A Note on Karmarkar's Projective Transformation

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Abstract. This note corrects a certain inaccuracy in the discussion of the projective transformation employed by Karmarkar in the reduction of a general linear programming to the form required for his algorithm.

In his famous paper [1], Karmarkar proposes (on page 386, Step 4) a reduction of a general problem to one on a subset of the unit simplex as follows. Given a linear programming problemof the form

$$\begin{aligned} \text{Minimize} \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \tag{1}$$

(where an optimal solution is assumed to exist and the optimal value is assumed to be zero), and a point $\boldsymbol{a} > \boldsymbol{0}$ such that $\boldsymbol{A}\boldsymbol{a} = \boldsymbol{b}$, define a transformation $\boldsymbol{x'} = (\boldsymbol{y}, \lambda) = T(\boldsymbol{x})$ by

$$y_i = \frac{x_i/a_i}{\sum_j (x_j/a_j) + 1} \qquad (i = 1, \cdots, n)$$

$$\lambda = 1 - \sum_{i=1}^{n} y_i \ .$$

It is then claimed that T maps the nonnegative orthant

$$P_+ = \{\boldsymbol{x} \in R^n : \boldsymbol{x} \geq \boldsymbol{0}\}$$

onto the simplex

$$\Delta = \{(\boldsymbol{y}, \lambda) \in R^{n+1} : \boldsymbol{y} \geq \boldsymbol{0} \ , \ \lambda \geq 0 \ , \ \sum_{i=1}^{n} y_i + \lambda = 1\} \ .$$

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Obviously, this claim is wrong since if $(\boldsymbol{y}, \lambda) = T(\boldsymbol{x})$ $(\boldsymbol{x} \in P_+)$, then $\lambda > 0$. The correct statement should be that $T(P_+)$ is the simplex Δ less the facet defined by $\lambda = 0$. This fact requires more care in the discussion of the reduction of the problem. The transformed problem considered in [1] is:

$$
\begin{aligned}
\text{Minimize} \quad & \boldsymbol{c}^T \boldsymbol{D} \boldsymbol{y} \\
\text{subject to} \quad & \boldsymbol{A} \boldsymbol{D} \boldsymbol{y} - \lambda \boldsymbol{b} = \boldsymbol{0} \\
& \boldsymbol{e}^T \boldsymbol{y} + \lambda = 1 \\
& \boldsymbol{y} \geq \boldsymbol{0} \ , \ \lambda \geq 0 \ ,
\end{aligned} \tag{2}
$$

where $\boldsymbol{D} \in R^{n \times n}$ is a diagonal matrix with a diagonal consisting of the coordinates of $\boldsymbol{a}$, and $\boldsymbol{e} = (1, \cdots, 1) \in R^n$. When this problem is solved, the output might be an optimal solution $(\boldsymbol{y}^0, \lambda^0)$ with $\lambda^0 = 0$, so $T^{-1}(\boldsymbol{y}^0, \lambda^0)$ is not well-defined. This means that there is a gap in the argument at the bottom of page 387 in [1]. [1].

A more precise argument is as follows. First, the optimal value of (2) is also zero, since it is equal to the infimum over the intersection of a subspace with the simplex less one facet, which in turn equals the optimal value of (1). Thus, for any optimal solution $\boldsymbol{x}^*$ of (1), the point $(\boldsymbol{y}^*, \lambda^*) = T(\boldsymbol{x}^*)$ (where $\lambda^* > 0$) is an optimal solution of (2). If $(\boldsymbol{y}^0, 0)$ is another optimal solution of (2) then the line segment $\sigma = [(\boldsymbol{y}^*, \lambda^*), (\boldsymbol{y}^0, 0))$ consists of optimal solutions of (2). Clearly, $T^{-1}(\sigma)$ is a ray whose endpoint is $\boldsymbol{x}^*$, so it is a set of points of the form $\boldsymbol{x}^* + t\boldsymbol{u}$ with $t \geq 0$. Obviously, $\boldsymbol{u} \geq \boldsymbol{0}$ (and $\boldsymbol{u} \neq \boldsymbol{0}$). It follows that

$$\boldsymbol{y}^0 = \frac{1}{\boldsymbol{e}^T \boldsymbol{D}^{-1} \boldsymbol{u}} \boldsymbol{D}^{-1} \boldsymbol{u}$$

or, equivalently,

$$\boldsymbol{u} = \boldsymbol{D} \boldsymbol{y}^0 \ .$$

In other words, the optimal solution $(\boldsymbol{y}^0, 0)$ of (2) defines a direction of a ray of optimal solutions of (1). In general, this direction alone does not determine an optimal solution of (1). With $\boldsymbol{u}$ at hand, since $\boldsymbol{c}^T \boldsymbol{u} = 0$ and $\boldsymbol{A}\boldsymbol{u} = \boldsymbol{0}$, the problem reduces to the following:

$$
\begin{aligned}
\underset{\boldsymbol{x}, t}{\text{Minimize}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\
\text{subject to} \quad & \boldsymbol{A} \boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{x} + t\boldsymbol{u} \geq \boldsymbol{0} \ , t \geq 0
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
\text{Minimize} \quad & \boldsymbol{c}^T \boldsymbol{x} \\
\text{subject to} \quad & \boldsymbol{A} \boldsymbol{x} = \boldsymbol{b} \\
& x_j \geq 0 \quad \text{for j such that } \mathrm{u_j} = 0
\end{aligned} \tag{3}
$$

[1] Mike Todd told me he had also pointed out the same gap in his review [2]

Geometrically, if $\boldsymbol{A} \in R^{m \times n}$, problem (1) is in dimension $n - m$ with n inequality constraints. If $(\boldsymbol{y}^0, 0)$ is a basic solution of (2) then at least $n - m - 1$ of the u_j's are zeros, which means that (3) is a problem in dimension $n - m$ with at least $n - m - 1$ inequality constraints. If (2) is primal-nondegenerate then m of the u_j's are positive so (3) is easy since it is in dimension $n - m$ with $n - m$ inequality constraints. If (2) is degenerate then problem (3) may be more difficult. Solving the problem in this way adds a factor of m to the time complexity.

At first sight, the difficulty raised above does not seem to cause a problem for interior point methods. Interior points of P_+ map one-to-one onto the interior of Δ. The final stage in any interior point method is to move from an approximately optimal interior solution to an optimal one. In view of the present note, it is essential that the move from an approximately optimal interior point to an optimal one be carried out in P_+ rather than Δ. However, consider the following problem with three variables:

$$\begin{aligned} \text{Minimize } \; & x_1 \\ \text{subject to } \; & x_1 + x_2 = 2 \\ & x_1, x_2, x_3 \geq 0 \ . \end{aligned}$$

Obviously, the optimal set is the ray defined by $x_1 = 0$, $x_2 = 2$, and $x_3 \geq 0$. Taking $\boldsymbol{a} = (1, 1, 1)$, the problem in the $(\boldsymbol{y}, \lambda)$ space is

$$\begin{aligned} \text{Minimize } \; & y_1 \\ \text{subject to } \; & y_1 + y_2 - 2\lambda = 0 \\ & y_1 + y_2 + y_3 + \lambda = 1 \\ & y_1, y_2, y_3 \geq 0 \ . \end{aligned}$$

Karmarkar's potential function in this case is:

$$\phi(\boldsymbol{y}, \lambda) = 4 \ln y_1 - \sum_{i=1}^{3} \ln y_i - \ln \lambda$$

Consider points of the form $y_1 = y_2 = \lambda$, $y_3 = 1 - 3y_1$ $(0 < y_1 < 1/3)$. When y_1 tends to zero, the value of ϕ at such points tends to $-\infty$ and indeed the point approaches an optimal solution. However, the inverse image, $\boldsymbol{x} = \boldsymbol{y}/\lambda$ is: $x_1 = x_2 = 1$, $x_3 = 1/y_1 - 3$, where the value of the objective function does not tend to the optimum.

Thus, the argument of potential reduction alone does not suffice for proving the claimed complexity of the algorithm for a general problem of the form (1). It is conceivable though that another argument might be used to prove it. Another idea, which was used already in the context of the ellipsoid method, is to add a constraint $\sum_j x_j \leq U$ which must be satisfied at every basic solution. If the size of the problem is L, and $\boldsymbol{x}$ is a basic solution, then $x_j \leq 2^L$. Thus, we can take $U = n2^L$ and the size of the new problem remains $O(L)$.

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References

[1] N. Karmarkar, "A new polynomial-time algorithm for linear programming", *Combinatorica* **4** (1984) 373–395.

[2] M. J. Todd, A review of Karmarkar's paper [1], *Computing Reviews*, Review No. 8602-0137, February 1986.