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ON THE NONMONOTONICITY OF THE BARGAINING SET, THE KERNEL AND THE NUCLEOLUS OF A GAME*

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Abstract. A solution concept for n-person games is called monotonic if the players, after committing themselves to payoff vectors that solve (in the sense of this concept) the game, can still assure that every individual's payment grows with the total amount paid. The bargaining set $\mathcal{M}_1^{(i)}$, the kernel, and the nucleolus are shown to be nonmonotonic.

A *characteristic function game* is a pair $(N; v)$ where $N = \{1, \cdots, n\}$ is a nonempty finite set and v is a real-valued function defined over the set of the nonempty subsets of N, 2^N. The elements of N are the *players* and the elements of 2^N are the *coalitions*.

An *imputation* in an n-person game $(N; v)$ is an n-tuple $x = (x_1, \cdots, x_n)$ of real numbers such that

$$x_i \geqq v(\{i\}), \qquad i = 1, \cdots, n, \tag{1}$$

and

$$\sum_{i \in N} x_i = v(N). \tag{2}$$

A *solution concept* is a mapping S from the set of characteristic functions such that S(v) is a set of imputations in $(N; v)$.

DEFINITION 1. (i) A characteristic function v^+ is *greater* than another function v (related to the same set of players N) if $v^+(N) > v(N)$ and $v^+(S) = v(S)$ for every $S \subset N$.

(ii) A solution concept S is called *monotonic* if for every pair of functions v, v^+ such that v^+ is greater than v and $x \in$ S(v), there is a $y \in$ S(v^+) such that $y_i \geqq x_i$ for each $i \in N$.

The Shapley value (see [7]) is a monotonic solution concept. If v^+ is greater than v and x is the Shapley value of $(N; v)$, then the Shapley value of $(N; v^+)$, y, satisfies

$$y_i = x_i + \frac{v^+(N) - v(N)}{n}, \qquad i = 1, \cdots, n. \tag{3}$$

The bargaining set $\mathcal{M}_1^{(i)}$ is a solution concept that was defined by Aumann and Maschler [1]. The kernel is another solution concept, defined by Davis and Maschler [3], which was proved to be a nonempty subset of the bargaining set. The nucleolus is a unique-point solution concept defined by Schmeidler [6]. Schmeidler proved that the nucleolus always exists and belongs to the kernel of the game. Charnes and Kortanek [2] and, independently, Kohlberg [4] proved that the nucleolus is a piecewise linear function of the characteristic function of the game. In this paper we prove that the bargaining set, the kernel and the nucleolus are not monotonic.

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The *excess* of a coalition S with respect to an imputation x is defined by

$$e(S, x) = v(S) - \sum_{i \in S} x_i. \tag{4}$$

Let $\theta(x)$ denote the $(2^n - 1)$-tuple of the numbers $e(S, x)$ arranged in decreasing order, i.e., $1 \leqq i \leqq j \leqq 2^n - 1$ implies $\theta_i(x) \geqq \theta_j(x)$.

DEFINITION 2. An imputation x in a game $(N; v)$ is said to belong to the *nucleolus* of $(N; v)$ if for every other imputation y, $\theta(x)$ precedes $\theta(y)$ in the lexicographical order on $R^{2^n - 1}$.

Schmeidler [4] proved that if

$$v(N) \geqq \sum_{i \in N} v(\{i\}), \tag{5}$$

the nucleolus of $(N; v)$ consists of exactly one point. Let us call the point itself the nucleolus.

Given an imputation x in a game $(N; v)$, denote

$$b_0(x, v) = \{\{i\} : x_i = v(\{i\})\}, \tag{6}$$

$$b_1(x, v) = \{S \in 2^N : (\forall T \in 2^N)(e(S, x) \geqq e(T, x))\}, \tag{7}$$

$$b_{k+1}(x, v) = \left\{S \in 2^N \setminus \bigcup_{i=1}^{k} b_i(x, v) : \left(\forall T \in 2^N \setminus \bigcup_{i=1}^{k} b_i(x, v)\right)(e(S, x) \geqq e(T, x))\right\}. \tag{8}$$

If r is the first index such that $\bigcup_{i=1}^{r} b_i(x, v) = 2^N$, then $b(x, v) = (b_0(x, v), b_1(x, v), \cdots, b_r(x, v))$ is called the *coalition array belonging to the pair* (x, v) (see Kohlberg [4]). For each coalition S, let e^S denote the characteristic vector of S. A collection of coalitions $\mathsf{A} = \{S_1, \cdots, S_p\}$ is said to be *balanced* if there are positive real numbers λ_i, $i = 1, \cdots, p$, such that

$$e^N = \sum_{i=1}^{p} \lambda_i e^{S_i}. \tag{9}$$

The coalition array $b(x, v)$ is said to be *array balanced* if for every k, $k = 1, \cdots, r$, $\bigcup_{i=0}^{k} b_i(x, v)$ is a balanced collection of coalitions.

THEOREM 3 (Kohlberg [4]). *An imputation x is the nucleolus of $(N; v)$ if and only if the coalition array belonging to (x, v) is array balanced.*

Example 4. Let $N = \{1, \cdots, 9\}$ and $x = (1, 1, 1, 2, 2, 2, 1, 1, 1)$. Let[1] $\mathsf{A} = \{123, 14, 24, 34, 15, 25, 35, 789\}$ and $\mathsf{B} = \{12367, 12368, 12369, 456\}$. Define a characteristic function v as follows:

$$v(S) = \begin{cases} 6 & \text{if } S \in \mathsf{A}, \\ 9 & \text{if } S \in \mathsf{B}, \\ 12 & \text{if } S = N, \\ \sum_{i \in S} x_i - 1 & \text{otherwise.} \end{cases} \tag{10}$$

[1] We write 123 instead of $\{1, 2, 3\}$, etc.

It can be easily verified that $b_0(x, v) = \varnothing$, $b_1(x, v) = \mathsf{A} \cup \mathsf{B}$, $b_2(x, v) = \{N, \varnothing\}$ and $\bigcup_{i=0}^{3} b_i(x, v) = 2^N$. $\mathsf{A} \cup \mathsf{B}$ is a balanced collection of coalitions. Let $\lambda_{456} = 1/2$, $\lambda_{789} = 5/6$, and for every $S \in \mathsf{A} \cup \mathsf{B} \setminus \{456, 789\}$, let $\lambda_S = 1/6$. Then

$$e^N = \sum_{S \in \mathsf{A} \cup \mathsf{B}} \lambda_S e^S. \tag{11}$$

Also, $b_1(x, v) \cup b_2(x, v)$ is a balanced collection and so is

$$b_1(x, v) \cup b_2(x, v) \cup b_3(x, v) = 2^N.$$

For each S let $\lambda_S = 1/2^{n-1}$; then

$$e^N = \sum_{S \in 2^N} \lambda_S e^S. \tag{12}$$

It follows that $b(x, v)$ is array balanced. Thus, according to Theorem 3, *x is the nucleolus of $(N; v)$*.

DEFINITION 5. (i) Let x be an imputation in a game $(N; v)$. An *objection* of player k against player l, with respect to x, is a pair $(\hat{y}; C)$, where C is a coalition such that $k \in C$ and $l \notin C$ and $\hat{y}$ is a vector whose indices are the members of C, satisfying $\sum_{i \in C} \hat{y}_i = v(C)$ and $\hat{y}_i > x_i$ for each $i \in C$.

(ii) A *counter objection* to the above objection is a pair $(\hat{z}; D)$, where D is a coalition such that $l \in D$ and $k \notin D$ and $\hat{z}$ is a vector whose indices are the members of D, satisfying $\sum_{i \in D} \hat{z}_i = v(D)$, $\hat{z}_i \geqq \hat{y}_i$ for $i \in D \cap C$, and $\hat{z}_i \geqq x_i$ for $i \in D \setminus C$.

(iii) An imputation x is said to belong to the *bargaining set* $\mathscr{M}_1^{(i)}$ of $(N; v)$ (for the grand coalition) if for any objection of one player against another with respect to x, there exists a counter objection.

DEFINITION 6. An imputation x is said to belong to the *kernel* (for the grand coalition) of $(N; v)$ if for every pair of distinct players i, j such that $x_j > v(\{j\})$,

$$\max\{e(S, x) : i \in S, j \notin S\} \leqq \max\{e(S, x) : i \notin S, j \in S\}. \tag{13}$$

The kernel is always a subset of the bargaining set $\mathscr{M}_1^{(i)}$ (Davis and Maschler [3]). The nucleolus always belongs to the kernel (Schmeidler [6]).

PROPOSITION 7. *Let $(N; v)$ be the game introduced in Example 4 and let $x = (1, 1, 1, 2, 2, 2, 1, 1, 1)$ be its nucleolus (see Example 4). Let v^+ denote a characteristic function over N such that $v^+(S) = v(S)$ for each $S \subsetneqq N$ and $v^+(N) = v(N) + 1$. Under these conditions, for every y in the bargaining set $\mathscr{M}_1^{(i)}$ of $(N; v^+)$ there exists i, $1 \leqq i \leqq 9$, such that $y_i < x_i$.*

Proof. Let y belong to the bargaining set $\mathscr{M}_1^{(i)}$ of $(N; v^+)$ and suppose, per absurdum, that $y_i \geqq x_i$ for every i, $i = 1, \cdots, 9$. Denote $z = y - x$. It is necessary[2] that $z_7 = z_8 = z_9$ (if, for example, $z_7 < z_8$, then player 7 can object against player 8, using 12367, and there is no counter objection to that by player 8). Also,

$$z_4 + \min(z_1, z_2, z_3) \geqq z_1 + z_2 + z_3 + z_6 + z_7 \tag{14}$$

(otherwise, player 4 has a justified[3] objection against player 6). On the other hand, since there is no justified objection of player 6 against player 4, necessarily,

$$z_4 + \min(z_1, z_2, z_3) \leqq z_1 + z_2 + z_3 + z_6 + z_7. \tag{15}$$

[2] See Maschler [5] for a clarification of the arguments in this proof.

[3] An objection to which there is no counter objection will be called *justified*.

Moreover, for each i, $i = 1, 2, 3$, there is no justified objection of i against 6 and therefore

$$z_i + z_5 \geqq z_4 + z_5 + z_6. \tag{16}$$

Thus,

$$z_i \geqq z_4 + z_6 = z_1 + z_2 + z_3 - \min(z_1, z_2, z_3) + 2z_6 + z_7. \tag{17}$$

Since $z_k \geqq 0, k = 1, \cdots, 9$, it follows from (17) that $z_6 = z_7 = 0, \min(z_1, z_2, z_3) = 0$ and $z_i = \max(z_1, z_2, z_3)$. But (17) is true for $i = 1, 2, 3$, so that $z_1 = z_2 = z_3 = 0$. Thus according to (15), $z_4 = 0$, and by symmetry considerations also $z_5 = 0$. On the other hand, $z_1 + \cdots + z_9 = 1$, hence—a contradiction.

COROLLARY 8. *The bargaining set $\mathcal{M}_1^{(i)}$, the kernel and the nucleolus are nonmonotonic solution concepts.*

Proof. The game $(N; v^+)$ (see Proposition 7) is greater than the game $(N; v)$. The nucleolus of $(N; v)$ belongs to the bargaining set $\mathcal{M}_1^{(i)}$ and to the kernel of this game. There does not exist a point in the bargaining set of $(N; v^+)$ (nor in the kernel and the nucleolus of $(N; v^+)$) that satisfies the monotonicity condition (see Definition 1).

Remark 9. The nucleolus of $(N; v^+)$ is the point $(1\frac{1}{9}, 1\frac{1}{9}, 1\frac{1}{9}, 2\frac{2}{9}, 2\frac{2}{9}, 1\frac{8}{9}, 1\frac{1}{9}, 1\frac{1}{9}, 1\frac{1}{9})$ and, indeed, player 6 gets less than in the game $(N; v)$.

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