will imply, by definition, that *cyclic ordering* is NP-complete.

The input of ST3 consists of clauses $x_\nu \vee y_\nu \vee z_\nu$ ($\nu = 1, \ldots, p$) where $\{x_\nu, y_\nu, z_\nu\} \subset U = \{u_1, \ldots, u_r, \bar{u}_1, \ldots, \bar{u}_r\}$. Without loss of generality assume that if $x_\nu \in \{u_i, \bar{u}_i\}$, $y_\nu \in \{u_j, \bar{u}_j\}$ and $z_\nu \in \{u_k, \bar{u}_k\}$ then $i < j < k$. With each u_τ ($\tau = 1, \ldots, r$) we associate a COT $\alpha_\tau\beta_\tau\gamma_\tau$, and with $\bar{u}_\tau$ we associate the reverse COT $\alpha_\tau\gamma_\tau\beta_\tau$. Let $A = \{\alpha_1, \beta_1, \gamma_1, \ldots, \alpha_r, \beta_r, \gamma_r\}$. It is assumed that the set A has exactly $3r$ distinct elements. With each clause $x \vee y \vee z$ ($\{x, y, z\} \subset U$) we associate a set Δ^0 of COT's as follows. Suppose that abc, def, ghi, are the COT's associated with x, y, z, respectively ($\{a, b, c, d, e, f, g, h, i\} \subset A$). Let $B = \{j, k, l, m, n\}$ be such that $A \cap B = \emptyset$ and assume that the B_ν-s that correspond to the various clauses $x_\nu \vee y_\nu \vee z_\nu$ are pairwise disjoint. Let

$$\Delta^0 = \{acj, bjk, ckl, dfj, ejl, flm, gik, hkm, imn, nml\}.$$

Lemma 1. *Let $S \subset U$ be such that $u_\tau \in S$ if and only if $\bar{u}_\tau \notin S$. Let $x \vee y \vee z$ be any clause. Let Δ be a set of* COT's *defined as follows. Every element of Δ^0 (the set of* COT's *associated with $x \vee y \vee z$) belongs to Δ; the* COT's *associated with the elements of $\{x, y, z\} \setminus S$ belong to Δ; if $\alpha\beta\gamma$ is a* COT *associated with an element of $\{x, y, z\} \cap S$ then $\alpha\gamma\beta$ belongs to Δ. Then, $S \cap \{x, y, z\} \neq \emptyset$ if and only if Δ is consistent.*

Proof. (Only if) The following table proves that Δ is consistent whenever $S \cap \{x, y, z\} \neq \emptyset$.

$S \cap \{x, y, z\}$	Δ	Every element of Δ is derived from
$\{x\}$	$\Delta^0 \cup \{acb, def, ghi\}$	*ackmbdefjlnghi*
$\{y\}$	$\Delta^0 \cup \{abc, dfe, ghi\}$	*abcjkdmflneghi*
$\{z\}$	$\Delta^0 \cup \{abc, def, gih\}$	*abcdefjklngimh*
$\{x, y\}$	$\Delta^0 \cup \{acb, dfe, ghi\}$	*ackmbdfejlnghi*
$\{x, z\}$	$\Delta^0 \cup \{acb, def, gih\}$	*ackmbdefjlngih*
$\{y, z\}$	$\Delta^0 \cup \{abc, dfe, gih\}$	*abcjkdmflnegih*
$\{x, y, z\}$	$\Delta^0 \cup \{acb, dfe, gih\}$	*acbjkdmflnegih*

(If) Notice that if $S \cap \{x, y, z\} = \emptyset$ then $\Delta = \Delta^0 \cup \{abc, def, ghi\}$. Thus, it is sufficient to show that $\Delta^0 \cup \{abc, def, ghi\}$ is inconsistent which would be a contradiction. To that end, consider the following chains of implications:

$$abc \xrightarrow{acj} bcj \xrightarrow{bjk} cjk \xrightarrow{ckl} jkl,$$

$$def \xrightarrow{dfj} efj \xrightarrow{ejl} fjl \xrightarrow{flm} jlm,$$

$$ghi \xrightarrow{gik} hik \xrightarrow{hkm} ikm \xrightarrow{imn} kmn,$$

$$jkl \xrightarrow{jlm} klm \xrightarrow{kmn} lmn.$$

These are interpreted as follows. Let C be any cyclic ordering of $\{a, b, c, \ldots, n\}$ from which all the elements of Δ^0 are derived. Thus, if abc is also derived from C then necessarily (since acj is derived from C) bcj is derived from C, and this implies that cjk is derived from C (since bjk is derived from C), etc. It can be observed that if every element in $\Delta^0 \cup \{abc, def, ghi\}$ is derived from C, then lmn is derived from C. However, this is absurd since $nml \in \Delta^0$. Thus, $\Delta^0 \cup \{abc, def, ghi\}$ is inconsistent and the proof is complete. □

Corollary 2. *Let S be as in Lemma* 1. *For every ν ($\nu = 1, \ldots, p$) let Δ_ν denote the set Δ that corresponds to the clause $x_\nu \vee y_\nu \vee z_\nu$. Under these conditions, $S \cap \{x_\nu, y_\nu, z_\nu\} \neq \emptyset$ for $\nu = 1, \ldots, p$ if and only if $\Delta_1 \cup \cdots \cup \Delta_p$ is consistent.*

Proof. The "if" part is immediate from Lemma 1. We shall prove the "only if" part. It follows from the "only if" part of Lemma 1 that each Δ_ν is derived from a cyclic ordering C_ν of the set of elements appearing in the COT's of Δ_ν. We claim that there is a cyclic ordering C_0 of the set A such that the restriction of each C_ν to elements of A is derived from C_0. Specifically, this cyclic ordering of A is $\delta_1\delta_2 \cdots \delta_{3r}$ where $(\delta_{3\tau-2}, \delta_{3\tau-1}, \delta_{3\tau}) = (\alpha_\tau, \beta_\tau, \gamma_\tau)$ if $\bar{u}_\tau \in S$ and $(\delta_{3\tau-2}, \delta_{3\tau-1}, \delta_{3\tau}) = (\alpha_\tau, \lambda_\tau, \beta_\tau)$ if $u_\tau \in S$. This follows from our choice of the ordering of variables in each clause, the specific orderings shown in our table, and the fact that $u_\tau \in S \iff \bar{u}_\tau \notin S$. Since the B_ν-s are pairwise disjoint and none of them intersects A, it follows that C_0 can be extended to a cyclic ordering C of $A \cup B_1 \cup \cdots \cup B_p$ such that every COT of $\Delta_1 \cup \cdots \cup \Delta_p$ is derived from C. □

Theorem 3. *Let Δ_ν^0 denote the set Δ^0 associated with the clause $x_\nu \vee y_\nu \vee z_\nu$ ($\nu = 1, \ldots, p$). Then the conjunction $(x_1 \vee y_1 \vee z_1) \wedge \cdots \wedge (x_p \vee y_p \wedge z_p)$ is satisfiable if and only if the set $\Delta_1^0 \cup \cdots \cup \Delta_p^0$ is consistent.*

Proof. (Only if) If the conjunction is satisfiable then, by definition, there exists an $S \subset U$ such that $u_\tau \in S \iff \bar{u}_\tau \notin S$ and $S \cap \{x_\nu, y_\nu, z_\nu\} \neq \emptyset$ for $\nu = 1, \ldots, p$. Corollary 2 implies that $\Delta_1^0 \cup \cdots \cup \Delta_p^0$ is consistent.

(If) Suppose that $\Delta_1^0 \cup \cdots \cup \Delta_p^0$ is consistent and let C be an appropriate cyclic ordering of $A \cup B_1 \cup \cdots \cup B_p$. Let $S \subset U$ be the set of all $x \in U$ such that the COT which is associated with x is not derived from C. Obviously, $u_\tau \in S \iff \bar{u}_\tau \notin S$.

Furthermore, it follows from Lemma 1 that for every ν ($\nu = 1, \ldots, p$) $S \cap \{x_\nu, y_\nu, z_\nu\} \neq \emptyset$, since not all the COT's associated with x_ν, y_ν, z_ν are derived from C. This proves that the conjunction is satisfiable. □

We have thus reduced ST3 to *cyclic ordering*. Note that for ST3 with p clauses the corresponding *cyclic ordering* has not more than $10p$ COT's.

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