Reprinted from:

MATHEMATICAL PROGRAMMING

Volume 12, No. 1, February 1977

ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS IN NONLINEAR COMPLEMENTARITY THEORY*

Nimrod MEGIDDO
Tel Aviv University, Tel Aviv, Israel

Masakazu KOJIMA
Tokyo Institute of Technology, Tokyo, Japan

pp. 110 – 130

NORTH-HOLLAND PUBLISHING COMPANY – AMSTERDAM

Mathematical Programming 12 (1977) 110–130.
North-Holland Publishing Company

ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS IN NONLINEAR COMPLEMENTARITY THEORY*

Nimrod MEGIDDO
Tel Aviv University, Tel Aviv, Israel

Masakazu KOJIMA
Tokyo Institute of Technology, Tokyo, Japan

Received 17 July 1975
Revised manuscript received 3 May 1976

A complementarity problem is said to be globally uniquely solvable (GUS) if it has a unique solution, and this property will not change, even if any constant term is added to the mapping generating the problem.

A characterization of the GUS property which generalizes a basic theorem in linear complementarity theory is given. Known sufficient conditions given by Cottle, Karamardian, and Moré for the nonlinear case are also shown to be generalized. In particular, several open questions concerning Cottle's condition are settled and a new proof is given for the sufficiency of this condition.

A simple characterization for the two-dimensional case and a necessary condition for the n-dimensional case are also given.

1. Introduction

We shall be dealing with the nonlinear complementarity problem which is defined as follows. Let $f: \mathbf{R}_+^n \to \mathbf{R}^n$ be a continuous mapping, where $\mathbf{R}^n$ is the n-dimensional euclidean space and $\mathbf{R}_+^n$ is the nonnegative orthant in $\mathbf{R}^n$.

Problem 1.1 (The Complementarity Problem (CP) associated with f). Find a vector $z \in \mathbf{R}_+^n$ such that $f(z) \in \mathbf{R}_+^n$ and $z^\mathrm{T} f(z) = 0$.

The complementarity problem has drawn much attention during the past decade since it has applications to many fields. Specifically, there are known applications of complentarity theory to linear and nonlinear programming, mathematical economics, game theory, and mechanics. There are many existence theorems for the nonlinear complementarity problem. Most of them can be derived from theorems that appeared in [6].

The goal of the present article is to present several results concerning the following property of complementarity problems.

* The research described in this paper was carried out while N. Megiddo was visiting Tokyo Institute of Technology under a Fellowship of the Japan Society for the Promotion of Science.

Definition 1.2. Let $f: \mathbf{R}_+^n \to \mathbf{R}^n$ be a continuous mapping. The CP associated with f is said to be *globally uniquely solvable* (GUS) if for any vector $q \in \mathbf{R}^n$ the CP associated with the function $f(\cdot)+q$ has a unique solution.

The GUS property of linear complementarity (i.e., complementarity problems associated with affine mappings) was, as a matter of fact, characterized by Samelson, Thrall, and Wesler [14]. Their characterization is the following.

Theorem 1.3 (The basic theorem on linear CP's). *Let $f(z) = Az + b$ be an affine mapping from $\mathbf{R}^n$ into itself. Then the* CP *associated with f is* GUS *if and only if all the principal minors of A are positive.*

The basic theorem has been partially generalized to the class of nonlinear CP's. The first generalization was given by Cottle [2].

Theorem 1.4 (Cottle's condition). *Let $f: \mathbf{R}^n \to \mathbf{R}^n$ be a continuously differentiable mapping such that the solutions (w, z) of $w - f(z) = 0$ are nondegenerate* [2]. *Suppose that there exists a $0 < \delta < 1$ such that for every principal minor $J_S(x)$ of the Jacobian matrix of f at x, $\delta < J_S(x) < \delta^{-1}$, for all $x \in \mathbf{R}_+^n$. Under these conditions the* CP *associated with f has a solution.*

The uniqueness of the solution in the case of Theorem 1.4 was proved by Moré [8]. The results of Moré will be discussed later. Another partial generalization of the basic theorem was later given by Karamardian [5].

Theorem 1.5 (Karamardian's condition). *Let $f: \mathbf{R}_+^n \to \mathbf{R}^n$ be a continuous mapping. Suppose that f is strongly monotone, i.e., there exists a $c > 0$ such that for all $x, y \in \mathbf{R}_+^n$*

$$(x-y)^{\mathrm{T}}[f(x) - f(y)] \geqq c\|x-y\|^2. \tag{1.1}$$

Under these conditions the CP *associated with f has a unique solution.*

Karamardian's condition implies in fact the GUS property. For differentiable mappings Karamardian's condition is equivalent to strong[1] positive-definiteness of the Jacobian matrix. Moré [7, 8] generalized Karamardian's condition.

Theorem 1.6 (Moré's condition). *Let $f: \mathbf{R}_+^n \to \mathbf{R}^n$ be a continuous mapping. Suppose that f is a uniform* P*-function, i.e., there exists a $c > 0$ such that for all $x, y \in \mathbf{R}_+^n$*

$$\operatorname*{Max}_i (x_i - y_i)[f_i(x) - f_i(y)] \geqq c\|x-y\|^2. \tag{1.2}$$

[1] A matrix-valued function $M(x)$ is said to be *strongly positive definite* if there exists a $c > 0$ such that for every y, $y^{\mathrm{T}}M(x)y \geqq c\|y\|^2$ for all x.

Under these conditions the CP *associated with f has a unique solution.*

If (1.2) is replaced by

$$\operatorname*{Max}_{i}(x_i - y_i)[f_i(x) - f_i(y)] > 0 \quad (x \neq y), \tag{1.3}$$

then the function f is said to be a P-function and it yields (see [8]) a CP with at most one solution. For differentiable functions, the property that all principal minors of the Jacobian matrix are positive everywhere, implies (1.3) (see [9]) and hence the CP has at most one solution in such a case. Even though Cottle's and Moré's conditions seem to be closely related, they are in fact independent (see Examples 6.1, 6.2).

So far we do not know of any generalization of the necessity part of the basic theorem. Certainly, even in the one-dimensional case, positiveness of the Jacobian is not necessary for the uniqueness of solutions of the CP. In higher dimensions even nonnegativity of the principal minors of the Jacobian matrix is not necessary (this is necessary in the one-dimensional case). The property of a P_0-function, i.e., that for every $x \neq y$ there is an i, $1 \leqq i \leqq n$, such that $x_i \neq y_i$ and $(x_i - y_i)[f_i(x) - f_i(y)] \geqq 0$ (and this means (see [9]) nonnegative principal minors in the Jacobian matrix in the differentiable case) is shown not to be necessary for the GUS property (see Example 6.3). However we do provide a generalization in this direction in section 4.

The present paper introduces generalizations of the basic theorem in both directions. In particular we provide a sufficient condition that covers both Cottle's and Moré's (and hence Karamardian's) conditions. We hope that the examples in section 6 clarify several characteristics of the GUS property.

2. Preliminaries

Let $N = \{1, \ldots, n\}$. For every nonempty subset S of N we denote by $\mathbf{R}^S$ an euclidean space whose coordinates are indexed by the elements of S. By $\mathbf{R}^S_+$ we denote the nonnegative orthant in $\mathbf{R}^S$. We shall sometimes regard $\mathbf{R}^S$ and $\mathbf{R}^S_+$ as subsets of $\mathbf{R}^N$. For every vector x in $\mathbf{R}^n$ and $S \subset N$ let $x^S = (x_i^S)_{i \in S}$ be the restriction of x to the coordinates of S. The null vector $x^{\emptyset}$ is assumed to be equivalent to both the number 0 and the empty set. The orthants of $\mathbf{R}^n$ will be denoted by Q^S $(S \subset N)$ where

$$Q^S = \{x \in \mathbf{R}^n \colon x^S \geqq 0, x^{N \setminus S} \leqq 0\}. \tag{2.1}$$

If $f \colon \mathbf{R}^n \to \mathbf{R}^n$ is any mapping and $S \subset N$ we define the S-principal subfunction[2] of f, $f_S \colon \mathbf{R}^S \to \mathbf{R}^S$, as follows. For any $y \in \mathbf{R}^S$ let $(y, 0)$ denote the extension of y into an n-dimensional vector by adding zero coordinates in the appropriate

[2] The reader who is familiar with the literature concerning nonlinear complementarity theory should notice the difference between Moré and Rheinboldt's [9, Def. 3.5] and our definition.

places. Then let $f_S(y)$ denote the restriction of $f(y, 0)$ to the coordinates in S, i.e., $f_S(y) = [f(y, 0)]^S$.

Throughout this paper the symbol f will be saved for the mapping of the complementarity problem under consideration, i.e., $f: \mathbf{R}^n_+ \to \mathbf{R}^n$. Under this convention we also save the symbol F for denoting the following extension $F: \mathbf{R}^n \to \mathbf{R}^n$ of f

$$F(x) = f(x^+) + x^- \tag{2.2}$$

where

$$x^+, x^- \in \mathbf{R}^n, \qquad x_i^+ = \begin{cases} x_i & \text{if } x_i \geqq 0, \\ 0 & \text{otherwise,} \end{cases} \qquad x_i^- = \begin{cases} x_i & \text{if } x_i \leqq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The following theorem suggests a clarifying interpretation for the GUS property of a complementarity problem.

Theorem 2.1: *Let $f: \mathbf{R}^n_+ \to \mathbf{R}^n$ be a continuous mapping. Then the* CP *associated with the mapping f is* GUS *if and only if the extension, F,* (see (2.2)) *of f is a homeomorphism of $\mathbf{R}^n$ onto itself.*

Proof. Obviously, F is continuous since f was assumed to be continuous. The CP associated with the mapping f is GUS if and only if for every $q \in \mathbf{R}^n$ there is a unique $x = x(q) \in \mathbf{R}^n_+$ such that $f(x) + q \in \mathbf{R}^n_+$ and $f_i(x) + q_i = 0$ for each i such that $x_i > 0$. The GUS property is thus equivalent to the existence of a unique $z = z(q) \in \mathbf{R}^n$ such that $f(z^+) + q = -z^-$. The latter equality is however equivalent to $F(z) = -q$, and therefore the CP has the GUS property if and only if F is a bijection of $\mathbf{R}^n$ (i.e., a one-to-one mapping from $\mathbf{R}^n$ onto itself). The remainder of the proof follows from the fact that the inverse of a continuous bijection of $\mathbf{R}^n$ is also continuous. We prove this assertion in the Appendix.

Corollary 2.2. *Let $f: \mathbf{R}^n_+ \to \mathbf{R}^n$ be a continuous mapping such that the* CP *associated with f is* GUS. *Then the solution of the* CP *associated with $f(\cdot) + q$ is a continuous function of the vector q for every $q \in \mathbf{R}^n$.*

This follows from the fact that if $z \equiv F^{-1}(-q)$, then $x = z^+$, is a solution for the CP associated with $f(\cdot) + q$.

Although Theorem 2.1 is just an observation, we are able at this point to provide the following necessary and sufficient condition.

Theorem 2.3. *Let $f: \mathbf{R}^n_+ \to \mathbf{R}^n$ be a continuous mapping. Then the* CP *associated with f is* GUS *if and only if F is norm-coercive on $\mathbf{R}^n$, i.e.,*

$$\lim_{\|x\| \to \infty} \|F(x)\| = \infty,$$

and locally univalent (i.e. univalent in the neighbourhood of every x).

The proof follows from [13; Th. 5.3.8] and Theorem 2.1.

3. On sufficient conditions for the GUS property

Obviously, the condition given in Theorem 2.3 generalizes all the sufficient conditions for existence and uniqueness (Theorems 1.4, 1.5, and 1.6). All these known conditions have the property that if the function f satisfies them then so do all its principal subfunctions. It is the goal of this section to present a sufficient condition that covers Cottle's [2], Moré's [7, 8], and Karamardian's [5] conditions and at the same time has that property concerning the principal subfunctions.

Theorem 3.1. *Let $f: \mathbf{R}^n_+ \to \mathbf{R}^n$ be a continuous mapping and assume that*

(i) *All the principal subfunctions of f are univalent.*

(ii) *For every $S \subset N$ ($S \neq \emptyset$) the principal subfunction f_S is norm-coercive on $\mathbf{R}^S_+$, i.e.,*

$$\lim_{x \in \mathbf{R}^S_+, \|x\| \to \infty} \|f_S(x)\| = \infty.$$

(iii) *For all $x, y \in \mathbf{R}^n$ and every i, $i = 1, \ldots, n$, if*

$$F_j(x_1, \ldots, x_n) = F_j(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_n)$$

for all $j \neq i$, and if $x_i > 0$, then

$$F_i(x_1, \ldots, x_n) > F_i(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_n).$$

Under these conditions the CP *associated with f is* GUS.

Proof. First we shall show that F is univalent on $\mathbf{R}^n$. Suppose that $F(x) = F(y)$ for some $x, y \in Q^S$ ($S \subset N$). Then we have

$$F(x) = f(x^+) + x^- = f(y^+) + y^- = F(y).$$

Since $x, y \in Q^S$, it follows from condition (i) that $x^+ = y^+$ and hence $x^- = y^-$. Thus, F is univalent on each orthant Q^S. We now consider the case that there is no $S \subset N$ for which $x, y \in Q^S$. In this case we can assume without loss of generality that there is an i, $1 \leqq i \leqq n$, such that $x_i > 0 > y_i$. If $F(x) = F(y)$ then

$$\begin{aligned} F_j(x) &= F_j(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_n) \quad \text{for } j \neq i, \\ F_i(x) &= F_i(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_n) + y_i \\ &< F_i(y_1, \ldots, y_{i-1}, 0, y_{i+1}, \ldots, y_n) \end{aligned}$$

and

$$x_i > 0.$$

This contradicts condition (iii). Hence $F(x) \neq F(y)$. Thus we have shown that F is univalent on $\mathbf{R}^n$.

It follows from Theorem A.1 that F is a homeomorphism of $\mathbf{R}^n$ into itself. On the other hand, condition (ii) implies the norm-coercivity of F. By Theorem 2.3 F is a homeomorphism from $\mathbf{R}^n$ onto itself. The GUS property follows from Theorem 2.1.

Remark 3.2. Most of the conditions stated in Theorem 3.1 are necessary for the GUS property. Univalence of F, which is necessary by Theorem 2.1, implies that all the principal subfunctions of f are univalent (see (2.2)). Also, F is necessarily norm-coercive (Theorem 2.3) and therefore f must be norm-coercive. However, it is not necessary for all of the principal subfunctions of f to be norm-coercive. Indeed, in Example 6.4 we present a function which induces a GUS complementarity problem and yet one of its principal subfunctions is not norm-coercive. Condition (iii) is also necessary for F to be univalent; if $F_j(x) = F_j(y)$ for all $j \neq i$, $y_i = 0 < x_i$, and $F_i(x) \leqq F_i(y)$, then

$$F(x) = F[y_1, \ldots, y_{i-1}, F_i(x) - F_i(y), y_{i+1}, \ldots, y_n].$$

We would like to draw attention to the fact that condition (ii) of Theorem 3.1 cannot be relaxed. Specifically, in Example 6.5 we present a function which satisfies conditions (i) and (iii) and which is also norm-coercive, but the CP associated with it is not GUS, due to the fact that one of the principal subfunctions is not norm-coercive.

The conditions given by Cottle and Moré (and Karamardian) have the property that if a function satisfies them then all of its principal subfunctions do the same. Thus, in view of Remark 3.2, the fact that Cottle's and Moré's (and Karamardian's) conditions imply our condition (i.e., the conditions listed in Theorem 3.1) can be proved indirectly using their sufficiency and the above mentioned property. We consider it interesting to provide a direct proof, especially in the case of Cottle's condition. It is also important to mention that Cottle's and Moré's conditions are independent, as demonstrated by Examples 6.1 and 6.2.

Cottle's theorem, which was given an algorithmic proof, requires two conditions, namely nondegeneracy of solutions and continuity of the Jacobian matrix, which are eliminated by the new proof that we give here.

Theorem 3.3. *If $f: \mathbf{R}^n_+ \to \mathbf{R}^n$ is a differentiable mapping, such that all the principal minors of the Jacobian matrix of f are bounded between δ and δ^{-1}, for some $0 < \delta < 1$, then the* CP *associated with f is* GUS.

Proof. (a) First, note the following properties of the extension F (see (2.2)) in our case. Since the mappings $x \to x^+$ and $x \to x^-$ are differentiable in points x such that $x_i \neq 0$ $(i = 1, \ldots, n)$, it follows that F is differentiable in the interior of every orthant of $\mathbf{R}^n$. In general, F satisfies something weaker than differentiability, which is the following. Given $y \in Q^S$, define a matrix $F^S = (F^S_{ij})_{1 \leqq i,j \leqq n}$ by

$$F^S_{ij} = \begin{cases} \partial f_i(y^+)/\partial x_j & \text{if } j \in S, \\ 1 & \text{if } i = j \notin S, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

It is easy to verify that for every Δy such that $y + \Delta y \in Q^S$

$$F(y + \Delta y) - F(y) = F^S \cdot \Delta y + \mathrm{o}(\Delta y) \tag{3.2}$$

where $\lim_{\|\Delta y\|\to 0} o(\Delta y)/\|\Delta y\| = 0$.

(b) We have to prove that F is a homeomorphism onto $\mathbf{R}^n$. The univalence of F was as a matter of fact proved by Moré in [8], even though he did not refer to F itself but to the uniqueness of solution in CP's associated with P-functions. In our approach, the univalence of F can be either derived from Gale and Nikaido's theorem on global univalence [4, 11] or proved directly by slightly modifying Gale and Nikaido's theorem. The two proofs are outlined in the Appendix. Thus, we only have to prove that $F(\mathbf{R}^n) = \mathbf{R}^n$. Our proof is based on Nikaido [12].

(c) We use induction on the dimension. First, if $n = 1$, then F is continuous and monotone and $\lim_{|x|\to\infty}|F(x)| = \infty$. This implies that every real number is attained by F. Suppose that $n > 1$ and assume, by induction, that the theorem is true in lower dimensional cases. Let $a = (a_1, \dots, a_n)$ by an arbitrary fixed vector. For any fixed value of x_n, the function $F^*(x_1, \dots, x_{n-1}) = (F_1(x), \dots, F_{n-1}(x))$ $(x = (x_1, \dots, x_n))$ is a homeomorphism onto $\mathbf{R}^{n-1}$, by the induction hypothesis. Thus, for every x_n there exists a unique vector $\phi(x_n)$ in $\mathbf{R}^{n-1}$ such that $F_j(\phi(x_n), x_n) = a_j$, $j = 1, \dots, n-1$. Define

$$t \equiv \psi(x_n) = F_n(\phi(x_n), x_n).$$

In order to complete the proof of $F(\mathbf{R}^n) = \mathbf{R}^n$, it suffices to show that there is an x_n such that $\psi(x_n) = a_n$.

(d) We shall prove that the mappings ϕ and ψ are continuous. Since F is univalent, it is inversible and F^{-1} is continuous (see Theorem A.1). It also follows that ψ is univalent,

$$\begin{aligned}\psi(x_n^1) = \psi(x_n^2) &\Rightarrow F^{-1}(a_1, \dots, a_{n-1}, \psi(x_n^1)) = F^{-1}(a_1, \dots, a_{n-1}, \psi(x_n^2)) \\ &\Rightarrow (\phi(x_n^1), x_n^1) = (\phi(x_n^2), x_n^2) \Rightarrow x_n^1 = x_n^2.\end{aligned}$$

Thus, ψ is inversible. Moreover, ψ^{-1} is continuous since $\psi^{-1}(t)$ is the n-th coordinate of $F^{-1}(a_1, \dots, a_{n-1}, t)$. Since F is a homeomorphism, $F(\mathbf{R}^n)$ is open, and therefore ψ^{-1} is defined on an open subset of $\mathbf{R}^1$. This implies that ψ is continuous (Theorem A.1). The continuity of ϕ follows from the equality

$$(\phi(x_n), x_n) = F^{-1}(a_1, \dots, a_{n-1}, \psi(x_n)).$$

(e) We shall prove that $\psi(\mathbf{R}^1) = \mathbf{R}^1$. A sufficient condition for this is that for every value of x_n

$$\liminf_{\Delta x\to 0}\, (1/\Delta x)[\psi(x_n + \Delta x) - \psi(x_n)] > \delta^2. \tag{3.3}$$

Let $\{h_k\}_{k=1}^{\infty}$ be any sequence of non-zero real numbers, such that $\lim_{k\to\infty} h_k = 0$. We use the following notation

$$\begin{aligned}
y^k &= (\phi(x_n + h_k), x_n + h_k), \qquad k = 1, 2, \dots, \\
y &= (\phi(x_n), x_n), \\
\Delta^k &= y^k - y, \qquad k = 1, 2, \dots, \\
g_{ij}^k &= \begin{cases} o_i(\Delta^k)(|\Delta_j^k|/|\Delta^k|)/\Delta_j^k & \text{if } \Delta_j^k \neq 0 \quad (1 \leqq i, j \leqq n, \quad k = 1, 2, \dots, \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

where $|\Delta^k| = \sum_{j=1}^n |\Delta_j^k|$),

$$g^k = (g_{ij}^k)_{1\le i,j\le n}, \qquad k = 1,2,\ldots.$$

Also, if $M \in \mathbf{R}^{n\times n}$ and $x \in \mathbf{R}^n$, we denote by $\tilde{M}$ and $\tilde{x}$ the principal submatrix of M and the subvector of x, respectively, that correspond to the indices $1,2,\ldots,n-1$.

Under this notation,

$$\tilde{F}(y^k) = \tilde{F}(y) = \tilde{a} \quad \text{for } k = 1,2,\ldots.$$

Assume that all the vectors y^k ($k = 1,2,\ldots$) belong to the same orthant Q^S. Since ϕ is continuous (see (d)), it follows that also $y \in Q^S$. It also follows from (3.2) that

$$F(y^k) - F(y) = (F^S + g^k)\Delta^k, \quad k = 1,2,\ldots.$$

The latter can be written also as follows.

$$0 = \tilde{F}(y^k) - \tilde{F}(y) = (\tilde{F}^S + \tilde{g}^k)\tilde{\Delta}^k + (F_{\cdot n}^S + g_{\cdot n}^k)\Delta_n^k$$

$$\psi(x_n + h_k) - \psi(x_n) = F_n(y^k) - F_n(y) = \sum_{j=1}^n (F_{nj}^S + g_{nj}^k)\Delta_j^k$$

where $F_{\cdot n}^S$ and $g_{\cdot n}^k$ are the n-th columns of F^S and g^k, respectively. Since $\Delta_n^k = h_k$ ($k = 1,2,\ldots$), we can write

$$(\tilde{F}^S + \tilde{g}^k)\tilde{\Delta}^k / h_k = -F_{\cdot n}^S - g_{\cdot n}^k \tag{3.4}$$

$$(1/h_k)(\psi(x_n + h_k) - \psi(x_n)) = \sum_{j=1}^{n-1} (F_{nj}^S + g_{nj}^k)\Delta_j^k / h_k + F_{nn}^S + g_{nn}^k. \tag{3.5}$$

Consider the definition of F^S (see (3.1)). It can be verified that the determinant of F^S is equal to the Jacobian of f_S evaluated at y^S. Also, the determinant of $\tilde{F}^S$ is equal to the Jacobian of $f_{S\setminus\{n\}}$ evaluated at $y^{S\setminus\{n\}}$. Thus, $\tilde{F}^S$ is non-singular. Moreover, since

$$\lim_{k\to\infty} g_{ij}^k = 0 \quad (1 \le i, j \le n)$$

it follows that $\tilde{F}^S + \tilde{g}^k$ is non-singular, for k sufficiently large. Hence (see (3.4)), the following limit exists

$$\tilde{\phi}^S \equiv \lim_{k\to\infty} \tilde{\Delta}^k / h_k = -(\tilde{F}^S)^{-1} F_{\cdot n}^S. \tag{3.6}$$

Substituting into (3.5), we have in the limit,

$$\lim_{k\to\infty} (1/h_k)(\psi(x_n + h_k) - \psi(x_n)) = \sum_{j=1}^{n-1} F_{nj}^S \phi_j^S + F_{nn}^S. \tag{3.7}$$

It can be verified that (3.6)–(3.7) implies

$$\lim_{k\to\infty} (1/h_k)(\psi(x_n + h_k) - \psi(x_n)) = \det(F^S)/\det(\tilde{F}^S) > \delta^2. \tag{3.8}$$

However, since there is only a finite number of orthants in $\mathbf{R}^n$, the latter implies (3.3).

This completes the proof.

The following theorem introduces an explicit condition, sufficient for the GUS property, which generalizes Cottle's, Moré's (and Karamardian's) conditions. It arises naturally from our proof of Theorem 3.3.

Theorem 3.4. *Let $f: \mathbf{R}^n_+ \to \mathbf{R}^n$ be a differentiable mapping. Let $J_S(x)$ denote the principal minor (corresponding to $S \subset N$) of the Jacobian matrix of f evaluated at $x \in \mathbf{R}^n_+$. Suppose that there exists $\epsilon > 0$ such that for every $x \in \mathbf{R}^n_+$, $S \subsetneq N$ and $i \in N \backslash S$*

$$J_{S\cup\{i\}}(x) \geqq \epsilon J_S(x) \tag{3.9}$$

(where $J_\emptyset(x) = 1$ for every x). Under these conditions, the CP *associated with f is* GUS.

The proof is essentially the same as that of Theorem 3.3. Here, ϵ plays the role of δ^2 in the proof of Theorem 3.3, and it is easy to observe that (3.9) suffices for establishing (3.8).

Remark 3.5. In fact, it is sufficient that (3.9) holds for every $x \in \mathbf{R}^n_+$, $S \subsetneq N$ and $i \in N \backslash S$ such that $i > j$ for every $j \in S$; only these combinations of S and i appear in the induction procedure in the proof of Theorem 3.3.

Proposition 3.6. *Let $f: \mathbf{R}^n_+ \to \mathbf{R}^n$ be a differentiable mapping. If f satisfies either Cottle's condition* (i.e., the condition specified in Theorem 3.3; even without continuity of the Jacobian matrix and nondegeneracy of solutions) *or Moré's condition* (Theorem 1.6) *then f satisfies the condition specified in Theorem* 3.4.

Proof. It is trivial that Cottle's condition implies the condition of Theorem 3.4. Assume that f is a differentiable uniform P-function. Let $c > 0$ be the appropriate constant (see (1.2)).

Let $i \in N$ be fixed, and let e^i denote the unit vector defined by $e^i_j = 0$ if $j \neq i$, $e^i_i = 1$. Define

$$g(x) = f(x) - cx_i e^i \quad \text{for every } x \in \mathbf{R}^n_+.$$

We shall first show that g is a P_0-function (see Section 1). Let $x, y \in \mathbf{R}^n_+$ be any two distinct vectors. Since f is a uniform P-function, there exists a $j \in N$ such that

$$(y_j - x_j)(f_j(y) - f_j(x)) \geqq c\|y - x\|^2.$$

If $j = i$, then

$$(y_j - x_j)(g_j(y) - g_j(x)) = (y_j - x_j)(f_j(y) - f_j(x)) - c(y_j - x_j)^2 \geqq 0.$$

If $j \neq i$, then

$$(y_j - x_j)(g_j(y) - g_j(x)) = (y_j - x_j)(f_j(y) - f_j(x)) \geqq 0.$$

Since in either case $y_j \neq x_j$, it has been proved that g is a P_0-function. Let $G_S(x)$ denote the principal minor (corresponding to $S \subset N$) of the Jacobian matrix of g evaluated at $x \in \mathbf{R}^n_+$ ($G_\emptyset(x) = 1$). By [9, Theorem 5.8] all these principal minors are non-negative,

$$G_S(x) \geqq 0, \qquad x \in \mathbf{R}^n_+.$$

This implies that for S such that $i \notin S$,

$$G_{S\cup\{i\}}(x) = J_{S\cup\{i\}}(x) - c \cdot J_S(x) \geqq 0 \quad (\text{for every } x \in \mathbf{R}^n_+).$$

Thus, f satisfies the condition specified in Theorem 3.4.

Remark 3.7. The condition specified in Theorem 3.4 implies the conditions of Theorem 3.1. This follows from the fact that (i) and (iii) are necessary and the property that all the principal subfunctions of a function f that satisfies the condition of Theorem 3.4 also satisfy this condition. Thus, such an f is norm-coercive and therefore all its principal subfunctions are norm-coercive.

Remark 3.8. In spite of the fact that Cottle's theorem can be proved without principal pivot operations (Theorem 3.3), it is not true that his positively boundedness can be weakened. Cottle showed that positively boundedness below is necessary for his theorem. On the other hand, Cottle [2, p. 155] expressed the desirability to have shown the same for boundedness above[3], even though he suggested the possibility that weaker conditions on the principal minors of the Jacobian matrix are sufficient. In Section 6 we give an example of a differentiable function f (see Example 6.6) such that all the principal minors of the Jacobian matrix of f are greater than the unit everywhere. Moreover, for every constant term q there is an x such that $x \geqq 0$ and $f(x) + q \geqq 0$. Yet, there is a q for which there is no x such that $x^{\mathrm{T}}[f(x) + q] = 0$, $f(x) + q \geqq 0$, and $x \geqq 0$. Also, the CP associated with $f(\cdot) + q$ satisfies the condition of nondegeneracy of solutions.

4. A necessary condition for the differentiable case

In this section we generalize a part of the basic theorem for linear CP's (Theorem 1.3). In fact we shall prove a necessary condition for a differentiable CP to have at most one solution for every constant term.

Theorem 4.1. *Let $f: \mathbf{R}^n_+ \to \mathbf{R}^n$ be a differentiable mapping such that for every $q \in \mathbf{R}^n$ the* CP *associated with $f(\cdot) + q$ has at most one solution. For every $\emptyset \neq S \subset N$ let $J_S(x)$ denote the Jacobian* (i.e., the determinant of the Jacobian

[3] Notice that in the one-dimensional case boundedness above is *not* necessary.

matrix) *of the S-principal subfunction*[4] f_S (see Section 2) *evaluated at the point* $x \in \mathbf{R}_+^n$. *Under these conditions, for every* $\phi \neq S \subset N$ *there exists a set* $X^S \subset \mathbf{R}_+^S$ *such that* $J_S(x) > 0$ *for* $x \in \mathbf{R}_+^S \backslash X^S$ *and* $J_S(x) = 0$ *for* $x \in X^S$; *if f is continuously differentiable then* X^S *is nowhere dense in* $\mathbf{R}_+^S$.

Proof. Sard [15] proved that the set of singular values of a continuously differentiable function $g: \mathbf{R}^n \to \mathbf{R}^n$ is of measure zero in $\mathbf{R}^n$. Thus, since the functions f_S are univalent (see Remark 3.2), it follows that the set of singular points of f_S is nowhere dense in the domain $\mathbf{R}_+^S$, if f is continuously differentiable.

Consider the extention F (see (2.2)) of f and its principal subfunctions F_S ($S \subset N$). Under the conditions specified in our theorem, it follows that F_S is a homeomorphism of $\mathbf{R}^S$ into itself. This implies that the local degree of F_S is constant over the whole $\mathbf{R}_S$ (see [1; Ch. XVI, §4]). This implies that the Jacobian of F_S is either nonnegative everywhere or nonpositive everywhere in $\mathbf{R}^S$ (see [1; Ch. XVI, §5.7, Exc. 1]). On the other hand, the restriction of F_S to the nonpositive orthant in $\mathbf{R}^S$ is the identity and therefore has positive Jacobian. It thus follows that F_S has a nonnegative Jacobian. This implies that $J_S(x)$ is positive everywhere except for a set in $\mathbf{R}_+^S$, over which it assumes the value zero. This set is nowhere dense if f is continuously differentiable.

Corollary 4.2. If f is linear and for every q the CP associated with $f(\cdot) + q$ has at most one solution then, according to Theorem 4.1, the Jacobian of f is a P-matrix. It follows from the basic theorem (Theorem 1.3) that for every q there is exactly one solution.

This result is by no means new and in fact appeared in Murty [10].

Remark 4.3. It is interesting to mention that positiveness of the Jacobians of the principal subfunctions is not sufficient to the uniqueness of solutions. This is demonstrated in Example 6.7.

5. Necessary and sufficient conditions

In Section 2 it was shown that (Theorem 2.1) a function f yields a GUS complementarity problem if and only if a certain extension F (see (2.2)) of f is a homeomorphism of the space onto itself. It is the goal of the present section to provide necessary and sufficient conditions for the same property which assume just the existence of some extention[5] G of f which turns out to be a homeomorphism of the space onto itself.

[4] Our condition is not necessary for all the principal subfunctions in the sense of Moré and Rheinboldt [9, Def. 3.5]. Example 6.3 demonstrates this fact.

[5] In some of the theorems G does not even have to exactly extend f.

We first state a theorem for the two-dimensional case.

Theorem 5.1. *Let $f: \mathbf{R}_+^2 \to \mathbf{R}^2$ be a continuous mapping such that $f(0) = 0$. Then the* CP *associated with f is* GUS *if and only if*:

(i) *The principal subfunctions $f_{\{1\}}, f_{\{2\}}$ are monotone increasing and f is univalent.*

(ii) *If $\lim_{x_1\to\infty} f_1(x_1, 0) \neq +\infty$, then $\lim_{x_1\to\infty} f_2(x_1, 0) = -\infty$ and if $\lim_{x_2\to\infty} f_2(0, x_2) \neq +\infty$, then $\lim_{x_2\to\infty} f_1(0, x_2) = -\infty$.*

(iii) *There exists a homeomorphism G of $\mathbf{R}^2$ onto itself such that G extends both f and the identity function of the nonpositive orthant $\mathbf{R}_-^2$.*

The proof is postponed. The following theorem provides a necessary and sufficient condition for any dimension.

Theorem 5.2. *Let $f: \mathbf{R}_+^n \to \mathbf{R}^n$ be a continuous mapping. Then the* CP *associated with f is* GUS *if and only if*

(i) *The principal subfunctions of f are univalent.*

(ii) *There exists a homeomorphism G of $\mathbf{R}^n$ onto itself such that*

$$G(Q^S) = f(\mathbf{R}_+^S) - \mathbf{R}_+^{N\setminus S} \equiv \{f(x) - y : x \in \mathbf{R}_+^S, y \in \mathbf{R}_+^{N\setminus S}\}$$

for all $S \subset N$ (see (2.1)).

Proof. Notice that $f(\mathbf{R}_+^S) - \mathbf{R}_+^{N\setminus S} = F(Q^S)$ (see (2.2)). Necessity is immediate since the extension F can serve as the homeomorphism G.

Suppose that conditions (i)–(ii) hold. It follows from condition (ii) that $F(\mathbf{R}^n) = \mathbf{R}^n$.

We shall show now that F is univalent. First, it can be easily verified that by condition (i) F is univalent in each orthant Q^S of $\mathbf{R}^N$. Notice that F is locally univalent in interior points of any orthant of $\mathbf{R}^N$. This implies that F is a local homeomorphism in such points (see Theorem A.1). It follows from this fact that $F(\operatorname{int} Q^S) \subset G(\operatorname{int} Q^S)$. For any $x \in \mathbf{R}^n$ denote

$$I(x) = \{i : x_i = 0\}. \tag{5.1}$$

Let $x, y \in \mathbf{R}^n$ be such that $F(x) = F(y)$. We shall prove by induction on $|I(x)| + |I(y)|$ that $x = y$. Suppose, first that $|I(x)| + |I(y)| = 0$. Then x and y are interior points of some orthants; suppose $x \in \operatorname{int} Q^S$ and $y \in \operatorname{int} Q^T$. Thus, there exist $u \in \operatorname{int} Q^S$ and $v \in \operatorname{int} Q^T$ such that $G(u) = F(x)$ and $G(v) = F(y)$. But G is univalent, so $u = v$. This obviously implies $S = T$ and, since F is univalent in each orthant, $x = y$.

Assume, by induction, that $F(x) = F(y)$ and $|I(x)| + |I(y)| \leqq k$ imply $x = y$. Consider any two points $x \neq y$ such that $F(x) = F(y)$ and $|I(x)| + |I(y)| = k + 1$. Without loss of generality, assume $|I(x)| \leqq |I(y)|$. We shall show that F is locally univalent in x. Let U denote a neighbourhood of x such that $y \notin U$ and $I(u) \subset I(x)$ for every u in U. We claim that F is univalent in U. Let u and v be two distinct points of U. We distinguish two cases. First, $I(u) = I(x)$. In this case there exists an orthant Q^S such that $x, u, v \in Q^S$. This implies $F(u) \neq F(v)$.

Second, $I(u) \subsetneq I(x)$. In this case $|I(u)|+|I(v)|<2|I(x)| \leqq k+1$, so that the induction hypothesis implies $F(u) \neq F(v)$. This proves that F is univalent in U.

It follows from the invariance theorem of domain ([3, Lemma 3.9, p. 303]) that $F(U)$ is an open set which contains $F(y)$. Since F is continuous at y, there exists $w \notin U$ such that $I(w)=\emptyset$ and $F(w) \in F(U)$. Thus, there is a u in U such that $F(w)=F(u)$. Since $|I(w)|+|I(u)|<k+1$, it follows from the induction hypothesis that $w=u$, and hence, a contradiction. This proves that F is globally univalent.

Thus, having proved that F is univalent and continuous mapping of $\mathbf{R}^n$ onto itself, we deduce (Theorem 2.1) that our complementarity problem is GUS.

Another necessary and sufficient condition for the n-dimensional case is given in the following theorem. We denote

$$Q_T^S = \{x \in \mathbf{R}^n \colon x_i \geqq 0 \text{ if } i \in S, x_i \leqq 0 \text{ if } i \in T, \text{ and } x_i = 0 \text{ otherwise}\}.$$

Notice that $Q^S = Q^S_{N \setminus S}$.

Theorem 5.3. *Let $f \colon \mathbf{R}^n_+ \to \mathbf{R}^n$ be a continuous mapping. Then the* CP *associated with f is* GUS *if and only if*:

(i) *The principal subfunctions of f are univalent.*

(ii) *There exists a homeomorphism G of $\mathbf{R}^n$ onto itself such that for all pairs of disjoint S, T*

$$G(Q_T^S) \subset f(\mathbf{R}_+^S) - \mathbf{R}_+^T.$$

Proof. Notice that $f(\mathbf{R}_+^S) - \mathbf{R}_+^T = F(Q_T^S)$ (see (2.2)). Thus, necessity is immediate since, as in Theorem 5.2, F can serve as the homeomorphism G.

Suppose that conditions (i), (ii) hold. As in Theorem 5.2, condition (ii) implies that $F(\mathbf{R}^n) = \mathbf{R}^n$ and it will suffice to show that F is also univalent. However, condition (i) implies that F is univalent in each orthant.

We shall prove that if $S \cap T = \emptyset$ then $G(Q_T^S) = F(Q_T^S)$. First, if $S = T = \emptyset$ then the above equality is trivial. We proceed by induction on $|S|+|T|$. Assume that $G(Q_T^S) = F(Q_T^S)$ for any S, T such that $S \cap T = \emptyset$ and $|S|+|T| = k$. Consider two disjoint sets S, T such that $|S|+|T| = k+1$. Let B and C denote the boundary and the interior, respectively, of Q_T^S with respect to $\mathbf{R}^{S \cup T}$. Thus, $Q_T^S = B \cup C$ and $B \cap C = \emptyset$. Since F is univalent in every orthant, $F(B) \cap F(C) = \emptyset$. Also, $G(B) \cap G(C) = \emptyset$. Since $B = \bigcup_{i \in S} Q_T^{S \setminus \{i\}} \cup \bigcup_{i \in T} Q^S_{T \setminus \{i\}}$ it follows from the induction hypothesis that $G(B) = F(B)$. But $G(B \cup C) \subset F(B \cup C)$, hence, $G(C) \subset F(C)$. The set $G(C)$ is closed with respect to $F(C)$ since G is a homeomorphism of $\mathbf{R}^n$ onto itself, $B \cup C = Q_T^S$ is closed, and $G(C) = F(C) \cap G(B \cup C)$. It should also be observed that F maps C homeomorphically because of the following. $F_{S \cup T}$ is univalent in Q_T^S (using condition (i) and (2.2)) and hence maps[6] $(\operatorname{relint} \mathbf{R}_+^S) \times (\operatorname{relint} \mathbf{R}_-^T)$ homeomorphically (see Theorem A.1). Also, $C = (\operatorname{relint} \mathbf{R}_+^S) \times (\operatorname{relint} \mathbf{R}_-^T) \times \{0\}$. Having proved that F maps C homeomorphically, we deduce that $F^{-1}(G(C))$ is closed in C and also the composition $F^{-1} \circ G$ maps C

[6] We denote the relative interior of a set P by relint P.

homeomorphically. This implies that $F^{-1}(G(C))$ is open in $\mathbf{R}^{S\cup T}$ ([3, Lemma 3.9, p. 303]) and since it is a subset of C, $F^{-1}(G(C))$ is open in C too. The connectedness of C implies that $C = F^{-1}(G(C))$, hence, $F(C) = G(C)$. Thus, $G(Q_T^S) = F(Q_T^S)$, and by Theorem 5.2, this suffices for the GUS property.

We shall now prove the theorem concerning the two-dimensional case.

Proof of Theorem 5.1. The necessity of conditions (i) and (iii) follows from arguments that have already been used several times in our proofs. We now prove the necessity of condition (ii). Suppose $\lim_{x_1\to\infty} f_1(x_1, 0) = t \neq +\infty$ (the limit exists because of condition (i)). Since G is a homeomorphism of $\mathbf{R}^2$ onto itself and extends f, it follows that $\lim_{x_1\to\infty} \|f(x_1, 0)\| = +\infty$. This implies that, as x_1 tends to infinity, $f_2(x_1, 0)$ tends either to $+\infty$ or to $-\infty$. Suppose, per absurdum, that $\lim_{x_2\to\infty} f_2(x_1, 0) = +\infty$. Using the Jordan-curve theorem ([1, Chapter II, §1.1]), we deduce that $L \equiv f(\mathbf{R}_+^{\{1\}}, 0) \cup \mathbf{R}_-^{\{2\}}$ is a separating curve and $f(\mathbf{R}_+^2)$ lies in one of the closed components generated by L. Since the extension F is univalent in our case, it follows that $f(\mathbf{R}_+^2) \cap [f(\mathbf{R}_+^{\{1\}}) - \mathbf{R}_+^{\{2\}}] = f(\mathbf{R}_+^{\{1\}})$. It can now be easily verified (see Fig. 1) that $F(\mathbf{R}^2)$ does not meet the set $\{x \in \mathbf{R}^2 \colon x_1 \geqq t\}$, and therefore F does not map the plane onto itself. This implies that the CP is not GUS in this case. The remainder of condition (ii) follows by symmetry.

We now prove the sufficiency of our conditions. We do so by showing that the extension F is a homeomorphism of the plane onto itself. The set $G(Q^{\{1\}})$ coincides with one of the closed components generated by the separating curve

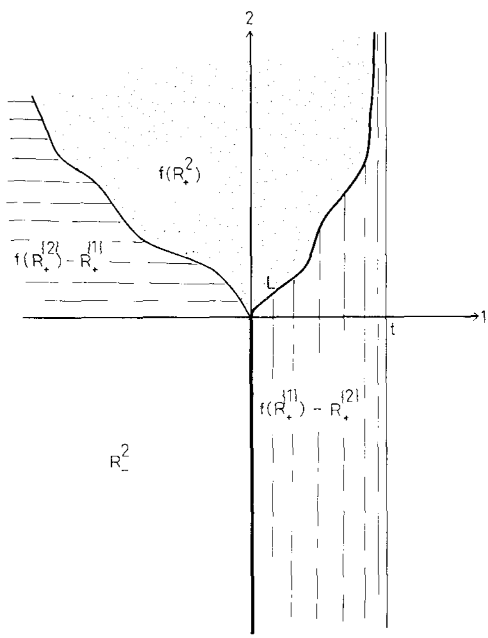

Fig. 1.

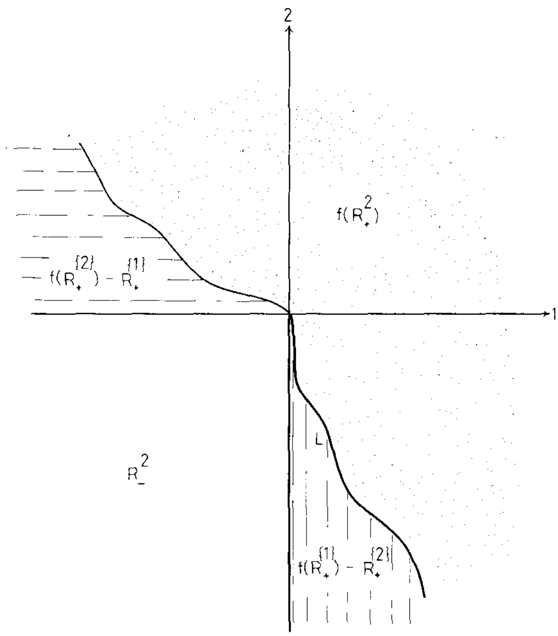

Fig. 2.

L. In view of our condition (ii) this implies that for every $y \in G(Q^{\{1\}})$ there is a vector $z \in \mathbf{R}^2_+$ such that $y + z \in f(\mathbf{R}^{\{1\}}_+)$ (Fig. 2). In other words, there is an x in $Q^{\{1\}}$ such that $y = F(x)$. It can now be easily verified that for every orthant Q^S of $\mathbf{R}^2$, if $y \in G(Q^S)$ then $y \in F(Q^S)$. Since we are concerned with the two-dimensional case, this implies that all the conditions of Theorem 5.3 are satisfied and therefore F is a homeomorphism.

6. Examples

Example 6.1 (Cottle's condition does not imply Moré's condition). Let $f(x_1, x_2) = (x_1 + x_2^2, x_2)$. All the principal minors of the Jacobian matrix are constant and equal to unit. Thus, Cottle's condition holds. The function f is not a uniform P-function since if $x^k = (1 - 2/k, k + 1/k^2)$ and $y^k = (1, k)$, $k = 2, 3, \ldots$, then

$$(x_1^k - y_1^k)[f_1(x^k) - f_1(y^k)] = -2/k^5 < 0,$$
$$(x_2^k - y_2^k)[f_2(x^k) - f_2(y^k)] = 1/(4k^2 + 1) \cdot \|x^k - y^k\|^2.$$

Example 6.2 (Moré's condition does not imply Cottle's condition). Let $f(x) = e^x - 1$. Obviously, f is a uniform P-function (Moré's condition) but the derivative of f is not bounded above, so that Cottle's condition is not satisfied. On the other hand, the positively boundedness below is not accidental. Indeed, if f is a uniform P-function and differentiable, then there is a positive c such that for any

$x \in \mathbf{R}^n_+$, $\Delta x \in \mathbf{R}^n$, and $\lambda > 0$, if $\|\Delta x\| = 1$ and $x + \lambda\Delta x \in \mathbf{R}^n_+$, then

$$\begin{aligned} c\lambda &\leqq \operatorname*{Max}_i\{\Delta x_i[f_i(x+\lambda\cdot\Delta x) - f_i(x)]\} \\ &\leqq \sum_{i=1}^n |\Delta x_i||f_i(x+\lambda\cdot\Delta x) - f_i(x)| \\ &\leqq \|f(x+\lambda\Delta x) - f(x)\|. \end{aligned}$$

It follows that $c \leqq \|J(x)\Delta x + o(\lambda)/\lambda\|$, where $\lim_{\lambda\to 0} o(\lambda)/\lambda = 0$. As λ tends to zero we have

$$\Delta x^{\mathrm{T}}[J(x)]^{\mathrm{T}} J(x)\Delta x \geqq c^2$$

which means that the Jacobian $|J(x)|$ is positively bounded below. The same is true for all the other principal minors of the Jacobian matrix since all the principal subfunctions (in the broad sense [9, Def. 3.5]) are also uniform P-functions.

Example 6.3 (Non-necessity of the P_0-function condition). Let $g(x_1, x_2) = (x_1 - 10, x_2 - 10)$,

$$\theta(x_1, x_2) = \begin{cases} \frac{3}{4}\pi \sin^2(x_1^2 + x_2^2) & \text{if } x_1^2 + x_2^2 \leqq \pi, \\ 0 & \text{otherwise,} \end{cases}$$

$$h(x_1, x_2) = (x_1, x_2)\begin{bmatrix} \cos\theta(x_1, x_2) & -\sin\theta(x_1, x_2) \\ \sin\theta(x_1, x_2) & \cos\theta(x_1, x_2) \end{bmatrix}$$

and $f = g^{-1} \circ h \circ g$. The function f (Fig. 3) is differentiable and its Jacobian is unit valued everywhere. Using Theorem 3.1, it follows that the CP associated with f is GUS. Yet f is not even a P_0-function, namely, if $x(10, 10)$ and $y = [10, 10 + \sqrt{\frac{1}{2}\pi}]$, then $x_1 = y_1$ and $(x_2 - y_2)[f_2(x) - f_2(y)] < 0$.

Example 6.4 (Non-necessity of principal subfunctions' norm-coercivity). Let $f(x_1, x_2) = (x_1 - x_2,\ x_1x_2 + 1 - e^{-x_2})$. The principal subfunction $f_{\{2\}}(x_2) = 1 - e^{-x_2}$ is *not* norm-coercive. Univalence of f and its principal subfunctions follows from the fact that f is a P-function. It is easy to verify (see Fig. 4), using Theorem 2.3, that the CP associated with f is GUS.

Example 6.5 (Principal subfunctions' norm-coercivity is vital for Th. 3.1). Let $f(x_1, x_2) = (x_1 + x_2, 1 - e^{-x_2})$. The function f is a P-function and therefore all of its principal subfunctions are univalent. Also it is norm-coercive on $\mathbf{R}^2_+$ and the extension F (see (2.2)) associated with it satisfies condition (iii) of Theorem 3.1. However, for $q = (0, -1)$, the CP associated with $f(\cdot) + q$ does not have a solution (see Fig. 5). This is due to the fact that $f_{\{2\}}$ is not norm-coercive.

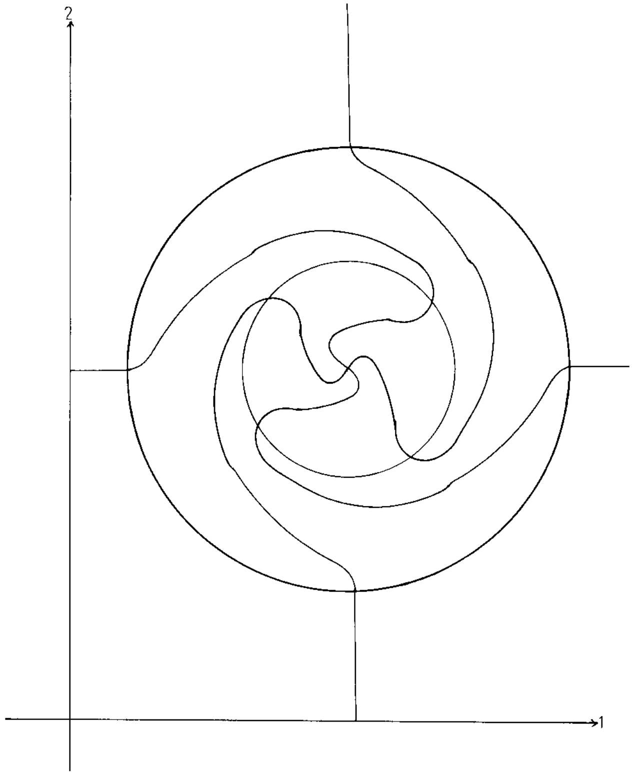

Fig. 3.

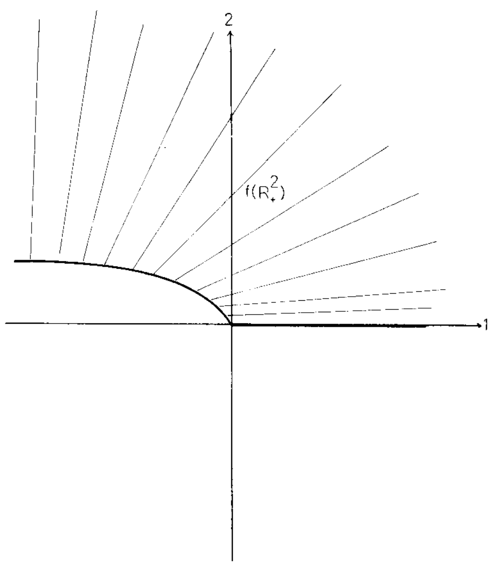

Fig. 4.

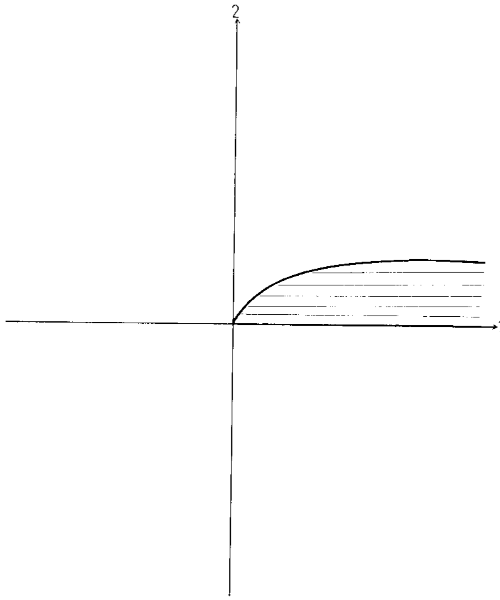

Fig. 5.

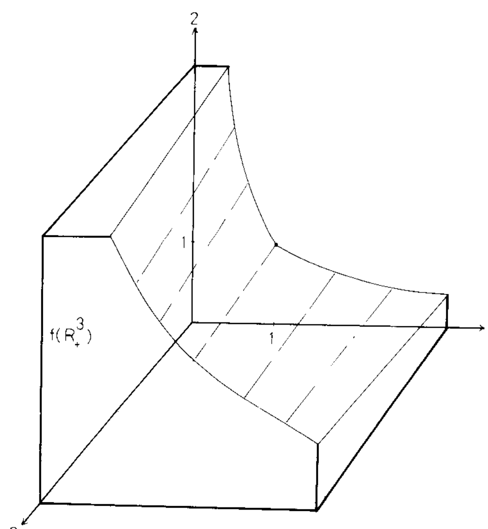

Fig. 6.

Example 6.6 (Boundedness above is necessary in Cottle's condition). Let f: $\mathbf{R}^3_+ \to \mathbf{R}^3$ be defined by

$$f_1(x_1, x_2, x_3) = (e^{x_1} - e^{x_2} - e^{-x_2} + 1)\, e^{x_3},$$
$$f_2(x_1, x_2, x_3) = (e^{x_2} - e^{x_1} - e^{-x_1} + 1)\, e^{x_3},$$
$$f_3(x_1, x_2, x_3) = x_3.$$

The principal minors of the Jacobian matrix of f are all bounded below by the unit. Cottle's condition is not satisfied since some principal minors are not bounded above. The complementarity problem associated with f is not GUS since f is not norm-coercive ($\lim_{t\to\infty} f(t, t, 1) = (\mathrm{e}, \mathrm{e}, 1)$) and the norm-coercivity condition is necessary according to Theorem 2.3. Particularly, the CP associated with $f(\cdot) - (\mathrm{e}, \mathrm{e}, 1)$ is not solvable (see Fig. 6). Notice that for any q in $\mathbf{R}^n$ the CP associated with $f(\cdot) + q$ is feasible[7] since $\lim_{t\to\infty} f(1, 1, t) = (\infty, \infty, \infty)$.

Example 6.7 (Positiveness of Jacobians of principal subfunctions does not suffice for the uniqueness of solutions). Let

$$g(x_1, x_2) = (\mathrm{e}^{x_1} \sin x_2, 1 - \mathrm{e}^{x_1} \cos x_2),$$

$$h(x_1, x_2) = (x_1 - x_2, \tfrac{1}{2}\pi\{(1 - \mathrm{e}^{-x_1})\, \mathrm{e}^{x_2} + (1 - \mathrm{e}^{-x_2})\, \mathrm{e}^{x_1}\})$$

and

$$f = g \circ h.$$

The Jacobian of g is equal to e^{x_1} and the Jacobian of h is equal to $\frac{1}{2}\pi(\mathrm{e}^{x_1} + \mathrm{e}^{x_2})$. Thus, the Jacobian of f is everywhere positive. The other principal subfunctions of f are

$$f_{\{1\}}(x_1) = \mathrm{e}^{x_1} \sin\{\tfrac{1}{2}\pi \cdot (1 - \mathrm{e}^{-x_1})\},$$

$$f_{\{2\}}(x_2) = 1 - \mathrm{e}^{-x_2} \cos[\tfrac{1}{2}\pi \cdot (1 - \mathrm{e}^{-x_1})]$$

and both of them have positive derivatives. However, letting $t_k = \log(2k + 1)$ we have $f(t_k, t_k) = 0$ for $k = 1, 2, \ldots$. Thus, f is not univalent and the CP associated with f has infinitely many solutions.

7. Concluding remarks

We have stated a sufficient condition for the GUS property of complementarity problems. This condition generalizes the other known sufficient conditions. We have also given other characterizations of the GUS property and, obviously, the sufficiency part of any characterization generalizes all other sufficient conditions.

The case of the plane seems to us clearer than the general n-dimensional one. Specifically, Theorem 5.1 tells us about the function f itself more than Theorems 5.2 and 5.3 do. We were able to prove a slight generalization of the necessity part of Theorem 5.1, for the n-dimensional case, but since it seemed rather unsatisfactory, we decided not to include that result in the present paper.

We conjecture that Theorems 5.2 and 5.3 can be somewhat unified in the following way. Assuming $f(0) = 0$, if condition (ii) of Theorem 5.2 or, equivalently, of Theorem 5.3 is replaced by:

(ii)* *There exists a homeomorphism G of $\mathbf{R}^n$ onto itself that coincides with f on*

[7] The CP associated with $f(\cdot) + q$ is feasible if there exists $z \geqq 0$ such that $f(z) + q \geqq 0$.

$\mathbf{R}^n_+$ *and with the identity function on* $\mathbf{R}^n$ *such that*

$$G(Q^S) \subset f(\mathbf{R}^S_+) - \mathbf{R}^{N\setminus S}_- \quad (S \subset N)$$

then, obviously, the condition is necessary. We conjecture that the modified condition (together with univalence of principal subfunctions) is also sufficient.

Positiveness of the Jacobians of all principal subfunctions (in the narrow sense; see Section 2) was shown to be insufficient for uniqueness while nonnegativity is necessary. Nevertheless, nonnegativity of the Jacobians of Moré and Rheinboldt's principal subfunctions (this is the broad sense; see [9, Def. 3.5]) was shown to be unnecessary for uniqueness, even though their positiveness is sufficient. It would be desirable to have a sufficient condition in terms of strict inequalities on the Jacobians, which will also be necessary if the strict inequality signs are replaced by nonstrict ones.

Finally, we suggest that the theory of nonlinear equations may be applicable to nonlinear complementarity theory in view of the observation given in Theorem 2.1, and the approximation used in the proofs of the Appendix.

Appendix

Theorem A.1. *Let* $f: G \to \mathbf{R}^n$ *be a continuous and univalent mapping, where* G *is an open subset of* $\mathbf{R}^n$. *Under these conditions* f *maps* G *homeomorphically.*

Proof. We use the invariance theorem of domain ([3, Lemma 3.9, p. 303]) which states as follows. *If* U *and* V *are homeomorphic subsets of* $\mathbf{R}^n$ *then* U *is open if* V *is open.* Let x be any point of G. Let $B = \{y \in \mathbf{R}^n: \|x-y\| \leqq \epsilon\}$ be a closed ball contained in G ($\epsilon > 0$). By compactness, f maps B homeomorphically. Using the invariance theorem of domain, we deduce that $f(\operatorname{int} B)$ is an open set. It follows that f^{-1} is continuous at $f(x)$. This completes the proof.

Proof (Uniqueness of solutions under Cottle's condition). (a) The extension F (see (2.2)) is differentiable everywhere except for points of the coordinates surfaces of $\mathbf{R}^n$. Therefore, Gale and Nikaido's theorem [4, 11] cannot be applied directly to F in order to deduce its univalence in $\mathbf{R}^n$. However, by slightly modifying their proof this can be proved. Essentially, one has to distinguish cases and deal with the different orthants separately. We omit the details.

(b) Another proof can be worked out by approximating F by differentiable functions $F_\alpha(x)$ and applying Gale and Nikaido's theorem to these functions. Specifically, for any $\alpha > 0$ and any real number ξ let

$$\gamma(\xi;\alpha) = \begin{cases} 0 & \text{if } \xi \leqq -\alpha, \\ (\xi+\alpha)^2/4\alpha & -\alpha < \xi \leqq \alpha, \\ \xi & \text{if } \alpha < \xi. \end{cases}$$

Denote $u_i(x;\alpha) = \gamma(x_i;\alpha)$, $i = 1, \ldots, n$, and $u = (u_1, \ldots, u_n)$ and define

$$F_\alpha(x) = f(u(x;\alpha)) - u(-x;\alpha). \tag{A.1}$$

It is easily verified that F_α is differentiable everywhere and its Jacobian matrix is everywhere a P-matrix. By Gale and Nikaido's theorem, F_α is univalent in $\mathbf{R}^n$. This implies that if x and y are interior points of orthants in $\mathbf{R}^n$ then $F(x) = F(y)$ implies $x = y$. The same implication can be proved for all pairs x, y in $\mathbf{R}^n$ by using induction on $|I(x)| + |I(y)|$ (see (5.1) and the proof of Theorem 5.2).

Another proof (of existence of solutions under Cottle's condition). Consider the approximations F_α of F (see (A.1)). It can be observed that if the principal minors of the Jacobian matrix of f lie between δ and δ^{-1} ($\delta > 0$), then so do the principal minors of the Jacobian matrices of the functions F_α. The result of Nikaido [12] can be applied to F_α and hence $F_\alpha(\mathbf{R}^n) = \mathbf{R}^n$. Thus, for any q in $\mathbf{R}^n$ there is a set $\{x^\alpha\}$ of points in $\mathbf{R}^n$ such that $F_\alpha(x^\alpha) = q$. If α is restricted to the interval $(0, 1]$ then the resulting set of x^α is bounded. This property can be shown to suffice to the existence of an x such that $F(x) = q$. Again, we omit the details.

References

[1] P.S. Aleksandrov, *Combinatorial topology* (Graylock Press, New York, 1960).

[2] R.W. Cottle, "Nonlinear programs with positively bounded jacobians", *SIAM Journal on Applied Mathematics* 14 (1966) 147–158.

[3] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology* (Princeton University Press, Princeton, N.J, 1952).

[4] D. Gale and H. Nikaido, "The jacobian matrix and global univalence of mappings", *Mathematische Annalen* 159 (1965) 81–93.

[5] S. Karamardian, "The nonlinear complementarity problem with applications I and II', *Journal of Optimization Theory and Applications* 4 (1969) 87–98 and 167–181.

[6] M. Kojima, "A unification of the existence theorems of the nonlinear complementarity problem", *Mathematical Programming* 9 (1975) 257–277.

[7] J.J. Moré, "Coercivity conditions in nonlinear complementarity problems", *SIAM Review* 16 (1974) 1–15.

[8] J.J. Moré, "Classes of functions and feasibility conditions in nonlinear complementarity problems", *Mathematical Programming* 6 (1974) 327–338.

[9] J.J. Moré and W. Rheinboldt, "On P- and S-functions and related classes of n-dimensional nonlinear mappings", *Linear Algebra and Its Applications* 6 (1973) 45–68.

[10] K.G. Murty, "On a characterization of P-matrices", *SIAM Journal on Applied Mathematics* 20 (1971) 378–383.

[11] H. Nikaido, *Convex structures and economic theory* (Academic Press, New York, 1968).

[12] H. Nikaido, "Relative shares and factor prices equalization", *Journal of International Economics* 2 (1972) 257–263.

[13] J.M. Ortega and W. Rheinboldt, *Iterative solution of nonlinear equations in several variables* (Academic Press, New York, 1970).

[14] H. Samelson, R.M. Thrall and O. Wesler, "A partition theorem for euclidean n-space", *Proceedings of the American Mathematical Society* 9 (1958) 805–907.

[15] A. Sard, "The measure of the critical values of differentiable maps", *Bulletin of the American Mathematical Society* 48 (1942) 883–890.