On the ϵ-perturbation Method for Avoiding Degeneracy

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Abstract. Although it is NP-complete to decide whether a linear programming problemis degenerate, the ϵ-perturbation method can be used to reduce in polynomial time any linear programming problemwith rational coefficients to a nondegenerate problem. The perturbed problem has the same status as the given one in terms of feasibility and unboundedness, and optimal bases of the perturbed problem are optimal in the given problem.

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1. Introduction

Degenerate problems cause some inconvenience in the practice as well as in the theory of linear programming. However, in this note we are interested only in the theoretical side. Many methods are known for avoiding the evils caused by degeneracy in the context of the simplex method (see, for example, Murty [3]). When a new algorithm is proposed, the analysis is often complicated by the need to address degeneracy, and results are sometimes proved under a nondegeneracy assumption.

Our aim here is to point out that for theoretical purposes degeneracy can easily be dispensed with in polynomial time. The basic idea is an old one and is known as the "ϵ-perturbation" method due to Charnes [2]. In the context of simplex-type methods there is no need to determine a precise value for ϵ. However, for a more general application, it is interesting to point out that an ϵ whose size is bounded by a polynomial in terms of the input size can be determined in polynomial time.

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2. Preliminaries

Consider the linear programming problem:

$$(P) \qquad \begin{array}{ll} \text{Maximize} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

where $\boldsymbol{A} \in \mathcal{Z}^{m\times n}$, $\boldsymbol{b} \in \mathcal{Z}^m$ and $\boldsymbol{c} \in \mathcal{Z}^n$ are integral, and $\boldsymbol{A}$ is of rank m. Assuming $\boldsymbol{A}$ has no zero row or column, the "size" of A is at least

$$\max(m, n) + \lceil \log_2(M+1) \rceil$$

where M is the maximum absolute value of any entry of $\boldsymbol{A}$.

The problem (P) is said to be primal degenerate if $\boldsymbol{b}$ is a linear combination of less than m columns of $\boldsymbol{A}$. The problem of recognizing primal degeneracy is NP-complete [1]. Denote

$$\boldsymbol{\epsilon} = (\epsilon, \epsilon^2, \cdots, \epsilon^m)^T.$$

It is well-known that there exists $\epsilon_0 > 0$ such that for every ϵ, $0 < \epsilon \leq \epsilon_0$, the perturbed problem, with the vector

$$\boldsymbol{b}(\epsilon) = \boldsymbol{b} + \boldsymbol{\epsilon}$$

replacing $\boldsymbol{b}$, is primal-nondegenerate. The proof follows by observing that for any basis (i.e., a nonsingular submatrix $\boldsymbol{B} \in \mathcal{Z}^{m\times m}$) of $\boldsymbol{A}$, the coordinates of the vector $\boldsymbol{B}^{-1}\boldsymbol{b}(\epsilon)$ are nonzero polynomials of degrees not greater than m in terms of ϵ, and hence do not vanish in some open interval $(0, \epsilon_0)$.

The dual problem of (P) is

$$(D) \qquad \begin{array}{ll} \text{Minimize} & \boldsymbol{b}^T\boldsymbol{y} \\ \text{subject to} & \boldsymbol{A}^T\boldsymbol{y} - \boldsymbol{v} = \boldsymbol{c} \\ & \boldsymbol{v} \geq \boldsymbol{0}\ . \end{array}$$

Dual nondegeneracy means that in every solution of

$$\boldsymbol{A}^T\boldsymbol{y} - \boldsymbol{v} = \boldsymbol{c}$$

the vector $\boldsymbol{v}$ has at least $n - m$ nonzeros. Dual degeneracy can be handled by an ϵ-perturbation of the vector $\boldsymbol{c}$. In fact, the resolution of primal degeneracy and dual degeneracy can be accomplished independently, so we concentrate here on the primal one.

3. The perturbation and its consequences

We first determine a valid value for ϵ_0.

Proposition 3.1. *For every $\epsilon > 0$ such that*

$$\epsilon < \epsilon_0 = \frac{1}{(m!)^2 M^{2m-1}} ,$$

the problem

$$(P(\epsilon)) \qquad \begin{array}{ll} \text{Maximize} & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} + \boldsymbol{\epsilon} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

is primal-nondegenerate.

Proof: Let $\boldsymbol{B}$ be any basis and let $\boldsymbol{B}_i^{-1}$ denote the ith row of $\boldsymbol{B}^{-1}$. Consider the polynomial

$$p(\epsilon) = a_0 + \sum_{j=1}^{m} a_j \epsilon^j = \boldsymbol{B}_i^{-1} \boldsymbol{b}(\epsilon) = \boldsymbol{B}_i^{-1} \boldsymbol{b} + \sum_{j=1}^{m} B_{ij}^{-1} \epsilon^j .$$

Now, $p(\epsilon)$ is not identically zero since $\boldsymbol{B}^{-1}$ is of course nonsingular. For every j ($j = 0, 1, \cdots, m$), if $a_j \neq 0$, then

$$|a_j| \geq \frac{1}{|\det(B)|} \geq \frac{1}{m! M^m} .$$

On the other hand, for $j = 1, \cdots, m$,

$$|a_j| \leq (m-1)! M^{m-1} .$$

Let k be the smallest index such that $a_k \neq 0$. For $\epsilon \leq \epsilon_0 < 1$, we have

$$|p(\epsilon)| \geq |a_k| \epsilon^k - \sum_{j=k+1}^{m} |a_j| \epsilon^j \geq |a_k| \epsilon^k - \epsilon^{k+1} \sum_{j=k+1}^{m} |a_j| \geq \frac{\epsilon^k}{m! M^m} - \epsilon^{k+1} \cdot m! M^{m-1} > 0 .$$

This estimate implies our claim. ∎

Note that the number ϵ_0 can be computed from the data in polynomial time.

Corollary 3.2.

(i) *A basis $\boldsymbol{B}$ is feasible in $(P(\epsilon))$ for some $\epsilon \in (0, \epsilon_0)$ if and only if it is feasible in $(P(\epsilon))$ for every $\epsilon \in (0, \epsilon_0)$.*

(ii) *If a basis* $\boldsymbol{B}$ *is a feasible in* $(P(\epsilon))$ *for some* $\epsilon \in (0, \epsilon_0)$ *then it is feasible in* (P).

Remark 3.3. A basis $\boldsymbol{B}$ may be feasible in (P) but infeasible in $(P(\epsilon))$ for all $\epsilon > 0$. A trivial example is the system $-x_1 = 0, x_1 \geq 0$. Thus, the result of Corollary 3.2 is not yet satisfactory since by solving $(P(\epsilon))$ instead of (P) we may reach a wrong conclusion that (P) is infeasible. This difficulty will be resolved below.

Denote $\boldsymbol{e} = (1, \cdots, 1)^T$ and let

$$K = (m!)^2 M^{2m-1} \ .$$

Consider the problem

$$\begin{aligned} \text{Maximize} \ \ & \boldsymbol{c}^T \boldsymbol{x} \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} + \boldsymbol{\epsilon} \\ & \boldsymbol{x} \geq -\epsilon K \boldsymbol{e} \end{aligned}$$

or, equivalently,

$$(\tilde{P}(\epsilon)) \qquad \begin{aligned} \text{Maximize} \ \ & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} \ \ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} + \epsilon K \boldsymbol{A}\boldsymbol{e} + \boldsymbol{\epsilon} \\ & \boldsymbol{x} \geq \boldsymbol{0} \ . \end{aligned}$$

Proposition 3.4. *The problem* (P) *is feasible if and only if there exists a basis* B *which is feasible in* $(\tilde{P}(\epsilon))$ *for every* ϵ *such*

$$0 \leq \epsilon \leq \epsilon_1 = \frac{1}{m^2 (m!)^4 M^{4m}} \ .$$

Proof: The 'if' part is obvious. Suppose (P) is feasible. Then there exists a basis $\boldsymbol{B}$ such that $\boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$. Obviously, for every i $(i = 1, \cdots, m)$

$$\boldsymbol{B}_i^{-1}(\boldsymbol{b} + \boldsymbol{\epsilon}) \geq -(m-1)! M^{m-1} \epsilon > -\epsilon K \ .$$

This implies that for every $\epsilon \geq 0$, $(\tilde{P}(\epsilon))$ is feasible. Thus, for every $\epsilon \geq 0$ there exists a basis $\boldsymbol{B}_\epsilon$ such that

$$\boldsymbol{B}_\epsilon^{-1}(\boldsymbol{b} + \epsilon K \boldsymbol{A}\boldsymbol{e} + \boldsymbol{\epsilon}) \geq \boldsymbol{0} \ .$$

Let $\boldsymbol{B}$ be any basis, let i be any index $(1 \leq i \leq m)$, and consider the polynomial

$$\begin{aligned} p(\epsilon) =& a_0 + \sum_{j=1}^{m} a_j \epsilon^j = \boldsymbol{B}_i^{-1}(\boldsymbol{b} + \epsilon K \boldsymbol{A}\boldsymbol{e} + \boldsymbol{\epsilon}) \\ =& \boldsymbol{B}_i^{-1}\boldsymbol{b} + (K \boldsymbol{B}_i^{-1} \boldsymbol{A}\boldsymbol{e} + B_{i1}^{-1})\epsilon + \sum_{j=2}^{m} B_{ij}^{-1} \epsilon^j \ . \end{aligned}$$

We first claim that $p(\epsilon)$ is not identically zero. The proof of this claim is as follows. There exists j $(1 \leq j \leq m)$ such that $B_{ij}^{-1} \neq 0$. If $j \neq 1$ then we are done. If $j = 1$ then if $\boldsymbol{B}_i^{-1}\boldsymbol{A}\boldsymbol{e} = 0$ we are done, and otherwise,

$$|a_1| = |K\boldsymbol{B}_i^{-1}\boldsymbol{A}\boldsymbol{e} + B_{i1}^{-1}| \geq K|\boldsymbol{B}_i^{-1}\boldsymbol{A}\boldsymbol{e}| - |B_{i1}^{-1}| \geq \frac{K}{m!M^m} - (m-1)!M^{m-1} > 1 \ .$$

As in the proof of Proposition 3.1, if $a_j \neq 0$, then

$$|a_j| \geq \frac{1}{m!M^m} \ .$$

Also, in this case for $j = 2, \cdots, m$,

$$|a_j| \leq (m-1)!M^{m-1}$$

and

$$|a_1| \leq Kmm!M^m + (m-1)!M^{m-1} < m(m!)^3M^{3m} \ .$$

It follows that for $\epsilon > 0$ such that $\epsilon \leq \epsilon_1$, we have $|p(x)| > 0$. This implies that there exists a basis $\boldsymbol{B}$ such that for every ϵ $(0 \leq \epsilon \leq \epsilon_1)$,

$$\boldsymbol{B}_i^{-1}(\boldsymbol{b} + \epsilon K\boldsymbol{A}\boldsymbol{e} + \boldsymbol{\epsilon}) \geq 0 \ .$$

∎

Corollary 3.5. *If a basis $\boldsymbol{B}$ is feasible in $(\tilde{P}(\epsilon))$ for any $\epsilon \in (0, \epsilon_1)$ then $\boldsymbol{B}$ is feasible in (P)*

Proof: It follows from the proof of Proposition 3.4 that if $\boldsymbol{B}$ is feasible in $(\tilde{P}(\epsilon))$ for some $\epsilon \in (0, \epsilon_1)$ then it is feasible in $(\tilde{P}(\epsilon))$ for all such ϵ and hence also in (P). ∎

Corollary 3.6. *For any $\epsilon \in (0, \epsilon_1)$ the problem $(\tilde{P}(\epsilon))$ is primal-nondegenerate.*

Given a basis $\boldsymbol{B}$, denote, as usual, by $\boldsymbol{c}_B$ the restriction of $\boldsymbol{c}$ to the coordinates corresponding to the columns in $\boldsymbol{B}$. A basis $\boldsymbol{B}$ is called *primal-optimal* in (P) if $\boldsymbol{c}_B^T\boldsymbol{B}^{-1}\boldsymbol{b}$ equals the maximum of (P). Without loss of generality, assume $\boldsymbol{c} \neq \boldsymbol{0}$.

Proposition 3.7. *If $\boldsymbol{B}$ is a primal-optimal basis in $(\tilde{P}(\epsilon))$ for any $\epsilon > 0$ such that*

$$\epsilon < \epsilon_2 = \frac{1}{2m^2(m!)^4M^{4m}\max_j\{|c_j|\}} \ ,$$

then it is also primal-optimal in (P).

Proof: If $(P(\epsilon))$ has an optimal solution for any $\epsilon \in (0, \epsilon_1)$, then by Corollary 3.5, (P) has a feasible solution. Moreover, by the duality theorem applied to $(\tilde{P}(\epsilon))$, the problem (D) is feasible and hence (P) has an optimal solution. Furthermore, (P) has a basic optimal solution. Suppose $\boldsymbol{B}$ is not optimal in (P) and let $\boldsymbol{C}$ be an optimal basis. Consider the polynomials

$$p(\epsilon) = \boldsymbol{c}_B^T \boldsymbol{B}^{-1}(\boldsymbol{b} + \epsilon K \boldsymbol{A}\boldsymbol{e} + \boldsymbol{\epsilon})$$
$$q(\epsilon) = \boldsymbol{c}_C^T \boldsymbol{C}^{-1}(\boldsymbol{b} + \epsilon K \boldsymbol{A}\boldsymbol{e} + \boldsymbol{\epsilon}) \ .$$

Since $q(0) > p(0)$, it follows that

$$q(0) \geq p(0) + \frac{1}{m! M^m} \ .$$

Moreover, for $\epsilon < 1$,

$$q(\epsilon) - p(\epsilon) \geq \frac{1}{m! M^m} - 2m^2 (m!)^3 M^{3m} \max_j \{|c_j|\} \epsilon \ .$$

It follows that for $\epsilon \in (0, \epsilon_2)$, $q(\epsilon) > p(\epsilon)$, which contradicts our assumptions. ∎

Proposition 3.8. *If $(\tilde{P}(\epsilon))$ is feasible and unbounded for any $\epsilon < \epsilon_1$ then also (P) is feasible and unbounded.*

Proof: Feasibility of (P) is claimed in Corollary 3.5. Unboundedness follows by the duality theorem, since the dual of $(P(\epsilon))$ is feasible if and only if (D) is. ∎

To conclude,

Theorem 3.9. *The problem $(\tilde{P}(\frac{1}{2}\epsilon_2))$ is nondegenerate, has polynomial size in terms of the size of (P), and is equivalent to (P) in the sense that both have the same status in terms of feasibility and boundedness. Moreover, every optimal basis of $(\tilde{P}(\frac{1}{2}\epsilon_2))$ is an optimal basis of (P).*

Note that (P) may have an optimal basis which not optimal in $(\tilde{P}(\frac{1}{2}\epsilon_2))$.

Remark 3.10. It is easy to see that using the ideas presented above, a perturbation can be applied to the objective function vector $\boldsymbol{c}$ so that the dual problem becomes nondegenerate. The perturbation itself depends only on the matrix $\boldsymbol{A}$. Note that in the case of the primal problem the estimate of an upper bound on ϵ, which guarantees that an optimal basis for the perturbed problem is optimal in the original problem, depends on the vector $\boldsymbol{c}$. When we perturb both $\boldsymbol{b}$ and $\boldsymbol{c}$, we get a problem which is primal- and dual-nondegenerate, and we would like to compute a suitable bound for ϵ. Such a nondegenerate problem has a unique basis which is both primal- and dual-optimal. As in Proposition 3.7, it is easy to find a value ϵ_3, depending on $\boldsymbol{b}$ and $\boldsymbol{c}$ such that if a basis $\boldsymbol{B}$ is primal- and dual-optimal in $(\tilde{P}(\epsilon))$ for any $\epsilon \in (0, \epsilon_3)$, then $\boldsymbol{B}$ is primal- and dual-optimal in (P).

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References

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