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# ON THE PARAMETRIC NONLINEAR COMPLEMENTARITY PROBLEM*

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A parametrized version of the nonlinear complementarity problem is formulated. The existence of a continuation of a solution is investigated and sufficient and necessary conditions for the monotonicity of such a continuation are given. The notions of strong and uniform monotonicity, originated in the linear theory, are discussed, and the theorems of the linear theory are generalized.

*Key words*: Nonlinear Complementarity, Parametric Solution, Strong Monotonicity, Uniform Monotonicity, Generalized Differentiability, Strong Positive-definite.

## 1. Introduction

The nonlinear complementarity problem (CP) is well-known. It can be stated as follows.

**Problem 1.1** (CP($f$)). *Given a continuous mapping* $f : \mathbf{R}^n_+ \to \mathbf{R}^n$, *find an* $n$*-vector* $z$ *such that*

$$f(z) \geqq 0, \qquad z \geqq 0, \qquad z^{\mathrm{T}} \cdot f(z) = 0.$$

A parametric version of linear CP-s (i.e., when $f$ is affine) was formulated by Maier [5]. The problem of monotonicity of solutions in the parametric linear CP was dealt with by Cottle [3] who assumed the matrix $M$ of the parametrized mapping $f(z; t) = Mz + q + tp$, either to be positive semi-definite, or else to have positive principal minors. Also, the vector $q$ was assumed to be non-negative. The results of Cottle have been recently generalized by Megiddo [6].

In this paper we shall be dealing with a parametric version of the nonlinear CP. We state the general form of this parametric complementarity problem (PCP) as follows.

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**Problem 1.2** (PCP($g$)). *Given a continuous mapping* $g : \mathbf{R}_+^{n+1} \to \mathbf{R}^n$, *solve the family* $\{g(\cdot\,; t) \colon t \geqq 0\}$ *of non-parametric* CP-*s*.

Maier [5] and Cottle [3] claim that the parametric linear CP is applicable in the context of elastoplastic structures. Cottle also suggested that a generalization of his results "would find applications in structural mechanics as well as economic equilibrium theory [2]".

The first problem explored in this paper is the following. Suppose that a PCP is solvable for some value $t^0$ of the parameter. Under what conditions is this solution extendable into a parametric solution over a neighbourhood of $t^0$, and when is such a solution monotone with respect to the parameter? We answer these questions in Section 3.

The notions of strong and uniform monotonicity were defined (in the context of parametric linear CP-s) by Cottle [3] and investigated by Cottle and by the author [6]. Cottle restricted his discussion to parametrized functions $g(z; t) = Mz + q + tp$ where $M$ is an $(n \times n)$-matrix whose principal minors are positive, $q$ is a non-negative $n$-vector, and $p$ is any $n$-vector. Under these conditions, strong monotonicity, which means monotone solution for every $p$, is characterized by

$$\tilde{M}^{-1}\tilde{q} \geqq 0,$$

for every principal submatrix $\tilde{M}$ of $M$ and corresponding subvector $\tilde{q}$ of $q$. Uniform monotonicity, which means monotone solution for every $p$ and every non-negative $q$, is characterized by $M$'s being a Minkowski matrix.

In our general nonlinear case, the roles of $q$ and $p$ are played by $g(0; 0)$ and $\partial g/\partial t$, respectively. Thus, one might like to define strong monotonicity as monotonicity which is independent of $\partial g/\partial t$. However, strong monotonicity should be treated as a global rather than local property because of the following fact. If $g(0; 0) > 0$, then the origin is a solution for $t^0 = 0$ and for $t$ in some neighbourhood of $t^0$. Thus, the strong monotonicity is "locally" satisfied. On the other hand, if $g(0; 0) \ngeqq 0$, then there exist values for $\partial g/\partial t$ for which a solution will not be monotone; this is the reason why $q$ was assumed to be non-negative in the linear case (Cottle [3]). Thus, it is interesting to investigate only the following problem: Suppose that $g(0; 0) \geqq 0$ and $z(t)$, $\underline{t} \leqq t \leqq \bar{t}$, is a parametric solution of PCP($g$). Under what conditions, independently of $\partial g/\partial t$, is this solution monotone? If we do not impose further conditions on $g$ then usually the strong monotonicity will not hold. This is due to the following fact. Given a function $g$ and a monotone solution $z(t)$, $0 \leqq t \leqq 1$, of PCP($g$), define $\theta(t) = \min(2t, 2 - 2t)$, and $g^*(z; t) = g(z; \theta(t))$, and the resulting solution $z^*(t) = z(\theta(t))$ of PCP($g^*$) is not monotone, unless $z(t)$ is constant.

The above discussion suggests that we restrict attention to linear dependency of $g$ on $t$. In other words, we shall assume in Section 4 that PCP($g$) is linearly parametrized, i.e.,

$$g(z; t) = f(z) + tp,$$

where $f : \mathbf{R}_+^n \to \mathbf{R}^n$ and $p$ is an $n$-vector. We also assume that $f$ is differentiable.

In order to avoid conceptual complications, resulting from non-uniqueness of solutions[1], we shall assume that all the principal minors of the Jacobian matrix of $f$ are positive everywhere. This suffices for uniqueness (see [8]) and also generalizes Cottle's assumptions in the linear case. Since this assumption does not suffice for existence, our sufficient conditions stated below imply global monotonicity in the following broad sense. A function defined on a subset $D$ of the real line is said to be *monotone* if it is monotone non-decreasing coordinatewise (in the usual sense) on every connected component of $D$.

In Section 4 we provide necessary and sufficient conditions for the properties of strong and uniform monotonicity; these generalize Cottle's theorems [3, Theorems 1, 2].

## 2. Preliminaries

Let $N = \{1, \dots, n\}$. For every subset $S$ of $N$, let $x^S$ denote the restriction[2] of an $n$-vector $x$ to the coordinates in $S$. The orthants of $\mathbf{R}^n$ will be denoted by $Q^S$ ($S \subset N$) where

$$Q^S = \{x \in \mathbf{R}^n: \quad x^S \geqq 0, x^{N \setminus S} \leqq 0\}.$$

We find it useful to adopt the following notation. If $f$ is a mapping of a certain CP, extend $f$ to the whole $\mathbf{R}^n$ by defining

$$F(x) = f(x^+) + x^-, \tag{2.1}$$

where $x^+, x^- \in \mathbf{R}^n$, $x_i^+ = x_i$ and $x_i^- = 0$ if $x_i \geqq 0$, and $x_i^+ = 0$ and $x_i^- = x_i$ if $x_i \leqq 0$. Similarly, if $g$ is a mapping of a certain PCP, extend $g$ to $\mathbf{R}^n \times \mathbf{R}^1_+$ by defining

$$G(x; t) = g(x^+; t) + x^-. \tag{2.2}$$

Using the notation above, we observe that the problem CP($f$) is solvable if and only if $0 \in F(\mathbf{R}^n)$. Moreover, if $F(x) = 0$ then $z = x^+$ is a solution to CP($f$) and conversely, if $z$ is a solution to CP($f$) then $F(x) = 0$ where $x_i = z_i$ if $z_i > 0$ and $x_i = -f_i(z)$ if $z_i = 0$. Analogous observations on the PCP can be made using the extension $G$ of $g$. The extension $F$ of $f$ has been thoroughly used in Megiddo and Kojima [7].

Throughout this paper, if $A$ is an $[n \times (n+m)]$-matrix ($m \geqq 0$), then for $S \subset \{1, \dots, n\}$, $A^S$ denotes the principal submatrix of $A$ corresponding to the set $S$.

## 3. Existence and monotonicity of solutions

In this section we deal with the existence and monotonicity of a parametric solution in a neighbourhood of some value $t^0$ of the parameter.

[1] See [6] for a treatment of the linear case without uniqueness assumptions.

[2] The null vector $x^\emptyset$ is assumed to be equivalent both to the empty set and to the real number zero.

First, we introduce our terminology and notation. A mapping $H: U \to \mathbf{R}^n$, where $U \subset \mathbf{R}^n \times \mathbf{R}^m_+$, is said to be *$Q^S$-differentiable* if for every $(y; t) \in (Q^S \times \mathbf{R}^m_+) \cap U$ there exists an $[n \times (n+m)]$-matrix $DH_S = DH_S(y; t)$ such that for every $\Delta y, \Delta t$, if $(y + \Delta y; t + \Delta t) \in (Q^S \times \mathbf{R}^m_+) \cap U$ then

$$H(y + \Delta y; t + \Delta t) = H(y; t) + DH_S \cdot (\Delta y; \Delta t) + o(\Delta y; \Delta t).$$

$H$ is said to be continuously $Q^S$-differentiable if $DH_S(y; t)$ is a continuous function of $(y; t)$. It can be observed that if $y$ lies in the interior of $Q^S$ then $Q^S$-differentiability in the neighbourhood of $(y; t)$ coincides with the standard differentiability. Also, if $H$ is $Q^S$-differentiable then $DH_S$ is the matrix of one-sided partial derivatives of $H$, where the "sides" are chosen so as to stay in $Q^S$. In view of this we denote by $D_yH_S$ the submatrix of $DH_S$ corresponding to the vector variable $y$. The following lemma is a generalization of the classical implicit function theorem [1].

**Lemma 3.1.** *Let $(y^0; t^0) = (y^0_1, \ldots, y^0_n; t^0_1, \ldots, t^0_m)$. Let $U$ be a neighbourhood of $(y^0; t^0)$ and let $H : U \to \mathbf{R}^n$ satisfy the following conditions.*

(i) $H(y^0; t^0) = 0$.

(ii) *$H$ is continuously $Q^S$-differentiable at any $(y; t) \in (Q^S \times \mathbf{R}^m_+) \cap U$.*

(iii) *For every $S$ such that $y^0 \in Q^S$, $\det[D_yH_S(y^0; t^0)] > 0$.*

*Under these conditions there exists a neighbourhood $T$ of $t^0$ and a unique function $y(t)$ defined in $T$ such that $y$ is continuous, $y(t^0) = y^0$, and $H[y(t); t] = 0$ for all $t \in T$.*

**Proof.** We prove the conclusions by induction on $n$.

(a) If $n = 1$ then it follows from our assumptions that there exists a neighbourhood $T$ of $t^0$ such that for every $t \in T$ the function $H(y; t)$ is strictly monotone increasing (in the variable $y$). This, together with continuity, implies the existence of a unique function $y(t)$ defined in $T$ such that $y(t^0) = y^0$ and $H[y(t); t] = 0$. The continuity of $y(t)$ follows as in the proof of the classical implicit function theorem.

(b) Suppose $n > 1$ and assume that all the conclusions hold for all cases of smaller values of $n$. If $y^0_i \neq 0$ for all $i = 1, \ldots, n$, then all the conditions of the classical implicit function theorem are satisfied in some neighbourhood of $(y^0; t^0)$ and hence all the conclusions hold. Thus, without loss of generality, assume that $y^0_n = 0$. Consider the first $n - 1$ equations

$$H_i(y_1, \ldots, y_{n-1}, y_n; t_1, \ldots, t_m) = 0 \quad (i = 1, \ldots, n-1).$$

It can be easily verified that this system (with $y_n$ considered as one of the independent variables) satisfies the assumptions of our lemma. Thus, the induction hypothesis can be applied. It follows that there exists a neighbourhood $V$ of $(y^0_n, t)$ and a unique continuous function $\phi(y_n, t) = (\phi_1(y_n, t), \ldots, \phi_{n-1}(y_n, t))$ such that $\phi(y^0_n, t^0) = (y^0_1, \ldots, y^0_{n-1})$, $(\phi(y_n, t), y_n; t) \in U$, and $H[\phi(y_n, t), y_n; t] = 0$ for all $(y_n, t) \in V$. Consider the following function.

$$h(y_n; t) = H_n[\phi(y_n, t), y_n; t].$$

We shall prove that there exists a neighbourhood $T$ of $t^0$ such that $h(y_n; t)$ is strictly monotone w.r.t. $y_n$ for every $t \in T$. To that end it suffices to show that

$$\liminf_{\Delta y \to 0} (1/\Delta y) \cdot [h(y_n + \Delta y; t) - h(y_n; t)] > 0, \tag{3.1}$$

for every $(y_n, t)$ in some neighbourhood of $(y_n^0, t^0)$. A similar argument appeared in Megiddo and Kojima [7, Th. 3.3]. Using essentially the same proof we can deduce that

$$\liminf_{\Delta y \to 0} (1/\Delta y) \cdot [h(y_n^0 + \Delta y; t^0) - h(y^0; t^0)] > 0.$$

Since $h$ is continuous w.r.t. both $t$ and $y_n$, the same can be proved for points in a neighbourhood of $(y_n^0, t^0)$, hence (3.1) is true. The remainder of the proof is carried out as in the case of the classical implicit function theorem.

Applying Lemma 3.1 to our PCP, we obtain

**Theorem 3.2.** *Let $g : \mathbf{R}_+^{n+1} \to \mathbf{R}^n$ be a continuous mapping and assume the following conditions.*

(i) *The problem* PCP($g$) *is solvable for the value $t^0$ of the parameter and $z^0$ is a solution of the resulting* CP$[g(\cdot; t^0)]$.

(ii) *The mapping $g$ is continuously $\mathbf{R}_+^{n+1}$-differentiable in some neighbourhood $U$ (w.r.t. $\mathbf{R}_+^{n+1}$) of $(z^0, t^0)$,*

(iii) *For every $S$, such that $\{i: z_i^0 > 0\} \subset S \subset \{i: g_i(z^0; t^0) = 0\}$, the corresponding principal minor of the Jacobian matrix of $g$ is positive, i.e.,* $\det(D_z g_N(z^0; t^0)) > 0$. *Under these conditions there exists a neighbourhood (w.r.t. $\mathbf{R}_+^1$), $T$ of $t^0$ and a unique continuous solution $z(t)$ of* PCP($g$) *over $T$ such that $z(t^0) = z^0$.*

**Proof.** Consider the extension $G$ of $g$ (see (2.2)). It is easy to verify that $G$ is continuously $Q^S$-differentiable (for every $S$) and

$$(D_x G_S(x; t))_{ij} = \begin{cases} \partial g_i(x; t)/\partial x_j, & \text{if } j \in S, \\ 1, & \text{if } i = j \notin S, \\ 0, & \text{if } i \neq j \notin S, \end{cases}$$

Moreover, Lemma 3.1 applied to the mapping $G$, implies the conclusion of the present theorem.

**Remark 3.3.** If condition (iii) of Theorem 3.2 is replaced by negativity of the same principal minors, then the same conclusion holds, since the same can also be done in Lemma 3.1.

The problem of monotonicity can be dealt with by using a derivation analogous to what is done in the implicit function theorem.

**Theorem 3.4.** *Let all the conditions of Theorem 3.2 be satisfied. Denote $\underline{S} = \{i: z_i^0 > 0\}$ and $\bar{S} = \{i: g_i(z^0; t^0) = 0\}$.*

(i) *If for every $S$, $\underline{S} \subset S \subset \bar{S}$,*

$$[(D_z g_N(z^0; t^0))^S]^{-1} \cdot (\partial g(z^0; t^0)/\partial t)^S < 0,$$

*then the parametric solution $z(t)$, assured by Theorem 3.2, is monotone non-decreasing (coordinate-wise) in some neighbourhood of $t^0$.*

(ii) *If the parametric solution $z(t)$, assured by Theorem 3.2, is monotone non-decreasing in some neighbourhood of $t^0$* then the set $S^* \equiv \underline{S} \cup \{i\colon z_i(t)$ is increasing in the neighbourhood of $t^0\}$ *satisfies*

$$[(D_z g_N(z^0; t^0))^{S^*}]^{-1} \cdot (\partial g(z^0; t^0)/\partial t)^{S^*} \leqq 0$$

*and if $t^0$ is positive, also*

$$[(D_z g_N(z^0; t^0))^{\underline{S}}]^{-1} \cdot (\partial g(z^0; t^0)/\partial t)^{\underline{S}} \leqq 0.$$

**Proof.** Let $T$ be the neighbourhood of $t^0$, assured by Theorem 3.2. For $t \in T$ define $x_i(t) = z_i(t)$ if $z_i(t) > 0$ and $x_i(t) = -g_i(z(t), t)$ if $z_i(t) = 0$. Let $x^0 = x(t^0)$. Obviously, $x = x(t)$ is the solution function of the equation $G(x; t) = 0$ (see (2.2)) in the neighbourhood of $(x^0; t^0)$.

(i) If $i \notin \bar{S}$ then $g_i(z(t); t) > 0$ for $t$ in some neighbourhood $T_1$ of $t^0$ $(T_1 \subset T)$. Hence, for such $t$, $z_i(t) = 0$. We shall prove that for $i \in \underline{S}$

$$\liminf_{\Delta t \to 0} (1/\Delta t) \cdot [z_i(t^0 + \Delta t) - z_i(t^0)] > 0. \tag{3.2}$$

Let $\{h_k\}$ be a sequence of non-zero real numbers which converges to zero. Then there is an index set $S$ and a subsequence of $\{h_k\}$, which we shall still call $\{h_k\}$, such that for all $k$, $z_i(t^0 + h_k) > 0$ if $i \in S$ and $z_i(t^0 + h_k) = 0$ if $i \notin S$. It follows that $x(t^0 + h_k) \in Q^S$ and that $\underline{S} \subset S \subset \bar{S}$. By the $Q^S$-differentiability, the limit

$$d_t x_S \equiv \lim_{k \to \infty} (1/h_k) \cdot [x(t^0 + h_k) - x(t^0)]$$

exists. Moreover, it can be observed (see [7, Theorem 3.3]) that this limit is independent of the particular sequence $\{h_k\}$ and

$$d_t x_s = -(D_x G_S)^{-1} \cdot \partial G / \partial t.$$

Restricting this vector equality to the coordinates in $S$, and returning to the original function $g$, we obtain

$$(d_t z_S)^S = -[(D_z g_N)^S]^{-1} \cdot (\partial g / \partial t)^S > 0. \tag{3.3}$$

Thus, (3.2) holds. This proves (3.2).

Suppose that $i \in \bar{S} \setminus \underline{S}$. We claim that there exists a neighbourhood $T_2 \subset T_1$ of $t^0$ such that for $t < t^0$ in $T_2$, $z_i(t) = 0$; otherwise, there is a sequence $\{h_k\}_{k=1}^{\infty}$ $(h_k < 0,\ \lim_{k\to\infty} h_k = 0)$ such that $z_i(t^0 + h_k) > 0$ and $x(t^0 + h_k) \in Q^S$ (where $\underline{S} \subset S \subset \bar{S}$ and $i \in S$), and this contradicts (3.3). If $i$ is such that $z_i(t) = 0$ for $t$ in some neighbourhood of $t^0$, then $z_i(t)$ is obviously monotone in such a neighbourhood. However, for every $t > t^0$ in some neighbourhood $T_3 \subset T_2$ of $t^0$, if $z_i(t) > 0$, then

$$\liminf_{\Delta t \to 0} (1/\Delta t) \cdot [z_i(t + \Delta t) - z_i(t)] > 0$$

(since this is analogous to (3.2)[3]. (The latter is also true for $i \in \underline{S}$.) This implies that $z_i(t)$ is monotone in some neighbourhood of $t^0$ (for all $i$).

(ii) Suppose that the parametric solution $z(t)$ is monotone non-decreasing. If $\{h_k\}_{k=1}^{\infty}$ is a sequence of positive numbers such that $\lim_{k\to\infty} h_k = 0$, then for all $k$ sufficiently large, $z_i(t^0 + h_k) > 0$ if $i \in S^*$, and if $i \notin S^*$ then $z_i(t^0 + h_k) = 0$. Thus, the corresponding function $x(t)$ satisfies $x(t^0 + h_k) \in Q^{S^*}$ for $k$ sufficiently large. Using the same reasoning as that of part (i), we deduce that $d_t x_{S^*}$ is well-defined and the monotonicity implies

$$(d_t z_{S^*})^{S^*} = -[(D_z g_N)^{S^*}]^{-1} \cdot (\partial g/\partial t)^{S^*} \geqq 0.$$

Similarly, assuming $t^0$ to be positive, if $\{h_k\}_{k=1}^{\infty}$ is an appropriate sequence of negative numbers, then for $k$ sufficiently large $x(t^0 + h_k) \in Q^{\underline{S}}$ (if $i \in \underline{S}$ then $x_i(t^0 + h_k) = z_i(t^0 + h_k) > 0$; if $i \notin \underline{S}$ then $z_i(t^0 + h_k) \leqq z_i(t^0) = 0$ and hence $x_i(t^0 + h_k) \leqq 0$). By similar arguments,

$$(d_t z_{\underline{S}})^{\underline{S}} = -[(D_z g_N)^{\underline{S}}]^{-1} \cdot (\partial g/\partial t)^{\underline{S}} \geqq 0.$$

This completes the proof of the present theorem.

The following theorem contains a variant of part (i) of Theorem 3.4. We present this variant because of its applicability to Section 4.

**Theorem 3.5.** *Let all the conditions of Theorem 3.2 be satisfied. For every $t$ such that $z(t)$ is defined, denote*

$$\underline{S}(t) = \{i: z_i(t) > 0\} \quad \textit{and} \quad \bar{S}(t) = \{i: g_i(z(t); t) = 0\}.$$

*If for every $t$ in some neighbourhood of $t^0$, and $S$, $\underline{S}(t) \subset S \subset \bar{S}(t)$,*

$$[(D_z g_N(z(t); t))^S]^{-1} \cdot (\partial g(z(t); t)/\partial t)^S < 0$$

*then $z(t)$ is monotone non-decreasing in some neighbourhood of $t^0$.*

**Proof.** The proof is essentially the same as that of Theorem 3.4 (part (i)) and is therefore omitted. It is based on the following fact. If $\xi(t)$ is a continuous real function such that

$$\liminf_{\Delta t\to 0} (1/\Delta t) \cdot [\xi(t + \Delta t) - \xi(t)] \geqq 0$$

for every $t$ in some (open and connected) neighbourhood of $t^0$, then $\xi(t)$ is monotone non-decreasing in that neighbourhood of $t^0$. This follows from the fact that $\hat{\xi}(t) = \xi(t) + \epsilon t$ (where $\epsilon > 0$) is obviously monotone; thus, if $t > t'$, then for every $\epsilon > 0$ $\xi(t) - \xi(t') \geqq \epsilon(t' - t)$ so that $\xi(t) \geqq \xi(t')$.

## 4. On the properties of strong and uniform monotonicity

The assumptions of this section are explained and justified in Section 1. The

[3]Note that $\underline{S} \subset \{i: z_i(t) > 0\} \subset \{i: g_i(z(t); t) = 0\} \subset \bar{S}$.

first theorem generalizes Cottle's theorem on strong monotonicity in parametric linear CP-s ([3, Theorem 1]). To simplify notation, let $J_S(z^0)$ denote the principal submatrix of the Jacobian matrix of $f$ (evaluated at $z^0$) corresponding to $S$.

**Theorem 4.1.** *Let $f:\mathbf{R}^n_+\to\mathbf{R}^n$ be continuously differentiable and such that $f(0)=0$. Assume that the principal minors of the Jacobian matrix of $f$ are positive everywhere in $\mathbf{R}^n_+$. Also, let $q$ be any non-negative $n$-vector.*

*Under these conditions, the solution function $z(t)\equiv z(t;q,p)$ of the problem* PCP$[f(z)+q+tp]$ *is monotone for every $p$ if and only if for every $z^0\geqq 0$ and $S$, $S\supset \underline{S}\equiv\{i:\ z_i^0>0\}$,*

$$J_S^{-1}(z^0)\cdot[q+f(z^0)]^S\geqq 0. \tag{4.1}$$

**Proof.** (a) We shall prove the sufficiency part. Given a point $z^0\geqq 0$, a direction $p\neq 0$ is relevant if there exists a positive $t^0$ such that $z^0$ solves the problem CP$[f(\cdot)+q+t^0p]$; the case $t^0=0$ is trivial since it implies $z^0=0$. Consider the problem PCP$[f(z)+q+tp]$. A relevant $p$ satisfies, together with the corresponding $t^0$,

$$\bar S\equiv\{i:\ f_i(z^0)+q_i+t^0p_i=0\}\supset\underline{S}.$$

Obviously, $t^0p^{\bar S}=-[q+f(z^0)]^{\bar S}$. Moreover, there exists a neighbourhood of $t^0$ in which $z(t)$ is defined (see Theorem 3.2) and for every $t$ in this neighbourhood

$$tp^{\bar S(t)}=-[q+f(z(t))]^{\bar S(t)}$$

(see Theorem 3.5 for the definition of $\bar S(t)$). It follows from (4.1) that for every such $t$ and $S$, $\underline{S}(t)\subset S\subset\bar S(t)$,

$$J_S^{-1}(z(t))\cdot p^S\leqq 0.$$

Applying Theorem 3.5, we deduce that the solution function $z(t)\equiv z(t;q,p)$ is monotone in a neighbourhood of $t^0$. It then follows that $z(t)$ is monotone in each connected component of its domain.

(b) We shall prove the necessity part. Suppose that for every $p$ the solution function $z(t)\equiv z(t;q,p)$ is monotone. Let $z^0\geqq 0$ be any point. Let $p=-q-f(z^0)$ and $t^0=1$. Obviously, $z^0$ solves the problem CP $[f(\cdot)+q+t^0p]$. By Theorem 3.4 (Part (ii) it follows that $J_{\underline{s}}^{-1}(z^0)\cdot[q+f(z^0)]^{\underline{s}}\geqq 0$. Let $S$ be any set such that $S\supset\underline{S}$. Define $z_i^k=z_i^0+1/k$ for $i\in S$ and $z_i^k=0$ otherwise, for $k=1,2,\dots$. According to what we have just proved $J_s^{-1}(z^k)\cdot[q+f(z^k)]^S\geqq 0$. Since $f$ is continuously differentiable and all the principal minors of its Jacobian matrix are positive, we have in the limit, as $k$ tends to infinity,

$$J_S^{-1}(z^0)\cdot[q+f(z^0)]^S>0$$

for all $S\supset\underline{S}$. This completes the proof.

***Remark* 4.2.** *If $f$* is linear, $f(z)=Mz$, then (4.1) reduces to

$$(M^S)^{-1}\cdot(q+Mz)^S\geqq 0$$

for every $z \geqq 0$ and $S \supset \underline{S}$. This is equivalent to Cottle's condition $-(M^S)^{-1} \cdot q^S \geqq 0$ for all $S$.

A necessary and sufficient condition for uniform monotonicity is now straightforward.

***Theorem* 4.3.** *Let $f : \mathbf{R}^n_+ \to \mathbf{R}^n$ be a continuously differentiable function which satisfies the conditions of Theorem 4.1. Under these conditions, the solution function $z(t) \equiv t; q, p)$ of the problem PCP$[f(z) + q + tp]$ is monotone for every $q \geqq 0$ and $p$, if and only if for every $z^0 \geqq 0$ and $S \supset \underline{S} \equiv \{i\colon z_i^0 > 0\}$*

$$J_S^{-1}(z^0) \geqq 0 \text{ and } J_S^{-1}(z^0) \cdot [f(z^0)]^S \geqq 0.$$

**Proof.** In view of Theorem 4.1, uniform monotonicity is equivalent to $J_S^{-1}(z^0) \cdot [q + f(z^0)]^S \geqq 0$ for every $q \geqq 0$, $z^0 \geqq 0$, and $S \supset \underline{S}$. Thus, (4.2) is obviously sufficient for uniform monotonicity. On the other hand, if $x$ is any positive $n$-vector, $z^0 \geqq 0$, and $S \supset \underline{S}$, then there exists a positive $t$ such that $q \equiv tx - f(z^0) \geqq 0$. Uniform monotonicity implies $J_S^{-1}(z^0) \cdot (tx)^S \geqq 0$. It is easily verified that since the latter holds for all positive $x$, all the entries of $J_S^{-1}(z^0)$ are necessarily non-negative. Also, strong monotonicity for $q = 0$ implies $J_S^{-1}(z^0) \cdot [f(z^0)]^S \geqq 0$. This completes the proof.

**Remark 4.4.** It is easily verified that if $f(z) = Mz$, then (4.2) is equivalent to $M$'s being a Minkowski matrix (see [4]), hence Theorem 4.3 extends [3, Th. 2].

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