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SHORT COMMUNICATION

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Given $f:\mathbf{R}_+^n \to \mathbf{R}^n$, a *feasible solution* is an $x \in \mathbf{R}_+^n$ such that $f(x) \in \mathbf{R}_+^n$. A *complementary solution* is a feasible solution x such that $x^{\mathrm{T}} \cdot f(x) = 0$. The mapping f is *monotone* [4] if $(x-y)^{\mathrm{T}} \cdot [f(x) - f(y)] \geqq 0$ for all $x, y \in \mathbf{R}_+^n$. The following is known.

(i) If $f(x) = Ax + b$ is affine and monotone (in which case A is positive semi-definite), then the existence of a feasible solution implies the existence of a complementary solution [1].

(ii) If f is continuous and monotone and if there is an $x \in \mathbf{R}_+^n$ such that $f_i(x) > 0$ $(i = 1, \dots, n)$, then there is a complementary solution [5, Theorem 3.2].

(iii) If f is continuous and strictly monotone, i.e. for all $x \neq y \in \mathbf{R}_+^n$ $(x-y)^{\mathrm{T}} \cdot [f(x) - f(y)] > 0$, then there is a complementary solution if there is a feasible one (According to [2, Theorem 3] this can be reduced to (ii)).

Moré [5] claims that it is not known whether the existence of a feasible solution implies the existence of a complementary one under a general assumption of monotonicity. In view of the following example this is not true. Consider

$$f(x_1, x_2) = (2x_1x_2 - 2x_2 + 1,\ -x_1^2 + 2x_1 - 1)^{\mathrm{T}}.$$

It can be easily verified that the set of feasible solutions is $\{x\colon x_1 = 1, x_2 \geqq 0\}$. No feasible solution x is complementary since $f_1(x) = 1$. The Jacobian matrix of f is

$$\begin{bmatrix} 2x_2 & 2x_1 - 2 \\ 2 - 2x_1 & 0 \end{bmatrix}$$

and since it is positive semi-definite at each $x \in \mathbf{R}_+^2$, it follows that f is monotone (see [3, Theorem 3.1]).

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