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AN INTERGENERATIONAL CAKE EATING GAME

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A game having a simple recursive structure is given. It consists of dividing a fixed stock of a resource among infinitely many future generations. An equilibrium solution consists of arranging the consumptions, a and b, of adjacent generations in the 'golden mean', $(a + b)/a = a/b = (1 + \sqrt{5})/2$.

A close reading of Hotelling's famous article on exhaustible resources reveals the following passage: 'If a mine-owner produces too rapidly, he will depress the price, perhaps to zero. If he produces too slowly, his profits, though larger, may be postponed farther into the future than the rate of interest warrants. *Where is his golden mean*? And how does this most profitable rate of production vary as exhaustion approaches?' [Hotelling (1931, p. 139)]. Clearly his reference to the 'golden mean' is only figurative, referring to the most profitable extraction rate. The literal definition of the 'golden mean' is given by the requirement that $(a + b)/a = a/b = (1 + \sqrt{5})/2$, where a and b are positive numbers. The esthetic and mathematical properties of this ratio, also referred to as the Golden Section, the Divine Proportion, the Divine Section, the Golden Ratio, and the Golden Proportion, have been studied for centuries [Runion (1972)] and it has found application in sequential search methods [Saaty (1970, pp. 27–30)].

In terms of the exhaustion of a fixed stock of a depletable resource, that we will call a 'cake', a 'golden mean' extraction rate is one in which the total amount extracted in any two successive periods is in the same proportion to the amount extracted in one of them as the amount extracted in that period is to the amount extracted in the other. The question we address is whether there exists a situation under which a golden mean extraction path would obtain. In what follows we describe a game played between succeeding generations in which the 'golden mean' extraction rate is a minmax solution.

The game. In the following game the players are generations that enter and leave the scene in a sequence (possibly doubly-infinite). The players divide a 'cake' according to the following rules.

When player i (call it the 'son') enters the game he is confronted with three pieces that constitute the remains of the cake. He sets one of them aside. Next, his 'father' (player $i - 1$) chooses one of the two remaining pieces for his own consumption and leaves the game. Now the son cuts either the piece he has set aside or the piece not selected by his father. The result is that again there remain three pieces, and at this point the 'grandson' (player $i + 1$) enters the game and so on.

The solution. The solution of this game is quite simple. Assuming that all players desire the cake, each one of them will select the biggest piece available before leaving the game. Moreover, it is a rational expectation that each player sets aside the biggest piece available when it is his turn to act like that. Thus, when a player has to make a cut, he will do it so as to maximize the second biggest piece (since his son will set aside the biggest). In general, there may be different ways to cut, that yield the same second biggest. Assume that the rule will be to select among these optimal cuts the one that makes the remains (i.e., the biggest and the smallest) as equal as possible. In other words, the primary objective of every player is to maximize his own share, but whenever possible he would like, as a secondary objective, to leave a fair legacy.

Example. Suppose that player i is confronted with pieces of sizes (50, 30, 18). The following steps in an optimal play will be:

(1) i sets 50 aside.
(2) $i - 1$ selects 30 and leaves the game.
(3) i cuts the 50 in half (25, 25, 18).
(4) $i + 1$ sets 25 aside.
(5) i selects 25 and leaves the game.
(6) $i + 1$ cuts the 25 into 18 and 7 (18, 18, 7).
(7) $i + 2$ sets 18 aside.
(8) $i + 1$ selects 18.

And so on.

To summarize, an optimal strategy is as follows. First, set the biggest piece aside. Next, if you have to cut and the pieces are A and a ($A \geqslant a$), then form $(\frac{1}{2}A, \frac{1}{2}A, a)$ if $\frac{1}{2}A \geqslant a$ and form $(a, a, A - a)$ if $\frac{1}{2}A \leqslant a$.

This game exhibits the following properties:

Proposition 1. If at some point $A = ((1 + \sqrt{5})/2)a$ (the 'golden ratio') then the same holds forever.

Proof. A pair $(((1 + \sqrt{5})/2)a, a)$ is followed in an optimal play by $(a, ((\sqrt{5} - 1)/2)a)$ and $(\sqrt{5} - 1)/2 = 2/(1 + \sqrt{5})$.

Proposition 2. *If at some point $A = a$, then in the next generation $a = 0$ and all the following generations cut the larger piece in half.*

Proof. From the property of maximizing the second biggest.

Proposition 3. *For any rational number $r \geqslant 1$, if at some point $A = ra$ then after a finite number of generations there will be a situation of two equal pieces (and then Proposition 2 applies).*

Proof. Since the evolution of the sequence depends only on the ratio of the pieces, we may assume without loss of generality that both A and a are integers. We prove our claim by double induction. The case $A = a = 1$ is of course trivial. Assume that the claim has been proved for all cases (of integer pieces) where the size of both remaining pieces is less than A. The case $A = a$ is trivial. If $\frac{1}{2}A \leqslant a < A$ then the next pair is $(a, A - a)$ and the induction hypothesis applies. If $a \leqslant \frac{1}{2}A$ then the next pair is $(\frac{1}{2}A, a)$, which may be fractional. However, $(\frac{1}{2}A, a)$ evolves like $(A, 2a)$ and now we apply induction with respect to a. The case $a = \frac{1}{2}A$ is obvious. Assume that the claim has been proved for all cases of the remaining smaller piece being greater than a. Obviously, since $2a > a$, the induction hypothesis applies to our case and that completes the proof.

Corollary. *If ratio of any 'father's' share to his 'son's' share is a rational number, then from a certain point and on this ratio will be precisely 2.*

Propositions 2 and 3 suggest that this game evolves into the familiar cake cutting problem [Saaty (1970, pp. 152–154)].

References

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