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THE KERNEL AND THE NUCLEOLUS OF A PRODUCT OF SIMPLE GAMES

BY
NIMROD MEGIDDO

ABSTRACT

The kernel and the nucleolus of a product of two simple games are given in terms of the kernels and the nucleoluses of the component games.

1. Introduction

The kernel of a characteristic function game was defined by M. Davis and M. Maschler in [1]. The nucleolus, which is closely related to the kernel was introduced by D. Schmeidler in [3]. The product of simple games is one of several forms of combination of games, which were defined by L. S. Shapley in [4].

The purpose of this paper is to characterize the kernel and the nucleolus of the product of two games (in the sense of Shapley) in terms of the kernels and the nucleolus of the component games. It turns out that the kernels compose in the same way the Von-Neumann and Morgenstern solutions do (see [5], Th. 1). The cores of the component games (in case they are non-empty) compose the same way, too. However, this is not true for he bargaining set $M_1^{(i)}$ which is known to contain the kernel. Bargaining compose in a more complicated way. The structure of the bargaining set of the product will be described in a forthcoming paper by the present author.

Section 2 provides the necessary definitions and introduces several notations. Section 3 deals with the kernel, and the nucleolus is treated in Section 4.

2. Definitions

A *simple game* is a pair $(N; \mathscr{W})$ where $N = \{1, 2, \cdots, n\}$ is a nonempty finite set, and $\mathscr{W}$ is a set of subsets of N. The members of N are called *players*; the subsets of N are called *coalitions*, while $\mathscr{W}$ is the set of all *winning coalitions*. For our

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purpose, it is not necessary to assume that the game is superadditive. We even do not assume that it is monotonic. Our results are trivial if $N \notin \mathscr{W}$ and this case is therefore omitted.

A player i is termed *veto player* if for every $\phi \neq S \in \mathscr{W}$, $i \in S$.

If $\Gamma_i = (N_i; \mathscr{W}^i)$, $i = 1,2$, are two simple games, such that $N_1 \cap N_2 = \phi$, their *product*, $\Gamma_1 \otimes \Gamma_2$, is defined to be the game $\Gamma = (N; \mathscr{W})$ where $N = N_1 \cup N_2$ and $\mathscr{W}$ consists of all $S \subseteq N$ such that both $S \cap N_1 \in \mathscr{W}^1$ and $S \cap N_2 \in \mathscr{W}^2$. Thus,

$$\mathscr{W} = \{S_1 \cup S_2 : S_i \in \mathscr{W}^i, i = 1,2\}. \tag{2.1}$$

Note that the assumption $N \in \mathscr{W}$ implies $N_i \in \mathscr{W}$, $i = 1,2$.

Let $\Gamma = (N; \mathscr{W})$ be a simple game. The *characteristic function* of Γ is the function v defined on the subsets of N by $v(S) = 1$ if $S \in \mathscr{W}$ and $v(S) = 0$ if $S \notin \mathscr{W}$. According to our assumption, it follows that

$$v(N) = 1. \tag{2.2}$$

If Γ is a simple game satisfying (2.2), such that

$$v(\{i\}) = 0, \; i = 1,2,\cdots,n \tag{2.3}$$

we define an *imputation* in Γ to be an n-tuple ($n = |N|$) $x = (x_1, \cdots, x_n)$ of real numbers that satisfies $x_i \geqq 0$, $i = 1,\cdots,n$, and $\sum_{i=1}^n x_i = 1$. If (2.3) is not taken into account, such x is a pseudo-imputation. The set of all imputations (or pseudo-imputations) is denoted by $X(\Gamma)$. For $\phi \neq S \subseteq N$ we denote $x(S) = \sum_{i \in S} x_i$ ($x(\phi)$ is defined to be equal to zero) and define the excess of S with respect to x to be

$$e(S,x) = v(S) - x(S). \tag{2.4}$$

Let i,j be two distinct players. We denote

$$\mathscr{T}_{ij} = \{S : S \subseteq N, i \in S, j \notin S\} \tag{2.5}$$

$$\mathscr{W}_{ij} = \mathscr{W} \cap \mathscr{T}_{ij} \tag{2.6}$$

and

$$s_{ij}(x) = \operatorname{Max}\{e(S,x) : S \in \mathscr{T}_{ij}\}. \tag{2.7}$$

$s_{ij}(x)$ is called the *maximum surplus* of i against j with respect to x. An imputation (or pseudo-imputation) is said to be *balanced* if for every pair of distinct players i,j,

$$[s_{ij}(x) - s_{ji}(x)] \cdot x_j \leqq 0, \tag{2.8}$$

If Γ is a game satisfying (2.2) and (2.3), the *kernel* of Γ (for the grand coalition), which is denoted by $\mathscr{K}(\Gamma)$, is the set of all balanced imputations. It is proved in [1] that the kernel is always non-empty. (If (2.3) does not hold, the pseudo-kernel is the set of all balanced pseudo-imputations; it is always non-empty.) We assume that our games satisfy (2.3) (as well as (2.2)). If they do not satisfy (2.3) we have only to deal with the pseudo-kernel of the respective game instead of its kernel and the same results hold.

For every imputation x let $\theta(x)$ be a 2^n-tuple whose components are the numbers $e(S,x)$, $S \subseteq N$, arranged according to their magnitude, i.e., $\theta_i(x) \geqq \theta_j(x)$ for $1 \leqq i \leqq j \leqq 2^n$. If Γ is a game satisfying (2.2) and (2.3), the *nucleolus* (for the grand coalition) of Γ is defined to be the set of all imputations x such for every $y \in X(\Gamma)$, $\theta(y)$ does not precede $\theta(x)$ in the lexico-graphical order in R^{2^n}. This set is denoted by $\mathscr{N}(\Gamma)$. It was proved by Schmeidler in [3] that the nucleolus consists of exactly one point, which, moreover, is contained in the kernel of the game. The pseudo-nucleolus is defined analogously in case (2.3) is not satisfied. In such a case we have to deal with the pseudo-nucleolus instead of the nucleolus of the respective game and the same results hold.

We shall denote the excess of a coalition by $e^i(S,x)$ and maximum surplus of player k against another player l by $s^i_{kl}(x)$, when they are taken with respect to the game Γ_i $(i = 1,2)$. Similarly, we denote by $\theta^i(x)$ the vector of excesses of coalitions of N_i with respect to Γ_i $(i = 1,2)$, whose coordinates form a non-decreasing sequence. If x is an imputation in Γ_i $(i = 1,2)$ let us denote $x^*_j = x_j$ for $j \in N_i$ and $x^*_j = 0$ for $j \notin N_i$. $x^* = (x^*_1, \cdots, x^*_n)$ is an imputation in Γ. We also use the following notations

$$(2.9) \qquad \mathscr{K}^*(\Gamma_i) = \{x^* : x \in \mathscr{K}(\Gamma_i)\} \qquad i = 1,2$$

$$(2.10) \qquad \eta(x,y) = \frac{\mathrm{Min}\{y(S) : \phi \neq S \in \mathscr{W}^2\}}{\mathrm{Min}\{x(S) : \phi \neq S \in \mathscr{W}^1\} + \mathrm{Min}\{y(S) : \phi \neq S \in \mathscr{W}^2\}}$$

where $x \in \mathscr{K}(\Gamma_1)$ and $y \in \mathscr{K}(\Gamma_2)$.

Let x be an imputation and let $S \subseteq N$ such that $x(S) \neq 0$. The *barycentric projection* of x on S is defined to be the imputation $B_S x$ that satisfies

$$(2.11) \qquad (B_S x)_i = \begin{cases} \dfrac{1}{x(S)}\, x_i & \text{for } i \in S \\ 0 & \text{for } i \notin S \end{cases}$$

Let Y_1, Y_2 be two sets of imputations. Given $0 \leqq \alpha \leqq 1$, we define

(2.12) $$\mathrm{Con}_2\alpha(Y_1, Y_2) = \{\alpha y^1 + (1-\alpha)y^2 : y^i \in Y_i,\ i = 1,2\}$$

and

(2.13) $$\mathrm{Con}_2(Y_1, Y_2) = \cup_\alpha\{\mathrm{Con}_2\alpha(Y_1, Y_2) : 0 \leqq \alpha \leqq 1\}.$$

3. The kernel of the product

We are dealing with the game $\Gamma = \Gamma_1 \otimes \Gamma_2$, where $\Gamma = (N; \mathscr{W})$, $\Gamma_i = (N_i; \mathscr{W}^i)$, $i = 1,2$ $(N_1 \cap N_2 = \phi)$. We assume that our games satisfy (2.2) and (2.3). We shall deal separately with the cases

(3.1)a $$\phi \notin \mathscr{W}^1 \cup \mathscr{W}^2$$

(3.1)b $$\phi \in \mathscr{W}^1 \cap \mathscr{W}^2$$

(3.1)c $$\phi \in \mathscr{W}^2 \setminus \mathscr{W}^1.$$

Given a general game $\Gamma = (N; v)$, the value of the empty set $v(\phi)$ is irrelevant for the kernel of the game, and sometimes ϕ is not defined to be coalition at all. However, the cases (3.1)a, (3.1)b, and (3.1)c may be considered as different product concepts in the following manner:

(3.2)a $$\mathscr{W} = \{S_1 \cup S_2 : S_i \in \mathscr{W}^i,\ i = 1,2\}$$

(3.2)b $$\mathscr{W} = \mathscr{W}^1 \cup \mathscr{W}^2 \cup \{S_1 \cup S_2 : S_i \in \mathscr{W}^i,\ i = 1,2\}$$

(3.2)c $$\mathscr{W} = \mathscr{W}^1 \cup \{S_1 \cup S_2 : S_i \in \mathscr{W}^i,\ i = 1,2\}.$$

THEOREM 3.1. *Let Γ_i, $i = 1,2$, be two simple games.*

i) *If $\phi \notin \mathscr{W}^1 \cup \mathscr{W}^2$ and if there are k_i veto players in Γ_i $(i=1,2)$ $0 \leqq k_i \leqq |N_i|$, $k_1 + k_2 \geqq 1$ and $\alpha = \dfrac{1}{k_1 + k_2}$, then*

(3.3) $$\mathscr{K}(\Gamma_1 \otimes \Gamma_2) = \mathrm{Con}_2\alpha[\mathscr{K}^*(\Gamma_1), \mathscr{K}^*(\Gamma_2)]$$

ii) *If $\phi \notin \mathscr{W}^1 \cup \mathscr{W}^2$ and if there are no veto players in the game Γ, then*

(3.4) $$\mathscr{K}(\Gamma_1 \otimes \Gamma_2) = \mathrm{Con}_2[\mathscr{K}^*(\Gamma_1), \mathscr{K}^*(\Gamma_2)]$$

iii) *If $\phi \in \mathscr{W}^1 \cap \mathscr{W}^2$ then*

(3.5) $$\mathscr{K}(\Gamma_1 \otimes \Gamma_2) = \{\eta(x,y)x^* + [1 - \eta(x,y)]y^* : x \in \mathscr{K}(\Gamma_1),\ y \in \mathscr{K}(\Gamma_2)\}$$

(see (2.10)).

iv) *If $\phi \in \mathscr{W}^2 \setminus \mathscr{W}^1$ then*

(3.6) $$\mathscr{K}(\Gamma_1 \otimes \Gamma_2) = \mathscr{K}^*(\Gamma_1).$$

Claim (i) follows from the fact that if there are k veto players in a game Γ ($1 \leqq k \leqq n$) then $\mathscr{K}(\Gamma)$ consists of exactly one point x, where for each veto player i, $x_i = 1/k$ and $x_i = 0$ for all other players (see [2], p. 591; note that if $i \in N_1$ then i is a veto player in Γ if and only if i is a veto player in Γ_1 and $\phi \notin \mathscr{W}^1$). It follows from claim (i) that if the product of simple games contains veto players, its kernel depends only on the sets of veto players of each component. Furthermore, if Γ_1 is a game without veto players, but Γ_2 contains a veto player then the kernel of $\Gamma_1 \otimes \Gamma_2$ does not depend on the kernel of Γ_1 (provided $\phi \notin \mathscr{W}^2$).

To prove claims (ii), (iii), and (iv) we have to introduce the following lemmas.

LEMMA 3.2. *If i,j are two distinct players in Γ_1, then $\mathscr{W}_{ij} = \phi$ if and only if $\mathscr{W}^1_{ij} = \phi$.*

The proof follows from (2.1).

LEMMA 3.3. *Let i,j be two distinct players. If $\mathscr{W}_{ij} = \phi$ then $s_{ij}(x) = -x_i$; if $\mathscr{W}_{ij} \neq \phi$ then $s_{ij}(x) = 1 - \mathrm{Min}\{x(S) : S \in \mathscr{W}_{ij}\}$.*

The proof is immediate.

For each imputation $x = (x_1, \cdots, x_n)$ such that $x(N_1) \neq 0$, let us denote by $\hat{x}$ the restriction of $B_{N_1}x$ (see (2.11)) to N_1. Clearly, $\hat{x}$ is an imputation (or a pseudo-imputation) in Γ_1.

LEMMA 3.4. *If i,j are two distinct players in N_1 and if x is an imputation satisfying $x(N_1) \neq 0$, then*

$$[s^1_{ij}(\hat{x}) - s^1_{ji}(\hat{x})] \cdot \hat{x}_j \leqq 0 \Leftrightarrow [s_{ij}(x) - s_{ji}(x)] \cdot x_j \leqq 0. \tag{3.7}$$

PROOF. Since $x_j = 0$ if and only if $\hat{x}_j = 0$, we may assume at once that $x_j > 0$. We now distinguish four cases:

I. $\mathscr{W}_{ij} = \phi = \mathscr{W}_{ji}$.

In this case $s_{ij}(x) = -x_i$, $s_{ji}(x) = -x_j$, $s^1_{ij}(\hat{x}) = -\dfrac{1}{x(N_1)}x_i$, $s^1_{ji} = \dfrac{1}{x(N_1)}x_j$ (see Lemmas 3.2 and 3.3) and the result follows.

II. $\mathscr{W}_{ij} \neq \phi = \mathscr{W}_{ji}$.

In this case $s_{ij}(x) = -x_i \leqq 0$, $s_{ji}(x) \geqq 0$ (see Lemma 3.3) and the result follows because, by Lemma 3.2, also $s^1_{ij}(\hat{x}) \leqq 0$ and $s^1_{ji}(\hat{x}) \geqq 0$.

III. $\mathscr{W}_{ij} = \phi \neq \mathscr{W}_{ji}$

As in II,

$$s_{ij}(x) \geqq 0 > s_{ji}(x) \qquad s^1_{ij}(\hat{x}) \geqq 0 > s^1_{ji}(\hat{x})$$

and the result follows.

IV. $\mathscr{W}_{ij} \neq \phi \neq \mathscr{W}_{ji}$.

Using Lemmas 3.2 and 3.3, we obtain :

$$\begin{aligned} s_{ij}(x) &= 1 - \operatorname{Min}\{x(S) : S \in \mathscr{W}_{ij}\} \\ &= 1 - \operatorname{Min}\{x(S) : S \in \mathscr{W}^1_{ij}\} - \operatorname{Min}\{x(S) : S \in \mathscr{W}^2\} \\ &= x(N_1) \cdot s^1_{ij}(\hat{x}) + x(N_2) - \operatorname{Min}\{x(S) : S \in \mathscr{W}^2\} \end{aligned} \tag{3.8}$$

and analogously

$$s_{ji}(x) = x(N_1) \cdot s^1_{ji}(\hat{x}) + x(N_2) - \operatorname{Min}\{x(S) : S \in \mathscr{W}^2\}. \tag{3.9}$$

This implies the result.

Our next lemma is of interest in the general theory of kernel of simple games.

LEMMA 3.5. *If Γ is a simple game satisfying* (2.2) *and* (2.3) *and if* $x \in \mathscr{K}(\Gamma)$, *then for each* $i \in N$ *which is not a veto player*

$$\operatorname{Min}\{x(S) : i \in S \in \mathscr{W}\} \geqq \operatorname{Min}\{x(S) : i \notin S \in \mathscr{W} \setminus \{\phi\}\} \tag{3.10}$$

and there is an equality in (3.10) *whenever* $x_i > 0$.

PROOF. The left-hand side of (3.10) is well -defined since $N \in \mathscr{W}$ and the right-hand side is well-defined since i is not a veto player.

a) Suppose

$$\operatorname{Min}\{x(S) : i \in S \in \mathscr{W}\} < \operatorname{Min}\{x(S) : i \notin S \in \mathscr{W} \setminus \{\phi\}\}. \tag{3.11}$$

There exists a winning coalition S_0 containing i such that

$$x(S_0) < \operatorname{Min}\{x(S) : i \notin S \in \mathscr{W} \setminus \{\phi\}\}. \tag{3.12}$$

Clearly, $S_0 \neq N$. Let j be a player outside S_0. By Lemma 3.3, and since obviously, $\mathscr{W}_{ij} \neq \phi$

$$\begin{aligned} s_{ij}(x) &= 1 - \operatorname{Min}\{x(S) : S \in \mathscr{W}_{ij}\} \\ &\geqq 1 - x(S_0) \\ &> 1 - \operatorname{Min}\{x(S) : i \notin S \in \mathscr{W} \setminus \{\phi\}\} \\ &\geqq 1 - \operatorname{Min}\{x(S) : S \in \mathscr{W}_{ji}\} \\ &\geqq s_{ji}(x) \end{aligned} \tag{3.13}$$

Since x belongs to the kernel, $x_j = 0$ must hold. But that is true for any player j outside S_0, so that $x(S_0) = 1$ in contradiction to (3.12).

b) Suppose

$$\operatorname{Min}\{x(S) : i \in S \in \mathscr{W}\} > \operatorname{Min}\{x(S) : i \notin S \in \mathscr{W} \setminus \{\phi\}\}. \tag{3.14}$$

As in a) we can show that (3.14) implies $x_i = 0$. This completes the proof of the Lemma.

LEMMA 3.6. *Let $\Gamma = \Gamma_1 \otimes \Gamma_2$ be a simple game without veto players satisfying (3.1)a. Let $x \in \mathscr{K}(\Gamma_1)$, $y \in \mathscr{K}(\Gamma_2)$ and $0 \leqq \alpha \leqq 1$. For every $i \in N_1$ and $j \in N_2$ the imputation $z = \alpha x^* + (1-\alpha)y^*$ satisfies*

$$[s_{ij}(z) - s_{ji}(z)] \cdot z_j \leqq 0 \tag{3.15}$$

$$[s_{ji}(z) - s_{ij}(z)] \cdot z_i \leqq 0. \tag{3.16}$$

PROOF. It is sufficient to provide a proof only to (3.15). Since j is not a veto player in Γ_2 there exists $T \in \mathscr{W}^2$ such that $j \notin T$. Clearly, $N_1 \cup T \in \mathscr{W}_{ij}$ (see (2.1) and (2.6)) so that $\mathscr{W}_{ij} \neq \phi$, and, analogously, $\mathscr{W}_{ji} \neq \phi$. Suppose $z_j > 0$ (this means $y_j > 0$). Since there are no veto players

$$\begin{aligned} s_{ij}(z) &= 1 - \mathrm{Min}\{z(S) : S \in \mathscr{W}_{ij}\} \\ &= 1 - \mathrm{Min}\{z(S) : i \in S \in \mathscr{W}^1\} - \mathrm{Min}\{z(S) : j \notin S \in \mathscr{W}^2\} \\ &= 1 - \alpha\,\mathrm{Min}\{x(S) : i \in \mathscr{W}^1\} - (1-\alpha)\,\mathrm{Min}\{y(S) : j \in S \in \mathscr{W}^2\} \end{aligned} \tag{3.17}$$

and analogously

$$s_{ji}(z) = 1 - \alpha\,\mathrm{Min}\{x(S) : i \notin S \in \mathscr{W}^1\} - (1-\alpha)\,\mathrm{Min}\{y(S) : j \in S \in \mathscr{W}^2\} \tag{3.18}$$

By Lemma 3.5

$$\mathrm{Min}\{x(S) : i \in S \in \mathscr{W}^1\} \geqq \mathrm{Min}\{x(S) : i \notin S \in \mathscr{W}^1\} \tag{3.19}$$

and since $y_j > 0$ and $\phi \notin \mathscr{W}^1$

$$\mathrm{Min}\{y(S) : j \in S \in \mathscr{W}^2\} = \mathrm{Min}\{y(S) : j \notin S \in \mathscr{W}^2\}. \tag{3.20}$$

By (3.17)–(3.20)

$$s_{ij}(z) \leqq s_{ji}(z) \tag{3.21}$$

and the present lemma is proved.

LEMMA 3.7. *If $x \in \mathscr{K}(\Gamma)$ and $\phi \neq S \in \mathscr{W}$ then $x(S) > 0$.*

PROOF. Suppose $x(S) = 0$. Let $i \in S$ and $j \notin S$ such that $x_j > 0$. Thus, $s_{ij}(x) = 1$, $s_{ji}(x) < 1$, and x cannot belong to the kernel since $x_j > 0$.

LEMMA 3.8. *If $x \in \mathscr{K}(\Gamma)$ and $x_i > 0$ then*

$$\mathrm{Min}\{x(S) : i \in S \in \mathscr{W}\} = \mathrm{Min}\{x(S) : \phi \neq S \in \mathscr{W}\}. \tag{3.22}$$

PROOF. Assume i is not a veto player. Obviously,

$$\begin{aligned} &\operatorname{Min}\{x(S) : \phi \neq S \in \mathscr{W}\} \\ (3.23) \qquad &= \operatorname{Min}[\operatorname{Min}\{x(S) : i \in S \in \mathscr{W}\}, \operatorname{Min}\{x(S) : i \notin S \in \mathscr{W} \setminus \{\phi\}\}] \end{aligned}$$

and the result follows from Lemma 3.5. If i is a veto player, we have nothing to prove.

LEMMA 3.9. *Let* $\Gamma = \Gamma_1 \otimes \Gamma_2$ *be a simple game satisfying* (3.1)b. *Let* $x \in \mathscr{K}(\Gamma_1)$, $y \in \mathscr{K}(\Gamma_2)$, *and* $0 \leqq \alpha \leqq 1$. *The imputation* $z = \alpha x^* + (1-\alpha)y^*$ *satisfies*

$$(3.24) \qquad [s_{ij}(z) - s_{ji}(z)] \cdot z_j \leqq 0$$

$$(3.25) \qquad [s_{ji}(z) - s_{ij}(z)] \cdot z_i \leqq 0$$

for every $i \in N_1$ *and* $j \in N_2$ *if and only if*

$$\alpha = \eta(x, y).$$

(see (2.10)).

PROOF. If $i \in N_1$ and $j \in N_2$ then

$$(3.26) \qquad s_{ij}(z) = 1 - \alpha \operatorname{Min}\{x(S) : i \in S \in \mathscr{W}^1\}$$

$$(3.27) \qquad s_{ji}(z) = 1 - (1-\alpha) \operatorname{Min}\{y(S) : j \in S \in \mathscr{W}^2\}.$$

The inequality (3.15) holds (for all i,j) if and only if for every $i \in N_1$ such that $x_i > 0$,

$$(3.28) \qquad \alpha \operatorname{Min}\{x(S) : i \in S \in \mathscr{W}^1\} \leqq (1-\alpha) \operatorname{Min}\{y(S) : j \in S \in \mathscr{W}^2\}$$

for every $j \in N_2$. The existence of the last inequality for every $j \in N_2$ is equivalent to

$$(3.29) \qquad \alpha \operatorname{Min}\{x(S) : i \in S \in \mathscr{W}^1\} \leqq (1-\alpha) \operatorname{Min}\{y(S) : \phi \neq S \in \mathscr{W}^2\}.$$

In view of Lemma 3.8, (3.29) is equivalent to

$$(3.30) \qquad \alpha \operatorname{Min}\{x(S) : \phi \neq S \in \mathscr{W}^1\} \leqq (1-\alpha) \operatorname{Min}\{y(S) : \phi \neq S \in \mathscr{W}^2\}$$

which means $\alpha \leqq \eta(x, y)$. Analogously, (3.25) holds if and only if $\alpha \geqq \eta(x,y)$ and this completes the proof.

LEMMA 3.10. *Let* $\Gamma = \Gamma_1 \otimes \Gamma_2$ *be a simple game satisfying* (3.1)c. *Let* $x \in \mathscr{K}(\Gamma_1)$, $y \in \mathscr{K}(\Gamma_2)$, *and* $0 \leqq \alpha \leqq 1$. *The imputation* $z = \alpha x^* + (1-\alpha)y^*$ *satisfies*

$$(3.31) \qquad [s_{ij}(z) - s_{ji}(z)]z_j \leqq 0$$

$$(3.32) \qquad [s_{ji}(z) - s_{ij}(z)]z_i \leqq 0$$

for every $i \in N_1$ *and* $j \in N_2$ *if and only if* $\alpha = 1$.

PROOF. If $i \in N_1$ is a veto player, then (3.31) holds for every $j \in N_2$ if and only if $\alpha = 1$ and (3.32) holds for all α. If $i \in N_1$ such that $x_i > 0$ and i is not a veto player, and if $j \in N_2$ such that $y_j > 0$ then (3.31) holds if and only if $\alpha = 1$ and (3.32) holds for all α since

$$s_{ij}(z) = 1 - \alpha \operatorname{Min}\{x(S) : i \in S \in \mathscr{W}^1\} \tag{3.33}$$

$$s_{ji}(z) = 1 - \alpha \operatorname{Min}\{x(S) : i \notin S \in \mathscr{W}^1\} - (1-\alpha)\operatorname{Min}\{y(S) : j \notin S \in \mathscr{W}^2\} \tag{3.34}$$

and by Lemma 3.5

$$\operatorname{Min}\{x(S) : i \in S \in \mathscr{W}^1\} = \operatorname{Min}\{x(S) : i \notin S \in \mathscr{W}^1\} \tag{3.35}$$

This concludes the proof of the present lemma.

The proof of Theorem 3.1(ii) follows from Lemmas 3.4 and 3.6. Note that if $x \in \mathscr{K}(\Gamma)$ and $x(N_2) = 0$ then $x \in \mathscr{K}^*(\Gamma_1)$ (see Lemma 3.4). In this case $x = 1 \cdot x + 0 \cdot y$ where y is an arbitrary point of $\mathscr{K}^*(\Gamma_2)$. Claim (ii) implies that the kernel of a product of two simple games satisfying (3.1)a is a connected set. The proof of claim (iii) follows from Lemmas 3.4 and 3.9 and (iv) follows from Lemmas 3.4 and 3.10.

4. The nucleolus of the product

We denote the unique imputation in $\mathscr{N}(\Gamma)$ by v and $\mathscr{N}(\Gamma_i) = \{v^i\}$, $i = 1,2$. In case Γ_i does not satisfy (2.3), $\mathscr{N}(\Gamma_i)$ is meant to be the pseudo-nucleolus of Γ_i, $i = 1,2$.

LEMMA 4.1. If $v(N_1) \neq 0$, then

$$v^{1*} = B_{N_1} v \tag{4.1}$$

(see (2.11)).

PROOF. Let $|N_i| = n_i$, $i = 1,2$, and assume that the restriction of $B_{N_1} v$ to the coordinates of N_1, which we denote by $\hat{v}$ does not coincide with the nucleolus of Γ_1. Thus, there exists an imputation x in Γ_1 such that $\theta^1(x)$ precedes $\theta^1(\hat{v})$ in the lexicographical order on R^{2^n}. Suppose they are

$$\theta^1(x) = (e^1(S_1, x), e^1(S_2, x), \cdots, e^1(S_{2^{n_1}}, x)) \tag{4.2}$$

$$\theta^1(\hat{v}) = (e^1(T_1, \hat{v}), e^1(T_2, \hat{v}), \cdots, e^1(T_{2^{n_1}}, \hat{v})). \tag{4.3}$$

If $|\mathscr{W}^1| = k$ we may assume that

$$\{S_1, \cdots, S_k\} = \{T_1, \cdots, T_k\} = \mathscr{W}^1 \tag{4.4}$$

since the excess of a winning coalition is always nonnegative and that of a losing one is always nonpositive.

Let i_0 be the first coordinate of unequal excesses, so that

(4.5) $$e^1(S_i, x) = e^1(T_i, \hat{v}) \text{ for } 1 \leqq i < i_0 \text{ and}$$

(4.6) $$e^1(S_{i_0}, x) < e^1(T_{i_0}, \hat{v}).$$

Since S_j wins if and only if T_j wins ($j = 1, \cdots, 2^{n_1}$), it is easily verified that for every $R \subseteq N_2$ and $1 \leqq j < i_0$

(4.7) $$e(S_j \cup R, \bar{x}) = e(T_j \cup R, v)$$

where $\bar{x} = v(N_1) \cdot x$ for $i \in N_1$ and $\bar{x}_i = v_i$ for $i \in N_2$.

Let us distinguish two cases:

I. $S_{i_0} \in \mathscr{W}^1$.

Let $R^* \in \mathscr{W}^2$ be a coalition having a maximal excess with respect to v (or, equivalently, with respect to $\bar{x}$) in $\mathscr{W}^2$. Thus, for every $R \subseteq N_2$ and $i_0 \leqq j \leqq 2^{n_1}$

(4.8) $$e(S_{i_0} \cup R^*, \bar{x}) \geqq e(S_j \cup R, \bar{x}) \text{ and}$$

(4.9) $$e(T_{i_0} \cup R^*, v) \geqq e(T_j \cup R, v).$$

Also, by (4.6),

(4.10) $$e(S_{i_0} \cup R^*, \bar{x}) < e(T_{i_0} \cup R^*, v).$$

It follows from (4.7)–(4.10) that $\theta(\bar{x})$ precedes $\theta(v)$ in the lexicographical order on R^{2^n}, and since $\bar{x}$ is an imputation in Γ, v is not the nucleolus of Γ.

II. $S_{i_0} \notin \mathscr{W}^1$.

In this case take $R^* = \phi$ and (4.8)–(4.10) are still true, we reach the same contradiction.

Corollary 4.2

(4.11) $$v = v(N_1) \cdot v^{1*} + v(N_2) \cdot v^{2*}.$$

Given v^1 and v^2, the characterization of v will be complete if we find what $v(N_1)$ and $v(N_2)$ must be.

Lemma 4.3. *If x is an imputation in simple game Γ such that for every $S \in \mathscr{W}$ $e(S,x) = 0$, then there must be at least one veto player in the game.*

Proof. If i is not a veto player, take $S \in \mathscr{W}$ such that $i \notin S$. Since $x(S) = 1$, it follows that $x_i = 0$. This, however, cannot be true for all the players in the game, hence there must be at least one veto player in the game.

LEMMA 4.4. *If* $S_i \in \mathscr{W}^i$, $i = 1,2$, *and* x, y *are imputations in* Γ_1 *and* Γ_2, *respectively, then for every* $0 \leqq \alpha \leqq 1$,

$$e(S_1 \cup S_2, \alpha x^* + (1-\alpha)y^*) = \alpha e^1(S_1, x) + (1-\alpha)e^2(S_2, y) \tag{4.12}$$

The proof is immediate.

Let $\mathscr{W}^1 = \{S_1, \cdots, S_k\}$ and $\mathscr{W}^2 = \{T_1, \cdots, T_l\}$ where the indices are arranged in such a way that

$$e^1(S_1, v^1) \geqq e^1(S_2, v^1) \geqq \cdots \geqq e^1(S_k, v^1) = 0 \tag{4.13}$$

$$e^2(T_1, v^2) \geqq e^2(T_2, v^2) \geqq \cdots \geqq e^2(T_l, v^2) = 0. \tag{4.14}$$

If $\phi \in \mathscr{W}^1 \cup \mathscr{W}^2$ or if either $e^1(S_1, v^1) = 0$ or $e^2(T_1, v^2) = 0$, then by Theorem 3.1 the nucleolus is known, since there is only one convex combination of v^{1*} and v^{2*} which belongs to $\mathscr{K}(\Gamma)$, and this, therefore, must be v (see Lemma 4.1). Thus, w.l.g., we assume that $\phi \notin \mathscr{W}^1 \cup \mathscr{W}^2$ and $e^1(S_1, v^1) > 0$, $e^2(T_1, v^2) > 0$. We denote

$$x^\alpha = \alpha v^{1*} + (1-\alpha)v^{2*}. \tag{4.15}$$

In order that x will be the nucleolus of the game Γ, $\theta(x^\alpha)$ must be minimal in the lexicographical order on $\theta([v^{1*}, v^{2*}])$. Since for every α, $0 \leqq \alpha \leqq 1$, and every pair (i,j)

$$e(S_1 \cup T_1), x^\alpha) \geqq e(S_i \cup T_j, x^\alpha) \tag{4.16}$$

it follows that $e(S_1 \cup T_1, x^\alpha)$ should be minimized first. If $e^1(S_1, v^1) > e^2(T_1, v^2)$ then α must be equal to 0 and it must be equal to 1 if $e^1(S_1, v^1) \leqq e^2(T_1, v^2)$. If $e^1(S_1, v^1) = e^2(T_1, v^2)$, then the minimax in which we are interested is not achieved in a unique α. Thus, we have to minimize the maximum excess over those coalitions $S_i \cup T_j$ for which $e(S_i \cup T_j, x^\alpha) \neq e(S_1 \cup T_1, x^\alpha)$. Let i_0, $2 \leqq i_0 \leqq 2^{n_1}$, be the first index for which $e^1(S_{i_0}, v^1) < e^1(S_1, v^1)$ and let j_0, $2 \leqq j_0 \leqq 2^{n_2}$, be the first index for which $e^2(T_{j_0}, v^2) < e^2(T_1, v^2)$. Clearly, for every α,

$$\begin{aligned} &\operatorname{Max}\{e(S_{i_0} \cup T_1, x^\alpha),\ e(S_1 \cup T_{j_0}, x^\alpha)\} \\ &\qquad = \operatorname{Max}\{e(S_i \cup T_j, x^\alpha) : \text{either } i \geqq i_0 \text{ or } j \geqq j_0\}. \end{aligned} \tag{4.17}$$

By Lemma 4.4, it follows that the number α_0 for which $\operatorname{Max}\{e(S_{i_0} \cup T_1, x^\alpha),$ $e(S_1 \cup T_{j_0}, x^\alpha)\}$ is minimized is given by

$$\alpha_0 = \frac{e^2(T_1, v^2) - e^2(T_{j_0}, v^2)}{e^1(S_1, v^1) - e^1(S_{i_0}, v^1) + e^2(T_1, v^2) - e^2(T_{j_0}, v^2)} \tag{4.18}$$

since this α_0 satisfies $e(S_{i_0} \cup T_1, x^{\alpha_0}) = e(S_1 \cup T_{j_0}, x^{\alpha_0})$. Thus, we have the following theorem,

THEOREM 4.5. *Let Γ_i, $i = 1,2$, be two simple games*

i) *If $\phi \notin \mathscr{W}^1 \cup \mathscr{W}^2$ and if there are veto players in the game, then the nucleolus coincides with the kernel and it is given by Theorem* 3.1(*i*).

ii) *If one of the following conditions is satisfied,*

a. $\phi \in \mathscr{W}^2 \setminus \mathscr{W}^1$

b. $\phi \notin \mathscr{W}^1 \cup \mathscr{W}^2$, *there are no veto players in the game and*

$$\mathrm{Min}\,\{v^1(S) : S \in \mathscr{W}^1\} > \mathrm{Min}\,\{v^2(T) : T \in \mathscr{W}^2\}$$

then

$$v = v^{1*} \tag{4.19}$$

iii) *If one of the following conditions is satisfied,*

a. $\phi \notin \mathscr{W}^1 \setminus \mathscr{W}^2$

b. $\phi \notin \mathscr{W}^1 \cup \mathscr{W}^2$, *there are no veto players in the game and*

$$\mathrm{Min}\,\{v^2(T) : T \in \mathscr{W}^2\} > \mathrm{Min}\,\{v^1(S) : S \in \mathscr{W}^1\}$$

then

$$v = v^{2*} \tag{4.20}$$

iv) *If $\phi \notin \mathscr{W}^1 \cup \mathscr{W}^2$ and there are no veto players and*

$$\mathrm{Min}\,\{v^1(S) : S \in \mathscr{W}^1\} = \mathrm{Min}\,\{v^2(T) : T \in \mathscr{W}^2\}$$

then

$$v = \alpha_0 v^{1*} + (1 - \alpha_0)v^{2*} \text{ where } \alpha_0 \text{ is given by (4.18).}$$

v) *If $\phi \in \mathscr{W}^1 \cap \mathscr{W}^2$ then*

$$v = \eta(v^1, v^2)v^{1*} + [1 - \eta(v^1, v^2)]v^{2*}. \tag{4.21}$$

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