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ON MONOTONICITY IN PARAMETRIC LINEAR COMPLEMENTARITY PROBLEMS

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This paper generalizes the answers that were given by R.W. Cottle to questions that were originally raised by G. Maier.

Essentially, we give necessary and sufficient conditions for some notions of monotonicity of solutions for the parametric linear complementarity problem. Our proofs are direct ones and not algorithmic, as Cottle's proofs are, and also cover a broader class of matrices.

1. Introduction

The linear complementarity problem, (LCP), which has already been investigated thoroughly (see [3, 4]), is stated as follows.

Problem 1.1 (LCP (M, q)). Given an $(n \times n)$-matrix M and an n-vector q, find an n-vector z such that

$$q + Mz \geq 0, \qquad z \geq 0, \qquad z^{\mathrm{T}}(q + Mz) = 0.$$

The parametric linear complementarity problem, (PLCP), formulated by Maier [5] and investigated by Murty [6] and Cottle [1], is the following.

Problem 1.2 (PLCP (M, q, p)). Given an $(n \times n)$-matrix M and n-vectors q, p, solve the family {LCP $(M, q + \alpha p)$: $a \geq 0$} of linear complementarity problems.

According to Maier [5], this problem is interesting in the context of elastoplastic structures. Monotonicity of solutions was claimed to be of special interest. Several results concerning monotonicity were obtained by Cottle [1].

Cottle developed a monotonicity-checking algorithm which requires q to be a non-negative vector and M either to be positive semi-definite, or else to have positive principal minors. This algorithm computes a particular parametric solution and decides whether this solution is monotone or not. Thus, in the case of non-unique solutions, a positive answer by that algorithm is not more than a sufficient condition for the existence of a monotone solution.

Using that algorithm, Cottle also proved the following.

Theorem 1.3 (Cottle [1, Th. 1]). *If M has positive principal minors and q is any n-vector, then the solution*[1] *of* PLCP (M, q, p) *is monotone for every n-vector p, if and only if*

$$\tilde{M}^{-1}\tilde{q} \geqslant 0$$

for every principal submatrix $\tilde{M}$ of M and corresponding subvector $\tilde{q}$ of q.

The property of (M, q) characterized in Theorem 1.3 is called *strong monotonicity*. A property of M, called *uniform monotonicity*, was characterized by Cottle in the following theorem.

Theorem 1.4 (Cottle [1, Th. 2]). *Let M be an $(n \times n)$-matrix whose principal minors are positive. Then the solution of* PLCP (M, q, p) *is monotone, for every p and every non-negative q, if and only if M is a Minkowski matrix (i.e., has positive principal minors and non-positive off-diagonal entries).*

Thus, Cottle's answers do not cover the positive semi-definite case.

It is the goal of the present paper to generalize Cottle's results to a broader class of matrices, revising the definition of monotonicity in the case of non-unique solutions.

2. A preliminary discussion

To begin, let $Z(\alpha) \equiv Z(\alpha; M, q, p)$ denote the set of all the solutions of LCP $(M, q + \alpha p)$, for $\alpha \geqslant 0$. Thus, Z is a point-to-set function (defined for every non-negative number α) whose values are subsets of $\mathbf{R}^n_+$, that may be empty. A definition of monotonicity of $Z(\alpha)$ is called for. One might like to call Z monotone if every point-to-point function z, such that $z(\alpha) \in Z(\alpha)$, is coordinate-wise monotone. Yet, such a definition is too restrictive and, unless Z is somewhat trivial (e.g., singleton-valued or empty-set-valued almost everywhere), there exist discontinuous functions z $(z(\alpha) \in Z(\alpha))$ which are not monotone.

It can be easily verified that the solution function Z is in our case lower-semi-continuous.[2] Accordingly, we call Z monotone if every *continuous* point-to-point function z (such that $z(\alpha) \in Z(\alpha)$), defined over a connected domain, is monotone. A necessary condition, which is related to non-degeneracy, for Z to satisfy that kind of monotonicity, can be derived by the following observation.

For every $(n \times n)$-matrix and $S \subset N = \{1, \ldots, n\}$, let A^S denote the submatrix of A, consisting of the columns corresponding to indices in S and let A^S_S denote the principal submatrix of A corresponding to the same set of indices. Similarly, let x^S denote the subvector of an n-vector x corresponding to the indices in S. It is easily verified that z is a solution of LCP (M, q) if and only if

[1] The existence and uniqueness of the solution was in fact proved by Samelson, Thrall and Wesler [7].

[2] $Z(\alpha)$ is *lower-semi-continuous* if $z_k \in Z(\alpha_k)$, $\alpha_k \to \alpha$ and $z_k \to z$ imply $z \in Z(\alpha)$.

$$M^S x^S - I^{\bar S} x^{\bar S} = -q,$$

where $S = \{i: z_i > 0\}$ and x is a non-negative n-vector such that $x^S = z^S$. A necessary condition for monotonicity is as follows.

Proposition 2.1. *Let $Z(\alpha) \equiv Z(\alpha; M, q, p)$ be the solution function of* PLCP *(M, q, p). Then, if Z is monotone, $Z(\alpha)$ is a finite set for every non-negative α, with the exception of at most a finite number of values of α.*

Proof. Assume that Z is monotone and suppose, on the contrary, that the condition is not satisfied. There is an infinite number of α's such that $Z(\alpha)$ is an infinite set. On the other hand, there is only a finite number of different S's. Thus, it follows that there exist $\alpha_2 > \alpha_1 \geq 0$, $S \subset N$, and n-vectors $y \geq x \geq 0$, $v \geq u \geq 0$ ($x \neq u$ and $y \neq v$) such that

$$\begin{aligned}
M^S x^S - I^{\bar S} x^{\bar S} &= -q - \alpha_1 p,\\
M^S y^S - I^{\bar S} y^{\bar S} &= -q - \alpha_2 p,\\
M^S u^S - I^{\bar S} u^{\bar S} &= -q - \alpha_1 p,\\
M^S v^S - I^{\bar S} v^{\bar S} &= -q - \alpha_2 p.
\end{aligned}$$

Without loss of generality, assume $x_1 < u_1$, $1 \in S$. If $0 < \epsilon \leq \alpha_2 - \alpha_1$ let

$$w(\alpha) = [(\alpha_2 - \alpha)x + (\alpha - \alpha_1)y]/(\alpha_2 - \alpha_1)$$

for α such that $\alpha_1 + \epsilon \leq \alpha \leq \alpha_2$ and let

$$\begin{aligned}
w(\alpha) = {} & [(\alpha - \alpha_1)/\epsilon] \cdot [(\alpha_2 - \alpha)x + (\alpha - \alpha_1)y]/(\alpha_2 - \alpha_1)\\
& + [(\epsilon - \alpha + \alpha_1)/\epsilon] \cdot [(\alpha_2 - \alpha)u + (\alpha - \alpha_1)v]/(\alpha_2 - \alpha_1)
\end{aligned}$$

for α such that $\alpha_1 \leq \alpha \leq \alpha_1 + \epsilon$.

Obviously,

$$M^S w^S(\alpha) - I^{\bar S} w^{\bar S}(\alpha) = -q - \alpha p \quad (\alpha_1 \leq \alpha \leq \alpha_2)$$

so that $z(\alpha) \equiv (w^S(\alpha), 0) \in Z(\alpha)$ for $\alpha_1 \leq \alpha \leq \alpha_2$. However, if ϵ is sufficiently small, then $z(\alpha)$ is not monotone, contradicting our assumption. This completes the proof of the present proposition.

Due to the necessary condition specified in Proposition 2.1, the graph of Z may be essentially viewed as a finite number of linear pieces. Monotonicity of Z in such a case is equivalent to the monotonicity of each linear piece. Henceforth, we shall be dealing with the latter property.

3. On the monotonicity of $Z(\alpha)$

Given a set P of n-vectors, we denote by lin P the linear space spanned by P. Also, let pos P and pos$^+$ P denote, respectively, the set of all non-negative linear combinations and the set of all positive linear combinations of all the vectors in P. We take the liberty of using the above notation also when P is a set of matrices (all having the same column length). In that case P is identified with the set of the column vectors of the matrices in P.

Definition 3.1. Let M, q, and p be given. We call a set $S \subset \{1, \dots, n\}$ *relevant for* (M, q, p) if the ray $\{-q - \alpha p: \alpha \geq 0\}$ meets the cone $\operatorname{pos}\{M^S, -I^{\bar{S}}\}$ at more than one point.

Lemma 3.2. *A set S is relevant for (M, q, p) if and only if there exists a set K of columns of $(M^S, -I^{\bar{S}})$ such that $-q \in \operatorname{pos}^+(K \cup \{p\})$ and $p \in \operatorname{lin} K$.*

Proof. (a) If S is relevant for (M, q, p), then there exist $\beta > \alpha \geq 0$ and non-negative n-vectors x, y such that

$$M^S x^S - I^{\bar{S}} x^{\bar{S}} = -q - \alpha p, \tag{3.1}$$

$$M^S y^S - I^{\bar{S}} y^{\bar{S}} = -q - \beta p. \tag{3.2}$$

If $(z, \gamma) = \frac{1}{2}(x, \alpha) + \frac{1}{2}(y, \beta)$, then z is non-negative, γ is positive and $M^S z^S - I^{\bar{S}} z^{\bar{S}} = -q - \gamma p$. Let K be the set of those columns, either of M^S or of $-I^{\bar{S}}$, corresponding to coordinates i such that $z_i > 0$. Obviously, $-q - \gamma p \in \operatorname{pos}^+ K$ and therefore $-q \in \operatorname{pos}^+(K \cup \{p\})$. If $z_i = 0$, then $x_i = y_i = 0$. That implies, by substracting equality (3.2) from equality (3.1), that $-p \in \operatorname{lin} K$.

(b) Assume that K is a set satisfying the conditions specified in the present lemma. It follows that there exists a positive number γ and a non-negative n-vector z such that $-q - \gamma p = M^S z^S - I^{\bar{S}} z^{\bar{S}}$, and $z_i = 0$ for every i whose corresponding column of $(M^S, -I^{\bar{S}})$ is not in K. Also, since $p \in \operatorname{lin} K$, there exists an $\epsilon > 0$ such that $-q - (\gamma \pm \epsilon) p \in \operatorname{pos}\{M^S, -I^{\bar{S}}\}$. This, of course, means that S is relevant for (M, q, p).

Definition 3.3. Given M, q, p, and S, we call p a *monotone direction* if there exists a vector-function $x(\alpha)$ such that

$$M^S x^S(\alpha) - I^{\bar{S}} x^{\bar{S}}(\alpha) = -q - \alpha p$$

and $x_i(\alpha)$ is a monotone non-decreasing function of α for all $i \in S$.

Lemma 3.4. *Let M, q, p, and S be such that S is relevant for (M, q, p). Under this condition, p is a monotone direction if and only if $-p^S \in \operatorname{pos} M_S^S$.*

Proof. (a) Suppose that p is a monotone direction and let $x(\alpha)$ be the function assured by Definition 3.3. If $\beta > \alpha \geq 0$, then

$$M^S x^S(\beta) - I^{\bar{S}} x^{\bar{S}}(\beta) = -q - \beta p,$$
$$M^S x^S(\alpha) - I^{\bar{S}} x^{\bar{S}}(\alpha) = -q - \alpha p.$$

This implies

$$-p = [M^S(x^S(\beta) - x^S(\alpha)) - I^{\bar{S}}(x^{\bar{S}}(\beta) - x^{\bar{S}}(\alpha))]/(\beta - \alpha). \tag{3.3}$$

Since $x^S(\beta) \geq x^S(\alpha)$, a restriction to the coordinates in S yields $-p^S \in \operatorname{pos} M_S^S$.

(b) Suppose that $-p^S \in \operatorname{pos} M^S_S$. There exists a non-negative n-vector a such that $-p = M^S a^S - I^{\bar{S}} a^{\bar{S}}$. Since S is relevant for (M, q, p), it follows from Lemma 3.2 that, in particular, $q \in \operatorname{lin}\{M^S, -I^{\bar{S}}\}$. Thus, there exists an n-vector b such that $-q = M^S b^S - I^{\bar{S}} b^{\bar{S}}$. Define $x(\alpha) = \alpha a + b$ and it follows that $x^S(\alpha)$ is monotone and $-q - \alpha p = M^S x^S(\alpha) - I^{\bar{S}} x^{\bar{S}}(\alpha)$. This proves the present lemma.

In view of the discussion in Section 2, we introduce the following regularity condition.

Definition 3.5. A parametric linear complementary problem PLCP (M, q, p) is called *regular* when every principal submatrix M^S_S of M, such that S is relevant, is nonsingular.

Theorem 3.6 (See Lemma 3.2). *Let* PLCP (M, q, p) *be a regular* PLCP. *Then the solution function* $Z(\alpha) \equiv Z(\alpha; M, q, p)$ *is monotone if and only if*

$$(M^S_S)^{-1} p^S \leqslant 0$$

for every S *which is relevant for* (M, q, p).

Proof. Under the regularity assumption, the solution function $Z(\alpha)$ is discrete-set-valued. For every S, which is relevant for (M, q, p), and $\alpha \geqslant 0$, there exists at most one non-negative n-vector x such that $M^S x^S - I^{\bar{S}} x^{\bar{S}} = -q - \alpha p$. Thus, monotonicity of Z is equivalent to p's being a monotone direction with respect to every relevant set S. In view of Lemma 3.4, this is equivalent to $-p \in \operatorname{pos} M^S_S$ or, using the regularity assumption again, $(M^S_S)^{-1} p^S \leqslant 0$. This proves our theorem.

Admittedly, the characterization of the relevance property and the resulting characterization of monotonicity of $Z(\alpha)$ involve a number of checkings that grows rapidly with the dimension. It is hoped, however, that the results of this section do provide some insight into the monotonicity problem and that some simpler sufficient (but not necessary) conditions for monotonicity can be easily derived. In any case these results will be used in the next section for characterizing strong and uniform monotonicity.

4. On the strong and uniform monotonicity

In this section we shall be mainly dealing with the strong monotonicity property of a pair (M, q), i.e., the property that for every p the problem PLCP (M, q, p) has a monotone solution function $Z(\alpha)$.

Definition 4.1. Let M and q be given. We call a set $S \subset \{1, \ldots, n\}$ *relevant for* (M, q) if there is an n-vector p such that S is relevant for (M, q, p) in the sense of Definition 3.1.

Lemma 4.2. *A set S is relevant for (M, q) if and only if $q^S \in \operatorname{lin} M_S^S$.*

Proof. By Lemma 3.2, it follows that S is relevant for (M, q) if and only if there is p and $K \subset (M^S, -I^{\bar{S}})$ such that $-q \in \operatorname{pos}^+(K \cup \{p\})$ and $-p \in \operatorname{lin} K$. This condition is obviously equivalent to the existence of a $K \subset (M^S, -I^{\bar{S}})$ such that $q \in \operatorname{lin} K$. The latter condition simply states that $q^S \in \operatorname{lin} M_S^S$.

As in the preceding section, we need a regularity condition (see also the discussion in Section 2).

Definition 4.3. A pair (M, q) is called *regular* if for every S which is relevant for (M, q) the principal submatrix M_S^S is non-singular.

Theorem 4.4. *Let (M, q) be a regular pair. Under this condition, the solution function $Z(\alpha) \equiv Z(\alpha; M, q, p)$ is monotone for every p if and only if*

$$(M_S^S)^{-1} q^S \geqslant 0$$

for every S such that $q^S \in \operatorname{lin} M_S^S$.

Proof. The pair (M, q) has the strong monotonicity property if and only if for every n-vector p and S which is relevant for (M, q, p), the direction p is monotone w.r.t. M, q, S. In the other words, that property is equivalent to the truth of the following implication. For all p, S, and $K \subset (M^S, -I^{\bar{S}})$, if $-q \in \operatorname{pos}^+(K \cup \{p\})$ and $-p \in \operatorname{lin} K$, then $-p^S \in \operatorname{pos} M_S^S$. Since $-q \in \operatorname{pos}^+(K \cup \{p\})$ is equivalent to $-p \in \operatorname{pos}^+(K \cup \{q\})$ and, also, $-p^S \in \operatorname{pos} M_S^S$ is equivalent to $-p \in \operatorname{pos}\{M^S, -I^{\bar{S}}, I^{\bar{S}}\}$, it follows that the strong monotonicity property is equivalent to the truth of the relation

$$\operatorname{lin} K \cap \operatorname{pos}^+(\mathrm{K} \cup \{\mathrm{q}\}) \subset \operatorname{pos}\{M^S, -I^{\bar{S}}, I^{\bar{S}}\} \tag{4.1}$$

for all $S \subset \{1, \ldots, n\}$ and $K \subset (M^S, -I^{\bar{S}})$. But if $q \notin \operatorname{lin} K$, then the left-hand-side of (4.1) is empty, and also if $q \in \operatorname{lin} K$, then that intersection coincides with $\operatorname{pos}^+(K \cup \{q\})$. Thus, the condition as a whole is equivalent to

$$\operatorname{pos}^+(K \cup \{q\}) \subset \operatorname{pos}\{M^S, -I^{\bar{S}}, I^{\bar{S}}\}$$

for all K such that $K \subset (M^S, -I^{\bar{S}})$ and $q \in \operatorname{lin} K$. This is equivalent to

$$q \in \operatorname{pos}\{M^S, -I^{\bar{S}}, I^{\bar{S}}\}$$

for all S such that $q \in \operatorname{lin}\{M^S, -I^{\bar{S}}\}$. Finally, using the regularity condition for strong monotonicity, this property is equivalent to

$$(M_S^S)^{-1} q^S \geqslant 0$$

for all S such that $q^S \in \operatorname{lin} M_S^S$. This completes our proof.

Remark 4.5. Using Theorem 4.4, a characterization of the uniform monotonicity property of a matrix M (i.e., monotone solution $Z(\alpha) \equiv Z(\alpha; M, q, p)$ for every p and every non-negative q) is straightforward. The regularity condition on M in this case insists that every principal submatrix of M is non-singular. Under this condition, uniform monotonicity is equivalent to

$$(M_S^S)^{-1}q^S \geqslant 0$$

for all $S \subset \{1, \ldots, n\}$ and every non-negative n-vector q. This is equivalent to M's being a Minkowski matrix [2, p. 247]. Notice that during the derivation of this characterization we were not assuming that M has positive principal minors, as in fact was assumed by Cottle. Thus, in particular, the positive semi-definite case is settled.

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