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OPTIMAL FLOWS IN NETWORKS WITH MULTIPLE SOURCES AND SINKS

Nimrod MEGIDDO
Tel Aviv University, Tel Aviv, Israel

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The concept of an optimal flow in a multiple source, multiple sink network is defined. It generalizes maximal flow in a single source, single sink network. An existence proof and an algorithm are given.

1. Introduction

In their famous book, Ford and Fulkerson [3] state that the situation in which there are multiple sources and sinks, with flow permitted from any source to any sink, presents nothing new. They claim that a multiple source, multiple sink network reduces to a single source, single sink network by the adjunction of a supersource, a supersink, and several arcs. That is of course true if only a maximization of the total flow through the network is desired. However, under certain circumstances, especially in economical applications, one wishes not only to maximize the total flow but also to distribute it "fairly" among the sinks or the sources. For instance, it might be desirable to maximize the minimum amount supplied to individual sinks [1] or, alternatively, delivered from individual sources. On the other hand, the labelling method for solving maximal flow problems [2] very often generates "unfair" flows, in the sense that most of the flow comes from one source, or goes to one sink. In fact, this is the nature of the labelling method.

In this paper, we introduce an "optimal" solution to the fair maximum flow problem in a network with several sources and sinks. It is based on lexicographical maximization of the vectors of individual flows, separately for the sources and the sinks.

[1] This problem is easily handled by making use of the supply–demand theorem [3, Theorem II, 1.1].

2. Definitions

A *network* is a triplet $\mathcal{N} = (N, \mathcal{A}, c)$, where N is a nonempty finite set whose elements are the *nodes*, $\mathcal{A}$ is a set of ordered pairs (called *arcs*) of nodes, and c is a function from $\mathcal{A}$ to the nonnegative reals, called the *capacity function*. The network is assumed to be *connected*, i.e., for every pair of nodes x, y there is a sequence $x = x_0, x_1, ..., x_m = y$ such that for every i, $1 \leqslant i \leqslant m$, either (x_{i-1}, x_i) or (x_i, x_{i-1}) is an arc.

We use the following conventional notation. For every pair of subsets X, $Y \subset N$,

$$(X, Y) = \{(x, y): x \in X, y \in Y, (x, y) \in \mathcal{A}\}. \tag{2.1}$$

A pair of the form $(X, N \setminus X)$ is called a *cut*. If g is a function from $\mathcal{A}$ to the reals, denote

$$g(X, Y) = \sum_{(x,y) \in (X,Y)} g(x, y), \tag{2.2}$$

and for every node x denote

$$\text{net}\,(g, x) = g(N, \{x\}) - g(\{x\}, N). \tag{2.3}$$

Let $S \subset N$ be the set of the *sources* and let $T \subset N \setminus S$ be the set of the *sinks* in the network. A *flow* is a function f from $\mathcal{A}$ to the nonnegative reals such that

$$f(x, y) \leqslant c(x, y) \tag{2.4}$$

for all arcs (x, y) and for every node x

$$\text{net}\,(f, x) \begin{cases} \leqslant 0 & \text{if } x \in S, \\ \geqslant 0 & \text{if } x \in T, \\ = 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

We denote the class of all the flows by $\mathcal{F}$.

Given a flow f, let $T(f)$ denote the $|T|$-tuple of the numbers net (f, t), $t \in T$, arranged in order of increasing magnitude. Analogously, let $S(f)$ denote the $|S|$-tuple of the numbers net (f, s), $s \in S$, arranged in order of decreasing magnitude.

Definition 2.1. (i) A flow f^* is called *sink-optimal* (*source-optimal*) if for every $f \in \mathcal{F}$, $T(f^*)$ ($S(f^*)$) is lexicographically greater (less) than or equal to $T(f)$ ($S(f)$).

(ii) A flow f^* is called *optimal* if it is both sink-optimal and source-optimal.

Definition 2.2. Let $\mathcal{N} = (N, \mathcal{A}, c)$ be a network and let X be a non-empty subset of N. The *X-condensed* network $\mathcal{N}/_X = (N^*, \mathcal{A}^*, c^*)$ is defined as follows. Let x^* be a new node ($x^* \notin N$) and define

$$N^* = (N \setminus X) \cup \{x^*\}, \tag{2.6}$$

$$\mathcal{A}^* = \{(x, y): (x, y) \in \mathcal{A}, x, y \notin X\} \cup \{(x^*, x): x \in N \setminus X\} \cup \cup \{(x, x^*): x \in N \setminus X\}. \tag{2.7}$$

$$c^*(x, y) = \begin{cases} c(x, y) & \text{if } x, y \in N \setminus X, \\ c(X, \{y\}) & \text{if } x = x^* \text{ and } y \in N \setminus X, \\ c(\{x\}, X\} & \text{if } y = x^* \text{ and } x \in N \setminus X. \end{cases} \tag{2.8}$$

Lemma 2.3. *Let f be a flow in $\mathcal{N} = (N, \mathcal{A}, c)$, suppose $X \supset S$, $N \setminus X \supset T$, and let $\bar{f}$ be a flow in $\mathcal{N}/_X$ defined by*

$$\bar{f}(x, y) = \begin{cases} f(x, y) & \text{if } x, y \in N \setminus X, \\ f(X, \{y\}) & \text{if } x = s \text{ and } y \in N \setminus X, \\ f(\{x\}, X) & \text{if } y = s \text{ and } x \in N \setminus X, \end{cases} \tag{2.9}$$

where s is the new node, being the single source of $\mathcal{N}/_X$. Under these conditions, f is a sink-optimal flow in $\mathcal{N}$ if and only if $\bar{f}$ is a sink-optimal flow in $\mathcal{N}/_X$.

Proof. The proof follows immediately from the fact that

$$T(f) = T(\bar{f}). \tag{2.10}$$

Obviously, an analogous result can be stated for source-optimal flows. Without loss of generality, whenever a sink-optimal flow is discussed, the network is assumed to have a single source s. It can be easily verified that the sink-optimal flow problem in a single source network is equivalent to the source-optimal flow problem in a single sink network.

Definition 2.4. The *characteristic function* of a network is a real-valued function $v : 2^T \to R$ such that for every $A \subset T$

$$v(A) = \max \{f(N, A) - f(A, N): f \in \mathcal{F}\}. \tag{2.11}$$

Also, if g is a $|T|$-tuple we denote

$$g(A) = \sum_{t \in A} g_t. \tag{2.12}$$

3. On the characteristic function

The network in this section is assumed to have a single source and a set T of sinks. An analogous theory can be developed for multiple source, single sink networks.

Remark 3.1. It can be easily derived from the max-flow min-cut theorem (see [3, p. 11]) that for every $A \subset T$

$$v(A) = \min \{c(X, N \setminus X): s \in X, A \subset N \setminus X\}, \tag{3.1}$$

i.e., $v(A)$ is the minimum capacity of a cut separating A from the source.

Lemma 3.2. *The characteristic function v is*

(i) *monotonic*:

$$A \subset B \Rightarrow v(A) \leqslant v(B). \tag{3.2}$$

(ii) *concave*:

$$v(A \cup B) \leqslant v(A) + v(B) - v(A \cap B). \tag{3.3}$$

Proof. (i) Suppose that $A \subset B \subset T$. Then,

$$\begin{aligned} v(B) &= \max \{f(N, B) - f(B, N): f \in \mathcal{F}\} \\ &= \max \{[f(N, A) - f(A, N)] + [f(N, B \setminus A) - f(B \setminus A, N)]: f \in \mathcal{F}\} \\ &\geqslant \max \{f(N, A) - f(A, N): f \in \mathcal{F}\} = v(A). \end{aligned} \tag{3.4}$$

(ii) Let A and B be any two subsets of T. Let $(X, N \setminus X)$ and $(Y, N \setminus Y)$ be minimal cuts separating the source from A and B, respectively. Thus, $s \in X \cap Y$, $A \subset N \setminus X$, $B \subset N \setminus Y$, $v(A) = c(X, N \setminus X)$, and $v(B) = c(Y, N \setminus Y)$. Denote $X \cap Y = P$, $X \setminus Y = Q$, $Y \setminus X = R$, and $N \setminus (X \cup Y) = W$. It follows that

$$\begin{aligned} v(A \cup B) &\leqslant c(P, N \setminus P) = c(P, Q \cup R \cup W) \\ &= c(P, Q) + c(P, R) + c(P, W) \end{aligned}$$

$$\begin{aligned}
&= [c(P,R)+c(P,W)+c(Q,R)+c(Q,W)] \\
&\quad + [c(P,Q)+c(P,W)+c(R,Q)+c(R,W)] \\
&\quad - [c(P,W)+c(Q,W)+c(R,W)]-[c(R,Q)+c(Q,R)] \\
&= c(X, N\setminus X)+c(Y, N\setminus Y) - c(X\cup Y, W) \\
&\quad - [c(R,Q)+c(Q,R)] \\
&\leqslant v(A)+v(B)-v(A\cap B). \qquad (3.5)
\end{aligned}$$

Corollary 3.3. (i) *If $A\subset B\subset T$ and $v(A)=v(B)$ then for every $C\subset T$*

$$v(A\cup C)=v(B\cup C). \qquad (3.6)$$

(ii) *If A and B are disjoint subsets of T such that*

$$v(A\cup B)=v(A)+v(B), \qquad (3.7)$$

then for every $A^*\subset A$ and $B^*\subset B$,

$$v(A^*\cup B^*)=v(A^*)+v(B^*). \qquad (3.8)$$

Proof. (i) For such A and B,

$$\begin{aligned}
v(A\cup C) &\leqslant v(B\cup C) \\
&\leqslant v(A\cup C)+v(B)-v(A\cup(B\cap C)) \\
&\leqslant v(A\cup C)+v(B)-v(A)=v(A\cup C) \qquad (3.9)
\end{aligned}$$

and that proves (3.6).

(ii) Suppose, per absurdum, that $A^*\subset A$ is a subset such that

$$v(A^*)+v(B)>v(A^*\cup B). \qquad (3.10)$$

Then

$$\begin{aligned}
v(A\cup B) &= v(B)+v(A) \\
&> v(A^*\cup B)-v(A^*)+v(A), \qquad (3.11)
\end{aligned}$$

in contradiction to the concavity proved in Lemma 3.2. The contradiction implies that for every $A^*\subset A$ and $B^*\subset B$

$$v(A^*)+v(B^*)=v(A^*\cup B^*). \qquad (3.12)$$

Remark 3.4. Lemma 3.2 and Corollary 3.3 simplify the computation of the characteristic function of a network. Since the characteristic function can be calculated by considering minimal cuts (Remark 3.1), it is known that simplified networks may be used [1 and 4]. Specifically, if $(X, N \setminus X)$ is a minimal cut separating $A \subset T$ from the source s and if $B \subset A \subset C \subset T$, then there are minimal cuts $(Y, N \setminus Y)$ and $(Z, N \setminus Z)$ separating the source from B and C, respectively, such that $Y \supset X \supset Z$ [4]. Moreover, in order to find Y, we may use the X-condensed network (see Definition 2.2) rather than the original network, and a similar simplification holds for Z. Also, the union of all the sets X such that $(X, N \setminus X)$ defines a minimal cut separating the source from A is also such a minimal cut [3, p. 13], and the same is true with the intersection of these sets. Thus, in order to find Y we may condense the union and in order to find Z we may condense the complement of the intersection. Notice that these extreme cuts can be found directly by an appropriate formulation of a labelling algorithm.

4. Optimal flows

Lemma 4.1. *Let $(g_t)_{t \in T}$ be a $|T|$-tuple of non-negative real numbers, and let v be the characteristic function of a single source, multiple sink network. A necessary and sufficient condition for the existence of a flow f such that for each $t \in T$*

$$g_t = \text{net}\,(f, t) \tag{4.1}$$

is that for every $A \subset T$

$$g(A) \leqslant v(A). \tag{4.2}$$

Proof. [2] Necessity is immediate since

$$v(A) = \max\,\{\sum_{t \in A} \text{net}\,(f, t) : f \in \mathcal{F}\,\}. \tag{4.3}$$

To prove sufficiency, we adjunct a super-sink as follows. Let t^* be a new node ($t^* \in N$) and let $\mathcal{N}^* = (N^*, \mathcal{A}^*, c^*)$ be a network, where

[2] This is also a special case of the second version of the supply–demand theorem [3, Corollary II, 1.2.].

$N^* = N \cup \{t^*\}$, $\mathcal{A}^* = \mathcal{A} \cup \{(t, t^*) \colon t \in T\}$, and

$$c^*(x, y) = \begin{cases} g_t & \text{if } (x, y) = (t, t^*), \\ c(x, y) & \text{if } (x, y) \in \mathcal{A} . \end{cases} \tag{4.4}$$

Suppose that t^* is the single sink in $\mathcal{N}^*$. We claim that $(N, \{t^*\})$ is a minimal cut in $\mathcal{N}^*$ separating s from t^*. For let $X \subset N^*$ be such that $s \in X$ and $t^* \in X$. It follows that

$$\begin{aligned} c^*(X, N^* \setminus X) &= c^*(X, N \setminus X) + c^*(X \cap T, \{t^*\}) \\ &= c(X, N \setminus X) + g(X \cap T) \geqslant v(T \setminus X) + g(X \cap T) \\ &\geqslant g(T) = c^*(N, \{t^*\}). \end{aligned} \tag{4.5}$$

Now let f^* be a maximal flow through $\mathcal{N}^*$. Necessarily,

$$f^*(t, t^*) = g_t \tag{4.6}$$

for every $t \in T$. Thus, if f is the restriction of f^* to $\mathcal{A}$, then for every $t \in T$

$$\begin{aligned} \text{net}\,(f, t) &= f^*(N, t) - f^*(t, N) = \text{net}\,(f^*, t) + f^*(t, t^*) \\ &= f^*(t, t^*) = g_t . \end{aligned} \tag{4.7}$$

Lemma 4.2. *Let f be a flow and for every $t \in T$ denote* net $(f, t) = g_t$. *If $A_1, ..., A_k$ are subsets of T such that*

$$g(A_i) = v(A_i), \qquad i = 1, ..., k, \tag{4.8}$$

and $A = \bigcup_{i=1}^{k}$ then also

$$g(A) = v(A). \tag{4.9}$$

Proof. Without loss of generality we assume that $k = 2$. It follows that

$$\begin{aligned} v(A) &\geqslant g(A) = g(A_1) + g(A_2) - g(A_1 \cap A_2) \\ &\geqslant v(A_1) + v(A_2) - v(A_1 \cap A_2) \geqslant v(A) \end{aligned} \tag{4.10}$$

and that of course implies (4.9).

Lemma 4.3. *A sink-optimal flow f^* is necessarily a maximal flow, i.e., for every $f \in \mathcal{F}$,*

$$\sum_{t \in T} \text{net}\,(f^*, t) \geqslant \sum_{t \in T} \text{net}\,(f, t). \tag{4.11}$$

Proof. [3] First, we claim that if f^* is a sink-optimal flow, then for every $u \in T$ there is a subset $A_u \subset T$ such that $u \in A_u$ and

$$\sum_{t \in A_u} \text{net}\,(f^*, t) = v(A_u). \tag{4.12}$$

For if this is not true, then according to Lemma 4.1 there is another flow f such that

$$\text{net}\,(f, u) > \text{net}\,(f^*, u) \tag{4.13}$$

and for every $t \in T$ $(t \neq u)$

$$\text{net}\,(f, t) = \text{net}\,(f^*, t), \tag{4.14}$$

in contradiction to the sink-optimality of f^*. Now, since $T = \bigcup_{u \in T} A_u$, it follows from Lemma 4.2 that

$$\sum_{t \in T} \text{net}\,(f^*, t) = v(T). \tag{4.15}$$

That, of course, proves maximality.

Theorem 4.4. *For every pair of a sink-optimal flow $f^{(T)}$ and a source-optimal flow $f^{(S)}$, through a network $\mathcal{N} = (N, \mathcal{A}, c)$*, there is an optimal *flow f^* through this network, such that for every $s \in S$*

$$\text{net}\,(f^*, s) = \text{net}\,(f^{(S)}, s), \tag{4.16}$$

and for every $t \in T$

$$\text{net}\,(f^*, t) = \text{net}\,(f^{(T)}, t). \tag{4.17}$$

Proof. [4] Let $(X, N \setminus X)$ be a minimal cut separating S from T. It follows from Lemma 2.3, Lemma 4.3, and the max-flow min-cut theorem that for every $(x, y) \in (X, N \setminus X)$,

$$f^{(S)}(x, y) = f^{(T)}(x, y) = c(x, y), \tag{4.18}$$

$$f^{(S)}(y, x) = f^{(T)}(y, x) = 0. \tag{4.19}$$

[3] This can also be deduced from the labelling procedure.
[4] This is also a consequence of the symmetric supply–demand theorem [3, Corollary II, 2.2].

Notice that the previous results could be stated for source-optimal flows as well. Now define for every $(x, y) \in \mathscr{A}$,

$$f^*(x, y) = \begin{cases} f^{(S)}(x, y) & \text{if } x \in X, \\ f^{(T)}(x, y) & \text{if } x \notin X. \end{cases} \tag{4.20}$$

It is easy to verify that f^* is a flow indeed (notice (4.18), (4.19)). Also, (4.20) implies (4.16), (4.17), and therefore f^* is both source-optimal and sink-optimal.

Remark 4.5. Denote

$$G = \left\{ g = (g_t)_{t \in T} \colon (\forall A \subset T) \left(\sum_{t \in A} g_t \leqslant v(A) \right), g_t \geqslant 0 \right\} \tag{4.21}$$

and let $\theta : R^{|T|} \to R^{|T|}$ be the function that rearranges the components of each vector in order of increasing magnitude. It follows from Lemma 4.1 that the calculation of a vector $g^* \in G$ such that $\theta(g^*)$ is lexicographically maximal in $\theta(G)$ yields a sink-optimal flow. Analogously, a calculation in the set $G' = \{g' = (g'_s)_{s \in S} : ...\}$ yields a source-optimal flow. An optimal flow can than be constructed via Theorem 4.4.

The next theorem suggests a direct method for obtaining this g^*.

Theorem 4.6. *Let $T_0 = \emptyset$ and for every $A \subset T$ let $w_0(A) = v(A)$. For every $k \geqslant 0$ such that $T_k \neq T$, define recursively*

$$\alpha_k = \min \{w_k(A)/|A| \colon 0 \neq A \subset T \setminus T_k\}, \tag{4.22}$$

$$T_{k+1} = T_k \cup \bigcup \{A : A \subset T \setminus T_k,\ w_k(A) = \alpha_k |A|\}, \tag{4.23}$$

$$g_t^* = \alpha_k \qquad (t \in T_{k+1} \setminus T_k), \tag{4.24}$$

$$w_{k+1}(A) = \min \{v(A \cup B) - g(B) \colon B \subset T_{k+1}\}$$
$$(A \subset T \setminus T_{k+1}). \tag{4.25}$$

Under these conditions there is k_0, $1 \leqslant k_0 \leqslant |T|$, such that $T_{k_0} = T$ (Hence g_t^* is well-defined for each $t \in T$) and $\theta(g^*)$ *is the lexicographical maximum of $\theta(G)$ (see Remark 4.5).*

Proof. Obviously,

$$\emptyset = T_0 \subsetneqq T_1 \subsetneqq \dots \subsetneqq T \tag{4.26}$$

so that there exists k_0 as specified in the theorem. First, we prove that $g^* \in G$. Let A be any nonempty subset of T. Let k be the greatest index such that $A_k \equiv A \cap (T_{k+1} \setminus T_k) \neq \emptyset$. Denote $B = A \setminus A_k$. It follows that

$$\begin{aligned} g^*(A) &= g^*(A_k) + g^*(B) = \alpha_k |A_k| + g^*(B) \\ &\leqslant w_k(A_k) + g^*(B) \leqslant v(A_k \cup B) = v(A). \end{aligned} \tag{4.27}$$

Thus, $g^* \in G$.

Note that $\{\alpha_k\}$ is an increasing sequence. For let $A \subset T \setminus T_{k+1}$, $B \subset T_{k+1} \setminus T_k$, and $C \subset T_k$ be subsets such that

$$\alpha_{k+1} = \frac{w_{k+1}(A)}{|A|} = \frac{v(A \cup B \cup C) - g^*(B \cup C)}{|A|}. \tag{4.28}$$

It follows that

$$\begin{aligned} \alpha_{k+1} &= \frac{v(A \cup B \cup C) - g^*(C) - g^*(B)}{|A|} \\ &\geqslant \frac{w_k(A \cup B) - g^*(B)}{|A|} > \frac{\alpha_k |A \cup B| - \alpha_k |B|}{|A|} = \alpha_k. \end{aligned} \tag{4.29}$$

Suppose, per absurdum, that there is $g \in G$ such that $\theta(g)$ is lexicographically greater than $\theta(g^*)$. For every $A \subset T$,

$$g(A) \leqslant v(A). \tag{4.30}$$

Thus,

$$\min \{g_t : t \in T\} \leqslant \min \{v(A)/|A| : \emptyset \neq A \subset T\} = \alpha_0. \tag{4.31}$$

On the other hand, $\theta(g)$ is lexicographically greater than $\theta(g^*)$ and, therefore,

$$\min \{g_t : t \in T\} \geqslant \min \{g_t^* : t \in T\} = \alpha_0. \tag{4.32}$$

Moreover, since

$$T_1 = \bigcup \{A : v(A) = \alpha_0 |A|\} \tag{4.33}$$

it follows from (4.30) and (4.33) that for every $t \in T_p$

$$g_t = \alpha_0 = g_t^*. \tag{4.34}$$

Assume, inductively, that for every $t \in T_k$,

$$g_t = g_t^*. \tag{4.35}$$

For every $A \subset T \setminus T_k$ and $B \subset T_k$,

$$g(A) \leqslant v(A \cup B) - g(B) = v(A \cup B) - g^*(B). \tag{4.36}$$

Thus,

$$\begin{aligned} &\min \{g_t : T \setminus T_k\} \leqslant \\ &\quad \leqslant \min \{(v(A \cup B) - g^*(B))/|A| : A \subset T \setminus T_k, B \subset T_k\} \\ &\quad = \alpha_k . \end{aligned} \tag{4.37}$$

On the other hand, since (4.35) holds for every $t \in T_k$, and $\theta(g)$ is lexicographically greater than $\theta(g^*)$, it follows that

$$\min \{g_t : t \in T \setminus T_k\} \geqslant \min \{g_t^* : t \in T_k\} = \alpha_k . \tag{4.38}$$

Moreover, it follows from (4.23) and (4.36) that for every $t \in T_{k+1} \setminus T_k$,

$$g_t = \alpha_k = g_t^*. \tag{4.39}$$

Thus, (4.35) holds for every $t \in T_{k+1}$ and therefore, inductively, for every $t \in T$. In other words, $\theta(g^*) = \theta(g)$ and hence, a contradiction.

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