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PARTIAL AND COMPLETE CYCLIC ORDERS

BY NIMROD MEGIDDO

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We show that, in contrast to a famous theorem on linear orders, not every partial cyclic order on $M = \{1, \ldots, m\}$ can be extended to a complete cyclic order. In fact, the complexity, in a certain sense, of sufficient conditions for such an extendability increases rapidly with m.

DEFINITION 1. (i) Two linear orders, $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_m)$, on M are called *cyclically equivalent* if there exists $k \in M$ such that $[j - 1 \equiv (i - 1 + k) \pmod{m}] \Rightarrow a_i = b_j$.

(ii) A *complete cyclic order* (CCO) on M is an equivalence class C of linear orders modulo cyclic equivalence; denote $a_1 a_2 \cdots a_m$ for the equivalence class containing $(a_1, a_2, \ldots, a_m)$.

DEFINITION 2. A *partial cyclic order* (PCO) on M is a set Δ of cyclically ordered triples (COTs) out of M such that:

(i) $xyz \in \Delta \Rightarrow zyx \notin \Delta$ ("antisymmetry"),

(ii) $\{xyz, xzw\} \subset \Delta \Rightarrow xyw \in \Delta$ ("transitivity");

since $xyz = zxy$, etc., also $yzw \in \Delta$ is implied.

THEOREM 3. (i) *If C is a CCO then the set Δ of all COTs derived from C is a PCO.* (ii) *If Δ is a saturated PCO, i.e., $\{x, y, z\} \in \binom{M}{3}$ & $xyz \notin \Delta \Rightarrow zyx \in \Delta$, then there exists a CCO from which all of Δ's COTs are derived*; *Δ is then said to be extendable to a CCO.*

COROLLARY 4. *A PCO is extendable to a CCO if and only if it is contained in a saturated PCO.*

It is natural to ask whether every PCO is extendable to a CCO (or, equivalently, is contained in a saturated PCO). In view of the following example, the answer is in the negative.

EXAMPLE 5. Let $M = \{a, b, \ldots, m\}$ be the set of the first thirteen letters, and let $\Delta = \{acd, bde, cef, dfg, egh, fha, gac, hcb, abi, cij, bjk, ikl, jlm, kma, lab, mbc, hcm, bhm\}$. Obviously, Δ is a PCO. Suppose that $\Delta^* \supset \Delta$ is a saturated PCO. If $abc \in \Delta^*$ then, since $acd \in \Delta^*$, also $bcd \in \Delta^*$. Then, also $cde \in \Delta^*$, and successive applications of transivity finally yield $acb \in \Delta^*$, which contra-

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dicts antisymmetry. Thus, $abc \notin \Delta^*$ and, therefore, by saturatedness, $acb \in \Delta^*$. Analogously, since $abi \in \Delta^*$, also $cbi \in \Delta^*$, and successive applications of transivity finally yield $abc \in \Delta^*$. Thus, antisymmetry is contradicted again. It follows that there is no saturated PCO that contains Δ.

The unextendability of Δ in Example 5 followed essentially from the fact that neither abc nor cba belongs to any PCO that contains Δ. That gives rise to the following definition.

DEFINITION 6. If $\tau = \{i, j, k\}$, $1 \leqslant i < j < k \leqslant m$, denote $\tau^+ = ijk$ and $\tau^- = kji$ for the two possible cyclic orderings of τ. A PCO Δ is said to satisfy *the nth order conditon* if for every $\tau_1, \ldots, \tau_n \in \binom{M}{3}$ there exists a PCO $\Delta^* \supset \Delta$ and $\epsilon_i \in \{+, -\}$ $(i = 1, \ldots, n)$, such that $\{\tau_1^{\epsilon_1}, \ldots, \tau_n^{\epsilon_n}\} \subset \Delta^*$.

Obviously, all the nth order conditions $(n = 0, 1, \ldots)$ are necessary for extendability to a CCO and, as n increases, the nth order condition becomes stronger. The conjunction of all the nth order conditions $(n = 0, 1, \ldots)$ is a sufficient condition for every m. It is natural to ask whether there exists an n such that the nth order condition suffices for every PCO Δ on a finite set M to be extendable at a CCO. Unfortunately, the answer to this question also is in the negative. A sequence of PCOs that prove this is constructed as follows.

EXAMPLE 7. Let $m_0 = 13$ and let Δ_0 be the PCO on $M_0 = \{1, \ldots, 13\}$ defined in Example 5 (identify a with 1, b with 2, etc.). As we have already seen, Δ_0 is not extendable to a CCO. However, since it is a PCO, it satisfies the 0th order condition. For the purpose of later use in induction, note that $\Delta_0 \setminus \{egh\}$ is extendable to the following complete cyclic ordering: *afbhcgdeijklm.* Suppose, by induction, that Δ_n is a PCO on $M_n = \{1, \ldots, m_n\}$, Δ_n satisfies the the nth order condition but is not extendable to a CCO. Suppose also that $xyz \in \Delta_n$ is such that $\Delta_n \setminus \{xyz\}$ is extendable to a CCO. We construct Δ_{n+1} as follows. Define $m_{n+1} = m_n + 15$ and $M_{n+1} = \{1, \ldots, m_{n+1}\}$ and denote $(u_1, \ldots, u_5, v_1, \ldots, v_5, w_1, \ldots, w_5) = (m_n + 1, \ldots, m_{n+1})$. Let

$$\begin{aligned}\Delta' = (\Delta_n \setminus \{xyz\}) \cup \{&zu_1u_2, yu_2u_3, u_1u_3u_4, u_2u_4u_5, u_3u_5x, u_4xy, xyv_1,\\ &u_1v_1v_2, yv_2v_3, v_1v_3v_4, v_2v_4v_5, v_3v_5x, v_4xy, v_5yu_1,\\ &u_5yw_1, zw_1w_2, yw_2w_3, w_1w_3w_4, w_2w_4w_5, w_3w_5u_5,\\ &w_4u_5y, w_5yz\}.\end{aligned}$$

Let Δ_{n+1} be the transitive closure of Δ', i.e., Δ_{n+1} is the intersection of all the transitive classes of COTs that contain Δ' (see Definition 2). It turns out that Δ_{n+1} is a PCO, but is not extendable to a CCO. Also, $\Delta_{n+1} \setminus \{w_3w_5u_5\}$ *is* extendible to a CCO. The proof of these facts follows from the analogous properties of Δ_n. The important property of Δ_{n+1} is that it satisfies the $(n + 1)$st order condition. A detailed proof will be given elsewhere. Here, we indicate that two cases are distinguished when a set of 3-element subsets of M_{n+1} is given.

First, when $|\tau_i \cap M_{n+1}| \geqslant 2$ for $i = 1, \ldots, n+1$, the $\epsilon_i - s$ are determined essentially by the CCO on M_n to which $\Delta_n \setminus \{xyz\}$ is extendible. Otherwise, the induction hypothesis is applied and the $\epsilon_i - s$ are determined essentially by a PCO Δ^* that contains Δ_n and n of the $\tau_i - s$.

In view of Example 7, an algorithm for extending a PCO to a CCO which is based on successive addings of COTs, cannot be polynomial. We conjecture that there is no polynomial algorithm for this problem; note that there seems to be an equivalence between our problem and that of the Hamiltonian path, from the point of view of complexity of computations.

DEPARTMENT OF STATISTICS, TEL AVIV UNIVERSITY, TEL AVIV, ISRAEL