Discrete Mathematics 19 (1977) 177–181.

MIXTURES OF ORDER MATRICES AND GENERALIZED ORDER MATRICES

Nimrod MEGIDDO
Department of Statistics, Tel Aviv University, Tel Aviv, Israel

Received 26 April 1976

This paper deals with matrix representations of linear orders, mixtures of order matrices and the non-integral solutions of the linear systems defining them.

1. Introduction

We shall be dealing with matrix representations of orders. A linear order R over a set $M = \{1, \ldots, m\}$ is represented by a 0, 1-matrix $x = (x_{ij})_{1 \le i,j \le m}$ where iRj if and only if $x_{ij} = 1$. We shall call x an *order matrix* and denote the class of order matrices over M by O_m. It can be easily verified that $x \in O_m$ *if and only if* x *is an integral solution of the following system*:

$$x_{ij} + x_{ji} = 1 \qquad (1 \le i < j \le m) \tag{1.1}$$

$$x_{ii} = 0 \qquad (i = 1, \ldots, m) \tag{1.2}$$

$$x_{ik} \le x_{ij} + x_{jk} \qquad (1 \le i, j, k \le m) \tag{1.3}$$

$$x_{ij} \ge 0 \qquad (1 \le i, j \le m). \tag{1.4}$$

We shall call the solutions of (1.1)–(1.4) *generalized order matrices* and denote the class of these matrices by G_m. Our present study is motivated by the following.

(a) The domain of a social choice function [1] consists of sequences of linear orders. Under the assumptions of equal-vote and independence of irrelevant alternatives, this domain may be replaced by $H_m \equiv \mathrm{conv}(O_m)$, since the function depends only on the relative frequency of those individuals preferring i to j (for every $i \ne j \in M$). A linear characterization of H_m seems to be useful for defining social choice functions.

(b) The integral solutions of the subsystem [(1.1), (1.2), (1.4)] are the tournament matrices [5]. The set of all solutions for [(1.1), (1.2), (1.4)], called generalized tournament matrices, coincides with the convex hull of the set of tournament matrices [6].

(c) Permutation matrices, which are closely related to order matrices, are defined to be the integral solutions of the system [(1.4), (1.5)],

$$\sum_i x_{ik} = \sum_j x_{kj} = 1 \qquad (k = 1, \ldots, m). \tag{1.5}$$

It is well known that the set of all solutions for (1.4)–(1.5) coincides with the convex hull of the set of permutation matrices.

(d) With a slight modification, namely $x_{ii} = 0.5$ instead of $x_{ii} = 0$, generalized order matrices appear in the literature of mathematical psychology as binary choice probabilities – x_{ij} being the probability of choosing i when being forced to choose from $\{i, j\}$. Marschak [3] claims that (1.3) is the weakest assumption needed.

(e) An interesting combinatorial problem is the following. Given a set T of cyclically ordered triples out of M (see [4]), find a cyclic order R over M such that (if possible) every $\tau \in T$ is derived from R. This is equivalent to finding an integral solution for (1.1)–(1.4) as well as

$$x_{ik} + 1 \leq x_{ij} + x_{jk} \qquad ((ijk) \in T). \tag{1.6}$$

Fieldman has conjectured, in view of computational experience, that this problem is solvable by linear programming. If this were true, then necessarily $H_m = G_m$ for each m.

Unfortunately, it is not true that $H_m = G_m$ for every m. For a counterexample we need $m = 13$. On the other hand, it can be shown that $H_m = G_m$ for $m \geq 4$.

2. On the classes O_m, H_m, P_m, G_m

Given an $x \in G_m$, the symbol $=(ijk)$ will stand for the equality $x_{ik} = x_{ij} + x_{jk}$. Similarly, $<(ijk)$ will stand for $x_{ik} < x_{ij} + x_{jk}$. The following lemma can be easily proved.

Lemma 2.1. *Let $x \in G_m$ and $i, j, k \in M$.*

(i) *If i, j, k are distinct and $=(ijk)$, then $=(kij)$, $=(jki)$, $<(kji)$, $<(ikj)$, $<(jik)$.*

(ii) *$=(ijk)$ and $=(ikl)$ imply $=(ijl$ and $=(jkl)$.*

Lemma 2.2. *If $x \in G_m$ then there is $i \in M$ such that for each $j \neq i$ $x_{ij} > 0$.*

Proof. Obviously, the lemma is true for $m \leq 2$. We proceed by induction on m. Assume $m > 2$. The induction hypothesis implies that for every $i \in M$ there is $k = k(i)$ such that $k \neq i$ and $x_{kj} > 0$ for each $j \in M \setminus \{i, k\}$. Suppose, per absurdum, that $x_{k(i),i} = 0$ for every $i \in M$. It follows that $i \to k(i)$ is a permutation of M. Obviously,

$$x_{k(k(i)),i} \leq x_{k(k(i)),k(i)} + x_{k(i),i} = 0.$$

Since $k(k(i)) \neq k(i)$, it follows that $k(k(i)) = i$. That implies $x_{i,k(i)} + x_{k(i),i} = 0$ and hence, a contradiction. This completes the proof.

Corollary 2.3. *If $x \in G_m$ then there exists a permutation matrix p such that $y = p^{\mathrm{T}} xp$ satisfies $y_{ij} > 0$ for $1 \leq i < j \leq m$.*

This follows by applying Lemma 2.2 to a decreasing sequence of principal submatrices of x.

Definition 2.4. A matrix $x \in G_m$ is *permutable* if there is a permutation matrix p such that the matrix $y = p^T x p$ satisfies $y_{ij} > 0$ $(1 \leq i < j \leq m)$ and $y_{ik} < y_{ij} + y_{jk}$ $(1 \leq i < j < k \leq m)$.

We denote the class of permutable matrices by P_m.

Theorem 2.5. *For every m $(m = 1, 2, \ldots)$, $H_m = G_m$ if and only if $P_m = G_m$.*

Proof. Notice that for every m $H_m \subset G_m$ and $P_m \subset G_m$.

(a) We shall prove that $H_m \subset P_m$. Let $b = \sum_{i=1}^{s} \lambda_i a^i$ where $a^i \in O_m$, $\lambda_i > 0$, $i = 1, \ldots, m$, and $\sum \lambda_i = 1$. For every $i \in M$ let $p(i) \in M$ be such that i is the $p(i)$-th greatest in the linear order represented by a^1. The mapping p is a permutation of M and, obviously, $a^1_{ij} > 0$ if and only if $p(i) < p(j)$. Moreover, if $p(i) < p(j) < p(k)$ then $a^1_{ik} < a^1_{ij} + a^1_{jk}$. Since $a^2, \ldots, a^s$ satisfy (1.3), it follows that $b_{ik} < b_{ij} + b_{jk}$. Also $b_{ij} > 0$ whenever $p(i) < p(j)$. This implies that $b \in P_m$. Obviously, $H_m = G_m$ implies $P_m = G_m$.

(b) We shall prove that $P_m = G_m$ implies $H_m = G_m$. Assume that $P_m = G_m$. Let $b \in G_m$ and assume that $b_{ij} = q_{ij}/r_{ij}$ where q_{ij} and r_{ij} are non-negative integers $(r_{ij} \neq 0)$. Let r denote the least common multiple of the numbers r_{ij}. If $r = 1$ then $b \in O_m \subset H_m$. We proceed by induction on r. Assume $r > 1$. The matrix $c = rb$ is integral. Since $b \in P_m$, let p be a permutation of M such that $p(i) < p(j)$ implies $b_{ik} < b_{ij} + b_{jk}$. Let $a \in O_m$ be defined by $a_{ij} = 1$ if and only if $p(i) < p(j)$. We shall show that $d = [1/(r-1)](c - a) \in G_m$. First, $d_{ij} \geq 0$ since whenever $a_{ij} = 1$, $p(i) < p(j)$ and therefore $c_{ij} \geq 1$. Also, $d_{ii} = 0$ and $d_{ij} + d_{ji} = 1$ for $i \neq j$. It can be also verified that d satisfies the triangle inequality (1.3). Thus, $d \in G_m$. The induction hypothesis applies to d and therefore $d \in H_m$. This implies that $b = ((r-1)/r)d + (1/r)a \in H_m$. It follows that every $b \in G_m$ also belongs to H_m.

3. Examples

Proposition 3.1. *For $m \leq 4$, $H_m = G_m$.*

Proof. The case $m \leq 2$ is trivial. Let $x \in G_3$ and we shall show that $x \in P_3$. Without loss of generality assume that x_{12}, x_{13}, $x_{23} > 0$ (Corollary 2.3). If $<(123)$ then x is obviously permutable. Otherwise, $<(132)$ (Lemma 2.1) and also $x_{32} = x_{31} + x_{12} > 0$.In the latter case $p(1,2,3) = (1,3,2)$ is a suitable permutation that implies $x \in P_3$. Thus, $G_3 = P_3$ and by Theorem 2.5, $H_3 = G_3$.

Let $x \in G_4$. Without loss of generality assume that x_{14}, x_{24}, $x_{34} > 0$ (Lemma 2.2). Also, since $G_3 = P_3$, we may assume that $<(123)$ and $x_{12}, x_{13}, x_{23} > 0$. Table 1 enumerates all possible cases and in each one of them a suitable permutation is indicated.

Table 1

1.	= (124), = (134)	(1, 4, 2, 3)
2.	= (142), = (134)	(1, 2, 4, 3)
3.	< (142), < (124), = (134)	(1, 2, 4, 3)
4.	= (124), = (143)	This contradicts < (123)
5.	= (142), = (143)	(1, 2, 3, 4)
6.	< (124), < (142), = (143)	(1, 2, 3, 4)
7.	= (124), < (134), < (143)	(1, 4, 2, 3)
8.	= (142), < (143), < (134), = (234)	(1, 2, 4, 3)
9.	= (142), < (143), < (134), < (234)	(1, 2, 3, 4)
10.	< (124), < (142), < (143), < (134), = (234)	(1, 2, 4, 3)
11.	< (124), < (142), < (143), < (134), < (234)	(1, 2, 3, 4)

The proof in case 1, for example, is as follows. $x_{14} = x_{12} + x_{24} > 0$ and by our assumptions $x_{12}, x_{13}, x_{42}, x_{43}, x_{23} > 0$. Also, < (142), < (143) (Lemma 2.1) and by our assumption < (123). If, per absurdum, = (423), then = (124) implies = (123) (Lemma 2.1) and hence a contradiction. Thus, < (423) and all the requirements are fulfilled.

It follows that $P_4 = G_4$ and hence $H_4 = G_4$.

Proposition 3.2. $H_{13} \neq G_{13}$.

Proof. Consider the following matrix.

$x =$

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*	*l*	*m*
a	0	.5	.5	.5	.5	.5	0	.5	.5	.5	.5	0	.5
b	.5	0	.5	1	.5	.5	.5	1	1	1	.5	.5	.5
c	.5	.5	0	1	1	.5	.5	.5	1	.5	.5	.5	1
d	.5	0	0	0	.5	.5	0	.5	.5	.5	.5	.5	.5
e	.5	.5	0	.5	0	.5	.5	.5	.5	.5	.5	.5	.5
f	.5	.5	.5	.5	.5	0	.5	1	.5	.5	.5	.5	.5
g	1	.5	.5	1	.5	.5	0	1	.5	.5	.5	.5	.5
h	.5	0	.5	.5	.5	0	0	0	.5	.5	.5	.5	.5
i	.5	0	0	.5	.5	.5	.5	.5	0	.5	.5	0	.5
j	.5	0	.5	.5	.5	.5	.5	.5	.5	0	.5	.5	.5
k	.5	.5	.5	.5	.5	.5	.5	.5	.5	.5	0	.5	1
l	1	.5	.5	.5	.5	.5	.5	.5	1	.5	.5	0	1
m	.5	.5	0	.5	.5	.5	.5	.5	.5	.5	0	0	0

We claim that $x \in G_{13}$. It can be inspected that $x_{ii} = 0$ and $x_{ij} + x_{ji} = 1$ $(i \neq j)$. To verify the triangle inequality, notice that a violation of it in this matrix can occur only in the following forms: $1 > 0.5 + 0$, $1 > 0 + 0.5$, $1 > 0 + 0$, $0.5 > 0 + 0$. In any case, there must be either a row or a column containing both 1 and an off-diagonal zero. This does not occur and hence $x \in G_{13}$.

We shall prove that $x \notin P_{13}$. First, it can be verified that the following equalities hold

$$\begin{aligned} &= (dca),\ = (edb),\ = (fec),\ = (gfd),\ = (hge),\ = (ahf),\ = (cag),\ = (bch),\\ &= (iba),\ = (jic),\ = (kjb),\ = (lki),\ = (mlj),\ = (amk),\ = (bal),\ = (cbm). \end{aligned} \tag{3.1}$$

Suppose, per absurdum, that $x \in P_{13}$. It follows that there exists a linear order R over M such that $\alpha R \beta R \gamma$ implies $<(\alpha\beta\gamma)$ for all $\alpha, \beta, \gamma \in M$. In view of Lemma 2.1, the same is true for every cyclic equivalent R' of R (see [4]). It follows that the following cyclically ordered triples are all derived from the cyclic order $[R]$.

acd, bde, cef, dfg, egh, fha, gac, hcb, abi, cij, bjk, ikl, jlm, kma, lab, mbc.

This implies that hcm and bhm are also derived from $[R]$. This contradicts what is proved in [4, Example 4], namely, there is no cyclic order from which all these triples are derived. Thus, $x \notin P_{13}$. It follows that $x \notin H_{13}$. This completes the proof.

Balas [2] has recently characterized the convex hull U_m of the set of permutation m-vectors, i.e. vectors that can be obtained by a permutation of the vector $(1, 2, \ldots, m)$. A vector $u \in R^m$ belongs to U_m if and only if

$$\sum_{i \in M} u_i = m(m+1)/2, \tag{3.2}$$

$$\sum_{i \in S} u_i \leq ms - s(s-1)/2 \qquad \text{for all } S \subset M \qquad (s = |S|). \tag{3.3}$$

It is easy to verify that (1.1)–(1.4) implies that $u = xe + e$ (where e is the summation vector) satisfies (3.2)–(3.3). In other words, $x \in G_m$ implies $xe + e \in U_m$ and this leaves the problem of characterizing H_m open.

References

[1] K.J. Arrow, Social choice and individual values, Cowles Commission Monograph 12 (John Wiley and Sons, New York, 1951).
[2] E. Balas, A linear characterization of permutation vectors, Management Science Research Report No. 364, Graduate School of Industrial Administration, Carnegie-Mellon University (May 1975).
[3] J. Marschak, Binary-choice constraints and random utility indicators, in: K.J. Arrow, S. Karlin and P. Suppes, eds., Mathematical Methods in Social Sciences, 1959 (Stanford University Press, Stanford, 1960) 312–329.
[4] N. Megiddo, Partial and complete cyclic orders, Bull. Amer. Math. Soc. 82 (2) (1976) 274–276.
[5] J.W. Moon, Topics on Tournaments (Holt, Rinehart and Winston, Inc., NY, 1968).
[6] J.W. Moon and N.J. Pullman, On generalized tournament matrices, SIAM Rev. 12 (1970) 384–399.