SIAM J. APPL. MATH.
Vol. 29, No. 3, November 1975

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TENSOR DECOMPOSITION OF COOPERATIVE GAMES*

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Abstract. A decomposition theory for n-person games is introduced. A "unique factorization" theorem is proved. In general, every monotonic game has a unique decomposition with a quotient that is either prime or absolutely decomposable. Finally, an application to reliability theory is suggested.

1. Introduction. The tensor composition of nonnegative characteristic function games is a generalization of Shapley's compound simple games [11], [12], [13], [14]. This composition was suggested by Owen in [9].

Shapley has proved a unique decomposition theorem with respect to his composition concept. This theorem was proved, independently, also by Birnbaum and Esary in [4]. In this paper we generalize the results of Shapley, namely, we prove a unique decomposition theorem with respect to Owen's composition. Particularly, Shapley's "committees" are generalized in a suitable way.

All the games in this paper are assumed to be monotonic. As a matter of fact, this assumption is not necessary for most of the theorems. It is essential only to the proof of Assertion 4.3a. Monotonicity was assumed also by Shapley, Birnbaum and Esary.

2. Definitions. A characteristic function game is a pair $\Gamma = (N; v)$, where $N = \{1, \cdots, n\}$ is a nonempty finite set and v is a real-valued function defined over the subsets of N. We usually assume that

$$v(N) = 1, \tag{2.1}$$

$$v(\varnothing) = 0 \tag{2.2}$$

and

$$v(S) \geqq 0, \qquad S \subset N. \tag{2.3}$$

The elements of N are called *players* and the subsets of N are called *coalitions*. The game is called *monotonic* if for every pair of coalitions S, T,

$$S \subset T \Rightarrow v(S) \leqq v(T). \tag{2.4}$$

A player i is termed *dummy* if for every coalition S,

$$v(S \cup \{i\}) = v(S). \tag{2.5}$$

A coalition D is said to be *inessential* if for every coalition $S \subset N \setminus D$,

$$v(S \cup D) = v(S). \tag{2.6}$$

Otherwise D is said to be *essential*.

* Received by the editors July 24, 1973.

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DEFINITION 2.1 (Owen). Let $\Gamma_0 = (M; u)$ and $\Gamma_i = (N_i; w_i)$, $i = 1, \cdots, m$, be games satisfying (2.1)–(2.3). Suppose that $M = \{1, \cdots, M\}$ and $N_i \cap N = \emptyset$ for every pair of distinct elements $i, j \in M$. The *tensor composition* of the *components* $\Gamma_1, \cdots, \Gamma_m$, with the *quotient* Γ_0, is defined to be the game

$$\Gamma \equiv (N; v) = \Gamma_0[\Gamma_1, \cdots, \Gamma_m], \tag{2.7}$$

where

$$N = N_1 \cup \cdots \cup N_m \tag{2.8}$$

and for every $S \subset N$,

$$v(S) = \sum_{T \subset M} \left\{ \prod_{i \notin T} w_i(S \cap N_i) \prod_{i \notin T} [1 - w_i(S \cap N_i)] \right\} u(T). \tag{2.9}$$

Sometimes we write $\Gamma_0[\Gamma_i : i \in M]$ for $\Gamma_0[\Gamma_1, \cdots, \Gamma_m]$.

It was proved by Owen [9] that $v(S)$ defined by (2.9) is the unique function that satisfies

$$v\left(\bigcup_{i \notin T} N_i \right) = u(T), \qquad T \subset M, \tag{2.10}$$

and

$$v(S \cup \{i\}) - v(S) = [w_j((S \cap N_j) \cup \{i\}) - w_j(S \cap N_j)][v(S \cup N_j) - v(S \setminus N_j)] \tag{2.11}$$

for every $S \subset N$ and $i \in N_j, j \in M$.

Justifications of Definition 2.1 were given in [9], [10]. A nongame-theoretic interpretation is given in Appendix B.

A game Γ is called *decomposable* if it can be represented as a composition of m components $(1 < m < |N|)$. Otherwise the game is called *prime*.

DEFINITION 2.2. Let $\Gamma = (N; v)$ be a game satisfying (2.1)–(2.3) and let C be a nonempty coalition in Γ. A game $\Gamma_C = (C; c)$ is called a *committee game* of Γ if for every $S \subset N$,

$$v(S) = c(S \cap C)v(S \cup C) + [1 - c(S \cap C)]v(S \setminus C). \tag{2.12}$$

The coalition C is then called a *committee* of Γ.

It is easy to verify that Definition 2.2 generalizes Shapley's committees [14, p. 6]. A committee C is said to be *proper* if it is a proper subset of N and contains at least two players.

The concept of the committee can be interpreted in the following way. Given a coalition C, define the relative contribution of a coalition $B \subset C$ to a coalition $T \subset N \setminus C$ (such that $v(T) \neq v(T \cup C)$) to be the fraction $(v(T \cup B) - v(T))/v(T \cup C) - v(T))$. Thus a committee is a coalition such that the relative contribution of every $B \subset C$ to T is independent of T.

We shall use the notation

$$\bar{v}(S) = 1 - v(S) \tag{2.13}$$

for every characteristic function v.

DEFINITION 2.3. Let $\Gamma = (N; v)$ be a game and let C be a coalition in Γ. Denote by i_C an element which does not belong to N and let $N_C = (N \setminus C) \cup \{i_C\}$. Define a characteristic function v_C over N_C as follows. For every $S \subset N_C$,

$$v_C(S) = \begin{cases} v((S \setminus \{i_C\}) \cup C) & \text{if } i_C \in S, \\ v(S) & \text{if } i_C \notin S. \end{cases} \tag{2.14}$$

The game $\Gamma/_C = (N_C; v_C)$ is called the *contraction* of Γ on the coalition C.

LEMMA 2.4. *Let $\Gamma = (N; v)$ and $\Gamma_C = (C; c)$ be games satisfying* (2.1)–(2.3) *and such that $C \subset N$. Then Γ_C is a committee game of Γ if and only if there is a representation of Γ as a composition (see Definition* 2.1), *where Γ_C is one of the components.*

Proof. (a) Suppose that Γ is the tensor composition $\Gamma = \Gamma_0[\Gamma_1, \cdots, \Gamma_m]$, where $\Gamma_0 = (M; u)$, $\Gamma_k = (N_k; w_k)$, $k = 1, \cdots, m$ (see Definition 2.1), and assume that $C = N_1$. Thus for every $S \subset N$ (see (2.13)),

$$\begin{aligned} v(S) &= \sum_{T \subset M} \left\{ \prod_{i \in T} w_i(S \cap N_i) \prod_{i \notin T} \bar{w}_i(S \cap N_i) \right\} u(T) \\ &= w_1(S \cap C) \sum_{1 \in T \subset M} \left\{ \prod_{\substack{i \in T \\ i \neq 1}} w_i(S \cap N_i) \prod_{i \notin T} \bar{w}_i(S \cap N_i) \right\} u(T) \\ &\quad + \bar{w}_1(S \cap C) \sum_{1 \notin T \subset M} \left\{ \prod_{i \in T} w_i(S \cap N_i) \prod_{\substack{i \notin T \\ i \neq 1}} \bar{w}_i(S \cap N_i) \right\} u(T) \\ &= w_1(S \cap C) v(S \cup N_1) + \bar{w}_1(S \cap C) v(S \setminus N_1), \end{aligned} \tag{2.15}$$

It follows that $\Gamma_1 = (C; w_1)$ is a committee game of Γ.

(b) Suppose that C is a committee of Γ with the characteristic function c. For each player $i \notin C$, let $\Gamma_i = (\{i\}; w_i)$ be a 1-player game where $w_i(\{i\}) = 1$. Denote $N_{i_C} = C$, $w_{i_C} = c$ and for each $i \neq i_C$, $N_i = \{i\}$. For every $S \subset N$,

$$\begin{aligned} v(S) &= c(S \cap C) v(S \cup C) + \bar{c}(S \cap C) v(S \setminus C) \\ &= c(S \cap C) v_C[(S \setminus C) \cup \{i_C\}] + \bar{c}(S \cap C) v_C(S \setminus C) \\ &= c(S \cap C) \sum_{i_C \in T \subset N_C} \left\{ \prod_{\substack{i \in T \\ i \neq i_C}} w_i(S \cap N_i) \prod_{i \notin T} \bar{w}_i(S \cap N_i) \right\} v_C(T) \\ &\quad + \bar{c}(S \cap C) \sum_{i_C \notin T \subset N_C} \left\{ \prod_{i \in T} w_i(S \cap N_i) \prod_{\substack{i \in T \\ i \neq i_C}} \bar{w}_i(S \cap N_i) \right\} v_C(T) \\ &= \sum_{T \subset N_C} \left\{ \prod_{i \in T} w_i(S \cap N_i) \prod_{i \notin T} \bar{w}_i(S \cap N_i) \right\} v_C(T). \end{aligned} \tag{2.16}$$

Thus

$$\Gamma = \Gamma/_C[\Gamma_i : i \in N_C], \tag{2.17}$$

where $\Gamma_{i_C} = (N_{i_C}; w_{i_C})$.

COROLLARY 2.5. *A game is decomposable if and only if it has a proper committee.*

3. Basic properties of committees.

LEMMA 3.1. *If C is an essential committee of $\Gamma = (N; v)$, then there is a unique function c over the subsets of C such that $\Gamma_C = (C; c)$ is a committee game of Γ.*

Proof. If C is an essential committee of Γ, then there exists $T \subset N \setminus C$ such that

$$v(T \cup C) \neq v(T). \tag{3.1}$$

If $\Gamma_C = (C; c)$ is a committee game, then for every $B \subset C$, necessarily

$$c(B) = \frac{v(B \cup T) - v(T)}{v(C \cup T) - v(T)}. \tag{3.2}$$

LEMMA 3.2. (i) *Let i be a dummy in the game Γ and let $C \subset N$ be a coalition in Γ. Then C is a committee of Γ if and only if the coalitions $C \cup \{i\}$ and $C \setminus \{i\}$ are committees of Γ.*

(ii) *If i is a dummy in the committee game $\Gamma_C = (C; c)$ of $\Gamma = (N; v)$, then it is a dummy in Γ too.*

The proof is immediate.

Henceforth we assume that all the players are nondummies. We say $\Gamma_C = (C; c)$ is an essential committee game of Γ if C is an essential committee of Γ.

LEMMA 3.3. *Let $\Gamma_C = (C; c)$ be an essential committee game of $\Gamma = (N; v)$ and let $D \subset C$ be a nonempty coalition. Then D is a committee of Γ if and only if it is a committee of Γ_C.*

Proof. (a) Suppose that $\Gamma_D = (D; d)$ is a committee game of Γ_C. For every $S \subset N$,

$$\begin{aligned} v(S) &= c(S \cap C)v(S \cup C) + \bar{c}(S \cap C)v(S \setminus C) \\ &= \{d(S \cap D)c[(S \cap C) \cup D] + \bar{d}(S \cap D)c[(S \cap C) \setminus D]\}v(S \cup C) \\ &\quad + \{1 - d(S \cap D)c[(S \cap C) \cup D] - \bar{d}(S \cap D)c[(S \cap C) \setminus D]\}v(S \setminus C) \\ &= d(S \cap D)\{c[(S \cap C) \cup D]v(S \cup C) + \bar{c}[(S \cap C) \cup D]v(S \setminus C)\} \\ &\quad + \bar{d}(S \cap D)\{c[(S \cap C) \setminus D]v(S \cup C) + \bar{c}[(S \cap C) \setminus D]v(S \setminus C)\} \\ &= d(S \cap D)v(S \cup D) + \bar{d}(S \cap D)v(S \setminus D). \end{aligned} \tag{3.3}$$

It follows that Γ_D is a committee game of Γ too.

(b) Suppose that $\Gamma_D = (D; d)$ is a committee game of Γ. Let $T \subset N \setminus C$ be a coalition such that $v(C \cup T) \neq v(T)$ (notice that C is essential). For every $S \subset C$,

$$c(S \cup D) = \frac{v(S \cup D \cup T) - v(T)}{v(C \cup T) - v(T)}, \tag{3.4}$$

$$c(S \setminus D) = \frac{v[(S \setminus D) \cup T] - v(T)}{v(C \cup T) - v(T)} \tag{3.5}$$

and

$$
\begin{aligned}
c(S) &= \frac{v(S \cup T) - v(T)}{v(C \cup T) - v(T)} \\
&= \frac{d(S \cap D)v(S \cup D \cup T) + \bar{d}(S \cap D)v[(S \setminus D) \cup T] - v(T)}{v(C \cup T) - v(T)} \\
&= d(S \cap D)\frac{v(S \cup D \cup T) - v(T)}{v(C \cup T) - v(T)} + \bar{d}(S \cap D)\frac{v[(S \setminus D) \cup T] - v(T)}{v(C \cup T) - v(T)} \\
&= d(S \cap D)c(S \cup D) + \bar{d}(S \cap D)c(S \setminus D).
\end{aligned} \tag{3.6}
$$

Thus Γ_D is a committee game of Γ_C too.

LEMMA 3.4. *Let $\Gamma_C = (C; c)$ be a committee game of $\Gamma = (N; v)$ and let $D \subset N \setminus C$ be a nonempty coalition. Then D is a committee of Γ if and only if it is a committee of the contraction game $\Gamma/_C$.*

Proof. (a) Suppose that $\Gamma_D = (D; d)$ is a committee game of Γ. Let $S \subset N_C$ be a coalition (see Definition 2.3). If $i_C \in S$, then

$$
\begin{aligned}
v_C(S) &= v[(S \setminus \{i_C\}) \cup C] \\
&= d[((S \setminus \{i_C\}) \cup C) \cap D]v[(S \setminus \{i_C\}) \cup C \cup D] \\
&\quad + \bar{d}[((S \setminus \{i_C\}) \cup C) \cap D]v[((S \setminus \{i_C\}) \cup C) \setminus D] \\
&= d(S \cap D)v_C(S \cup D) + \bar{d}(S \cap D)v_C(S \setminus D).
\end{aligned} \tag{3.7}
$$

If $i_C \notin S$, then

$$
\begin{aligned}
v_C(S) &= v(S) \\
&= d(S \cap D)v(S \cup D) + \bar{d}(S \cap D)v(S \setminus D) \\
&= d(S \cap D)v_C(S \cup D) + \bar{d}(S \cap D)v_C(S \setminus D).
\end{aligned} \tag{3.8}
$$

It follows from (3.7)–(3.8) that Γ_D is a committee game of $\Gamma/_C$ too.

(b) Suppose that $\Gamma_D = (D; d)$ is a committee game of $\Gamma/_C$. For every $S \subset N$,

$$
\begin{aligned}
v(S) &= c(S \cap C)v(S \cup C) + \bar{c}(S \cap C)v(S \setminus C) \\
&= c(S \cap C)v_C[(S \setminus C) \cup \{i_C\}] + \bar{c}(S \cap C)v_C(S \setminus C) \\
&= c(S \cap C)\{d(S \cap D)v_C[(S \setminus C) \cup \{i_C\} \cup D] \\
&\qquad + \bar{d}(S \cap D)v_C[((S \setminus C) \cup \{i_C\}) \setminus D]\} \\
&\quad + \bar{c}(S \cap C)\{d(S \cap D)v_C[(S \setminus C) \cup D] + \bar{d}(S \cap D)v_C[(S \setminus C) \setminus D]\} \\
&= d(S \cap D)\{c(S \cap C)v(S \cup D \cup C) + \bar{c}(S \cap C)v[(S \cup D) \setminus C]\} \\
&\quad + \bar{d}(S \cap D)\{c(S \cap C)v[(S \setminus D) \cup C] + \bar{c}(S \cap C)v[(S \setminus D) \setminus C]\} \\
&= d(S \cap D)v(S \cup D) + \bar{d}(S \cap D)v(S \setminus D).
\end{aligned} \tag{3.9}
$$

Thus Γ_D is a committee game of Γ too.

LEMMA 3.5. *Let $\Gamma_C = (C; c)$ be a committee game of $\Gamma = (N; v)$ and let $D \subset N$ be a coalition such that $C \subset D$. Denote $D_C = (D \setminus C) \cup \{i_C\}$ (see Definition 2.3). Under these conditions, D is a committee of Γ if and only if D_C is a committee of the contraction game $\Gamma/_C$.*

Proof. (a) Suppose that $\Gamma_D = (D;d)$ is a committee game of Γ. For every $B \subset D_C$ define

$$d_C(B) = \begin{cases} d(B) & \text{if } i_C \notin B, \\ d[(B \setminus \{i_C\}) \cup C] & \text{if } i_C \in B. \end{cases} \tag{3.10}$$

Let $S \subset N_C$ be a coalition. If $i_C \notin S$, then

$$\begin{aligned} v_C(S) &= v(S) \\ &= d(S \cap D)v(S \cup D) + \bar{d}(S \cap D)v(S \setminus D) \\ &= d_C(S \cap D_C)v_C(S \cup D_C) + \bar{d}_C(S \cap D_C)v_C(S \setminus D_C). \end{aligned} \tag{3.11}$$

Similarly, if $i_C \in S$, then

$$\begin{aligned} v_C(S) &= v[(S \setminus \{i_C\}) \cup C] \\ &= d[((S \setminus \{i_C\}) \cup C) \cap D]v[(S \setminus \{i_C\}) \cup D] \\ &\quad + \bar{d}[((S \setminus \{i_C\}) \cup C) \cap D]v[(S \setminus \{i_C\}) \setminus D] \\ &= d_C(S \cap D_C)v_C(S \cup D_C) + \bar{d}_C(S \cap D_C)v_C(S \setminus D_C). \end{aligned} \tag{3.12}$$

It follows from (3.11)–(3.12) that $(D_C;d_C)$ is a committee game of $\Gamma/_C$.

(b) Suppose that $\Gamma_{D_C} = (D_C;d_C)$ is a committee game of $\Gamma/_C$. For every $B \subset D$ define

$$d(B) = c(B \cap C)d_C[(B \setminus C) \cup \{i_C\}] + \bar{c}(B \cap C)d_C(B \setminus C). \tag{3.13}$$

Let $S \subset N$ be a coalition and denote $B = S \cap D$. Thus

$$\begin{aligned} v(S \cup C) &= v_C[(S \setminus C) \cup \{i_C\}] \\ &= d_C[((S \setminus C) \cup \{i_C\}) \cap D_C]v_C[((S \setminus C) \cup \{i_C\}) \cup D_C] \\ &\quad + \bar{d}_C[((S \setminus C) \cup \{i_C\}) \cap D_C]v_C[((S \setminus C) \cup \{i_C\}) \setminus D_C] \\ &= d_C[(B \setminus C) \cup \{i_C\}]v(S \cup D) + \bar{d}_C[(B \setminus C) \cup \{i_C\}]v(S \setminus D). \end{aligned} \tag{3.14}$$

Also,

$$\begin{aligned} v(S \setminus C) &= v_C(S \setminus C) \\ &= d_C[(S \setminus C) \cap D_C]v_C[(S \setminus C) \cup D_C] \\ &\quad + \bar{d}_C[(S \setminus C) \cap D_C]v_C[(S \setminus C) \setminus D_C] \\ &= d_C(B \cap C)v(S \cup D) + \bar{d}_C(B \cap C)v(S \setminus D). \end{aligned} \tag{3.15}$$

It follows from (3.14)–(3.15) that

$$\begin{aligned} v(S) &= c(B \cap C)v(S \cup C) + \bar{c}(B \cap C)v(S \setminus C) \\ &= c(B \cap C)\{d_C[(B \setminus C) \cup \{i_C\}]v(S \cup D) + \bar{d}_C[(B \setminus C) \cup \{i_C\}]v(S \setminus D)\} \\ &\quad + \bar{c}(B \cap C)\{d_C(B \setminus C)v(S \cup D) + \bar{d}_C(B \setminus C)v(S \setminus D)\} \\ &= d(B)v(S \cup D) + \bar{d}(B)v(S \setminus D). \end{aligned} \tag{3.16}$$

Thus $\Gamma_D = (D;d)$ is a committee game of Γ.

4. Intersecting committees. In this section we shall be dealing with dummy-free monotonic games. It is a consequence of these assumptions that *every non-empty coalition is essential.*

We use the following notation. $\Gamma = (N;v)$ is a game and $\Gamma_C = (C;c)$ and $\Gamma_D = (D;d)$ are two committee games of Γ. Denote

(4.1a, b, c, d) $\quad E = C \cup D, \quad E_1 = C \setminus D, \quad E_2 = C \cap D, \quad E_3 = D \setminus C.$

We assume that E_1, E_2, and E_3 are all nonempty.

LEMMA 4.1. *The intersection, $E_2 = C \cap D$, of the committees C, D is also a committee of Γ.*

Proof. Let $T \subset N \setminus E_2$ be a coalition such that

$$v(E_2 \cup T) \neq v(T). \tag{4.2}$$

(Notice that E_2 is assumed to be nonempty and, therefore, essential.) Denote $T_0 = T \setminus E$, $T_i = T \cap E_i$, $i = 1, 2, 3$ (see (4.1)). For every $B \subset E_2$ define

$$e_2(B) = \frac{v(B \cup T) - v(T)}{v(E_2 \cup T) - v(T)}. \tag{4.3}$$

The committeehood of C implies

$$\begin{aligned} e_2(B) &= \frac{[c(B \cup T_1) - c(T_1)][v(C \cup T) - v(T \setminus C)]}{[c(E_2 \cup T_1) - c(T_1)][v(C \cup T) - v(T \setminus C)]} \\ &= \frac{c(B \cup T_1) - c(T_1)}{c(E_2 \cup T_1) - c(T_1)}. \end{aligned} \tag{4.4}$$

Thus the definition of $e_2(B)$ (see (4.3)) is independent of T_0 and T_3 (provided (4.2) is satisfied). Analogously, the committeehood of D implies

$$e_2(B) = \frac{d(B \cup T_3) - d(T_3)}{d(E_2 \cup T_3) - d(T_3)}, \tag{4.5}$$

and thus the definition of $e_2(B)$ is also independent of T_1 (provided (4.2) holds). Moreover, if $T', T'' \subset N \setminus E_2$ satisfy (4.2), i.e., $v(E_2 \cup T') \neq v(T')$ and $v(E_2 \cup T'') \neq v(T'')$, then the same is true for $(T' \cap E_1) \cup (T'' \setminus E_1)$ and $(T' \cap E_3) \cup (T'' \setminus E_3)$. Thus $\Gamma_{E_2} = (E_2;e_2)$ is a committee game of Γ.

LEMMA 4.2. *Let $\Gamma = (N;v)$ be a dummy-free game satisfying* (2.1)–(2.4), *and let C, D be committees of Γ such that E_1, E_2 and E_3 are all nonempty* (see (4.1)). *Under these conditions, $E_1 = C \setminus D$ is a committee of Γ.*

We first prove the following.

ASSERTION 4.2a. *If $d(E_3) = 0$, then there exists a coalition $T \subset N \setminus E$ such that*

$$v(E \cup T) \neq v(C \cup T). \tag{4.6}$$

Proof. It follows from the equality $d(E_3) = 0$ that for every $T \subset N \setminus E$,

$$v(E_3 \cup T) = v(T). \tag{4.7}$$

Suppose, per absurdum, that for every $T \subset N \setminus E$,

$$v(E \cup T) = v(C \cup T). \tag{4.8}$$

On the other hand, for every $C^* \subset C$,

$$v(C^* \cup T) = c(C^*)v(C \cup T) + \bar{c}(C^*)v(T) \tag{4.9}$$

and

$$v(C^* \cup E_3 \cup T) = c(C^*)v(E \cup T) + \bar{c}(C^*)v(E_3 \cup T). \tag{4.10}$$

It follows from (4.7)–(4.10) that

$$v(C^* \cup T) = v(C^* \cup E_3 \cup T). \tag{4.11}$$

Thus E_3 is inessential, in contradiction to our assumption.

ASSERTION 4.2b. *If there is a coalition $T \subset N \setminus E$ such that $v(E_3 \cup T) \neq v(E \cup T)$, then there exist real numbers λ, μ such that $\lambda^2 + \mu^2 \neq 0$, and for every $B \subset E_1$,*

$$\lambda c(B) = \mu[c(B \cup E_2) - c(E_2)]. \tag{4.12}$$

Proof. For every $B \subset E_1$ and $T \subset N \setminus E$,

$$\begin{aligned} v(B \cup E_3 \cup T)\, c(B)v(E \cup T) + \bar{c}(B)v(E_3 \cup T) &\\ = c(B)[v(E \cup T) - v(E_3 \cup T)] &\\ + d(E_3)v(D \cup T) + \bar{d}(E_3)v(T) &\\ = c(B)[v(E \cup T) - v(E_3 \cup T)] &\\ + d(E_3)\{[v(E \cup T) - v(E_3 \cup T)]c(E_2) + v(E_3 \cup T)\} &\\ + \bar{d}(E_3)v(T). & \end{aligned} \tag{4.13}$$

On the other hand,

$$\begin{aligned} v(B \cup E_3 \cup T) = d(E_3)\{[v(E \cup T) - v(E_3 \cup T)]c(B \cup E_2) + v(E_3 \cup T)\} &\\ + \bar{d}(E_3)\{[v(C \cup T) - v(T)]c(B) + v(T)\}. & \end{aligned} \tag{4.14}$$

Define

$$\lambda = v(E \cup T) - v(E_3 \cup T) - \bar{d}(E_3)[v(C \cup T) - v(T)] \tag{4.15}$$

and

$$\mu = d(E_3)[v(E \cup T) - v(E_3 \cup T)]. \tag{4.16}$$

If $d(E_3) = 0$, then

$$\lambda = v(E \cup T) - v(C \cup T) \tag{4.17}$$

(see (4.7)). According to Assertion 4.2a we can choose T such that $\lambda \neq 0$. If $d(E_3) \neq 0$, then we can choose T such that $\mu \neq 0$ (this is actually our assumption). It follows from (4.13)–(4.16) that (4.12) is satisfied.

The proof of Lemma 4.2. Let $S \subset N \setminus E_1$ be a coalition such that $v(S \cup E_1) \neq v(S)$ (notice that E_1 is essential). For every $B \subset E_1$ define

$$e_1(B) = \frac{v(B \cup S) - v(S)}{v(E_1 \cup S) - v(S)}. \tag{4.18}$$

We shall prove that this definition is independent of S (provided $v(E_1 \cup S) \neq v(S)$).

Denote $D^* = S \cap D$ and $T = S \setminus E$. A short calculation yields

(4.19)

$$e_1(B) = \frac{d(D^*)[c(B \cup E_2) - c(E_2)][v(E \cup T) - v(E_3 \cup T)] + \bar{d}(D^*)c(B)[v(C \cup T) - v(T)]}{d(D^*)\bar{c}(E_2)[v(E \cup T) - v(E_3 \cup T)] + \bar{d}(D^*)c(E_1)[v(C \cup T) - v(T)]}.$$

If for every coalition $T \subset N \setminus E$,

$$v(E \cup T) = v(E_3 \cup T), \tag{4.20}$$

then (4.19) is reduced to

$$e_1(B) = c(B)/(c(E_1)). \tag{4.21}$$

This is obviously independent of S. Suppose that $T \subset N \setminus E$ is a coalition such that

$$v(E \cup T) \neq v(E_3 \cup T). \tag{4.22}$$

It follows from Assertion 4.2b that $e_1(B)$ is independent of the numbers

$$d(D^*)[v(E \cup T) - v(E_3 \cup T)]$$

and

$$\bar{d}(D^*)[v(C \cup T) - v(T)].$$

Thus $e_1(B)$ is independent of D^* and T (provided $v(S \cup E_1 \neq v(S))$ and hence E_1 is a committee of Γ.

LEMMA 4.3. *Under the conditions of Lemma 4.2, the union $E = C \cup D$ is a committee of Γ.*

We first prove the following.

ASSERTION 4.3a. *Under the above conditions, either $d(E_3) \neq 1$ or $c(E_2) \neq 0$.*

Proof. Suppose, per absurdum, that both

$$d(E_3) = 1 \tag{4.23}$$

and

$$c(E_2) = 0. \tag{4.24}$$

Let $T \subset N \setminus E$ be any coalition and denote

$$v_0 = v(T), \tag{4.25}$$

$$v_i = v(E_i \cup T), \qquad i = 1, 2, 3, \tag{4.26}$$

$$v_{ij} = v(E_i \cup E_j \cup T), \qquad 1 \leqq i < j \leqq 3, \tag{4.27}$$

and

$$v_{123} = v(E \cup T). \tag{4.28}$$

It follows from (4.23)–(4.24) that

$$v_2 = v_0, \tag{4.29}$$

$$v_{23} = v_3, \tag{4.30}$$

$$v_{13} = v_{123}, \tag{4.31}$$

$$v_2 = d(E_2)v_{23} + \bar{d}(E_2)v_0. \tag{4.32}$$

It follows from (4.29) and (4.32) that

$$d(E_2)(v_{23} - v_0) = 0. \tag{4.33}$$

If $d(E_2) = 0$, then E_2 is inessential. This can be deduced from Lemmas 3.3 and 4.2 since they imply for $B \subset E_3$, $d(B) = d(E_3)e_3(B) + \bar{e}_3(B)\,d(\varnothing) = e_3(B)$, and $d(B \cup E_2) = d(D)e_3(B) + \bar{e}_3(B)\,d(E_2) = e_3(B)$. That is, E_2 is inessential in Γ_D, whence inessential in Γ. Thus necessarily $d(E_2) \neq 0$, and (4.33) implies

$$v_{23} = v_0. \tag{4.34}$$

Also,

$$v_{13} = c(E_1)v_{123} + \bar{c}(E_1)v_3. \tag{4.35}$$

If $c(E_1) = 0$, then by (4.35) $v_{13} = v_3$, so that (4.30), (4.31), and (4.34) imply

$$v_{123} = v_0. \tag{4.36}$$

Thus E is inessential. If $c(E_1) = 1$, then E_2 can be seen to be inessential from Lemmas 3.3 and 4.2 as above. If $0 < C(E_1) < 1$, then (4.31), (4.34) and (4.35) imply (4.36) (notice that (4.34) implies $v_3 = v_0$ by monotonicity) and, again, E is inessential. Thus Assertion 4.3a is proved.

The proof of Lemma 4.3. For every $S \subset N$,

$$v[(S \setminus C) \cup D] = c(E_2)v(S \cup E) + \bar{c}(E_2)v[(S \cup D)\setminus C], \tag{4.37}$$

$$v[(S \cup D)\setminus C] = d(E_3)v[(S\setminus C) \cup D] + \bar{d}(E_3)v(S\setminus E). \tag{4.38}$$

Consider (4.37)–(4.38) as a system of two simultaneous linear equations for the unknowns $v[(S\setminus C) \cup D]$ and $v[(S \cup D)\setminus C]$. Assertion 4.3a implies that this system has a unique solution, namely,

$$v[(S \cup D)\setminus C] = \frac{c(E_2)\,d(E_3)v(S \cup E) + \bar{d}(E_3)v(S\setminus E)}{1 - d(E_3)\bar{c}(E_2)}, \tag{4.39}$$

$$v[(S\setminus C) \cup D] = \frac{c(E_2)v(S \cup E) + \bar{c}(E_2)\,\bar{d}(E_3)v(S\setminus E)}{1 - d(E_3)\bar{c}(E_2)}. \tag{4.40}$$

Symmetrically,

$$v[(S \cup C)\setminus D] = \frac{d(E_2)c(E_1)v(S \cup E) + \bar{c}(E_1)v(S\setminus E)}{1 - c(E_1)\bar{d}(E_2)}, \tag{4.41}$$

$$v[(S\setminus D) \cup C] = \frac{d(E_2)v(S \cup E) + \bar{d}(E_2)\bar{c}(E_1)v(S\setminus E)}{1 - c(E_1)\bar{d}(E_2)} \tag{4.42}$$

For every $S \subset N$,

(4.43)

$$\begin{aligned} v(S) &= c(S \cap C)v(S \cup C) + \bar{c}(S \cap C)v(S\setminus C) \\ &= c(S \cap C)\{d[(S \cup C) \cap D]v(S \cup E) + \bar{d}[S \cup C) \cap D]v[(S \cup C)\setminus D]\} \\ &\quad + \bar{c}(S \cap C)\{d[(S\setminus C) \cap D]v[(S\setminus C) \cup D] + \bar{d}[(S\setminus C) \cap D]v(S\setminus E)\}. \end{aligned}$$

For every $B \subset E$ define

$$e(B) = c(B \cap C)\left\{d[(B \cup C) \cap D] + \frac{\bar{d}[B \cup C) \cap D]d(E_2)c(E_1)}{1 - c(E_1)\bar{d}(E_2)}\right\} + \frac{\bar{c}(B \cap C)d[(B \setminus C) \cap D]c(E_2)}{1 - d(E_3)\bar{c}(E_2)}. \tag{4.44}$$

Substituting (4.40) and (4.41) in (4.43), we find that

$$v(S) = e(S \cap E)v(S \cup E) + \bar{e}(S \cap E)v(S \setminus E). \tag{4.45}$$

Thus, $(E; e)$ is a committee game of Γ.

LEMMA 4.4. *Under the conditions of Lemma* 4.2, $E_1 \cup E_3 = (C \setminus D) \cup (D \setminus C)$ *is also a committee of* Γ.

Proof. (a) Let Γ_3 be a 3-player game satisfying (2.1)–(2.4) and denote the value of a coalition $S \subset \{1, 2, 3\}$ in Γ_3 by v_S. It is easy to verify (see Lemma 3.1) that $\{1, 2\}$ is a committee of Γ_3 if and only if both

$$v_{12}(v_{13} - v_3) = v_1(1 - v_3) \tag{4.46}$$

and

$$v_{12}(v_{23} - v_3) = v_2(1 - v_3). \tag{4.47}$$

(Equalities (4.46), (4.47) are necessary and sufficient for defining a value to $\{1\}$, $\{2\}$, respectively, in a committee game over $\{1, 2\}$, and these two coalitions are the significant ones since $\{1, 2\}$ must have a unit value.) Analogously, $\{2, 3\}$ is a committee of Γ_3 if and only if both

$$v_{23}(v_{12} - v_1) = v_2(1 - v_1) \tag{4.48}$$

and

$$v_{23}(v_{13} - v_1) = v_3(1 - v_1). \tag{4.49}$$

Suppose that both $\{1, 2\}$ and $\{2, 3\}$ are committees of Γ_3. We shall prove that also $\{1, 3\}$ is a committee of Γ_3. Indeed, if $v_{12} = 0$, then $v_1 = v_2 = 0$ and by (4.49), $v_{23}v_{13} = v_3$. Thus both

$$v_{13}(v_{12} - v_2) = v_1(1 - v_2) \tag{4.50}$$

and

$$v_{13}(v_{23} - v_2) = v_3(1 - v_2). \tag{4.51}$$

If $v_3 = 1$, then $v_{13} = v_{23} = 1$. In this case, (4.51) holds and (4.48) implies (4.50). If both $v_{12} \neq 0$ and $v_3 \neq 1$, then (4.46)–(4.47) imply

$$v_2(v_{13} - v_3) = v_1(v_{23} - v_3). \tag{4.52}$$

Equalities (4.49) and (4.52) then imply (4.51).

Symmetrically, if either $v_{23} = 0$ or $v_1 = 1$, then both (4.50) and (4.51) hold, and if both $v_{23} \neq 0$ and $v_1 \neq 1$, then (4.48)–(4.49) imply

$$v_3(v_{12} - v_1) = v_2(v_{13} - v_1). \tag{4.53}$$

Equalities (4.46) and (4.53) then imply (4.50). Thus in every case, (4.50)–(4.51) hold and, therefore, $\{1, 3\}$ is a committee of Γ_3.

(b) In Lemmas 4.1–4.3 we have proved that E_1, E_2, E_3, and E are committees of Γ. Lemma 3.3 implies that they are all committees of the committee game $\Gamma_E = (E; e)$ too. Also, $E_1 \cup E_2$ and $E_2 \cup E_3$ are committees of Γ_E. The game Γ_E can be represented as

$$\Gamma_E = \Gamma_3[\Gamma_{E_1}, \Gamma_{E_2}, \Gamma_{E_3}], \tag{4.54}$$

where Γ_{E_i}, $i = 1, 2, 3$, are the appropriate committee games and Γ_3 is a 3-player game of which $\{1, 2\}$ and $\{2, 3\}$ are committees. According to part (a) of the present proof, $\{1, 3\}$ is a committee of Γ_3. Thus, following Lemma 3.5, $E_1 \cup E_3$ is a committee of Γ_E whence (Lemma 3.3) of Γ.

5. Perfect compositions.

DEFINITION 5.1. (i) A tensor composition $\Gamma = \Gamma_0[\Gamma_1, \cdots, \Gamma_m]$ (see Definition 2.1) is called *perfect*[1] if for every nonempty coalition $T \subset M$ the coalition $N^T = \bigcup_{i \in T} N_i$ is a committee of Γ.

(ii) An *absolutely decomposable* game is a game every nonempty coalition of which is a committee.

(iii) A composition (or decomposition) is called *prime* if the quotient game is prime.

LEMMA 5.2. *A composition* $\Gamma = \Gamma_0[\Gamma_1, \cdots, \Gamma_m]$ (see Definition 2.1) *is perfect if and only if the quotient game* Γ_0 *is absolutely decomposable.*

The proof follows from successive applications of Lemma 3.5.

Example 5.3. Let $\otimes^m$ be the m-player ($m \geqq 1$) unanimity[2] game. Obviously, $\otimes^m$ is absolutely decomposable (for every nonempty coalition $T \subset M$ the $|T|$-player unanimity game over T is a committee game of $\otimes^m$). Thus the *product* $\otimes^m[\Gamma_1, \cdots, \Gamma_m]$ of the games $\Gamma_1, \cdots, \Gamma_m$, in which

$$v(S) = \prod_{i=1}^{m} w_i(S \cap N_i), \quad S \subset N, \tag{5.1}$$

is a perfect composition.

Example 5.4. Let $\oplus^m = (M; u)$ be an m-player game, where $u(T) = 1$ for every nonempty $T \subset M$ and $u(\varnothing) = 0$. Obviously, $\otimes^m$ is absolutely decomposable and therefore the *sum* $\oplus^m[\Gamma_1, \cdots, \Gamma_m]$ of the games $\Gamma_1, \cdots, \Gamma_m$ is a perfect composition. It follows from [9, Lemma 1, p. 314] that for every $S \subset N$,

$$v(S) = \sum_{\varnothing \neq T \subset M} (-1)^{|T|+1} \prod_{i \in T} w_i(S \cap N_i). \tag{5.2}$$

Remark 5.5. According to our definitions, every 2-person game is both prime and absolutely decomposable. Moreover, this property characterizes the games of at most two players. Thus every composition of two components is perfect.

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[1] We relate the word "perfect" both to compositions and decompositions.

[2] $\otimes^m = (M; u)$, where $u(M) = 1$ and for every $T \subsetneqq M$, $u(T) = 0$.

Remark 5.6. Every additive[3] game is absolutely decomposable (with additive committee games). A composition with an additive quotient is therefore perfect. A composition $\Gamma = \Gamma_0[\Gamma_1, \cdots, \Gamma_m]$, where Γ_0 is additive, is a "convex combination" of the games $\Gamma_1, \cdots, \Gamma_m$ in the sense that for every $S \subset N$,

$$v(S) = \sum_{i=1}^{m} u(\{i\}) w_i(S \cap N_i). \tag{5.3}$$

Example 5.7. Let $\Gamma_3 = (M; u)$ be a 3-player game, where $u(\{i\}) = 1/7$, $i = 1, 2, 3$, $u(\{i, j\}) = 3/7$, $1 \leqq i < j \leqq 3$, $u(\varnothing) = 0$ and $u(M) = 1$. It is easily verified that Γ_3 is a nonadditive absolutely decomposable game which is different from $\otimes^3$ and $\oplus^3$.

LEMMA 5.8. *A dummy-free game $\Gamma = (N; v)$ has a perfect decomposition if and only if there is a partition of N into m ($m \geqq 2$) disjoint committees $N_1, \cdots, N_m$ such that for every pair i, j, $1 \leqq i < j \leqq m$, $N_i \cup N_j$ is also a committee of Γ.*

Proof. Necessity follows from Lemmas 5.2 and 2.4. Suppose that there exists a partition of N as specified in the Lemma. We shall prove, by induction on $|T|$, that for every $T \subset \{1, \cdots, m\}$ the coalition $N^T = \bigcup_{i \in T} N_i$ is a committee. If $|T| \leqq 2$, then N_T is assumed to be a committee. If $|T| > 2$, then N^T is a union of two intersecting coalitions, N^{T_1} and N^{T_2}, where $|T_1| = |T_2| = |T| - 1$. N^{T_1} and N^{T_2} are committees by the induction assumption. Lemma 4.3 implies then that also N^T is a committee. It follows that by contracting Γ on the committees $N_1, \cdots, N_m$ successively, we finally reach an absolutely decomposable game Γ_0 such that $\Gamma = \Gamma_0[\Gamma_1, \cdots, \Gamma_m]$ (where Γ_i is the respective committee game over N_i, $i = 1, \cdots, m$). Thus Γ has a perfect decomposition.

DEFINITION 5.9. Let $\Gamma = (N; v)$ be a game and let $N = C_1 \cup \cdots \cup C_r = D_1 \cup \cdots \cup D_q$ be two perfect decompositions of Γ (see Lemma 5.8). If $r > q$ and if for every i, $i = 1, \cdots, r$, there is j, $1 \leqq j \leqq q$, such that $C_i \subset D_j$, then the decomposition $N = C_1 \cup \cdots \cup C_r$ is called a *refinement* of the decomposition $N = D_1 \cup \cdots \cup D_q$. A perfect decomposition which has no refinements is called an *unrefinable perfect decomposition.* Notice that if Γ is an absolutely decomposable game, then the trivial decomposition, with Γ itself as the quotient game, is an unrefinable perfect decomposition.

LEMMA 5.10. *Let $\Gamma = (N; v)$ be a dummy-free game satisfying* (2.1)–(2.4) *and let $N = N_1 \cup \cdots \cup N_m$ be an unrefinable perfect decomposition of Γ. If C is a committee of Γ, then for every i, $i = 1, \cdots, m$, either $C \cap N_i = \varnothing$, or $C \subset N_i$ or $N_i \subset C$.*

Proof. Suppose, per absurdum, that our statement is false. Without loss of generality, assume that $C \cap N_1 \neq \varnothing$, $N_1 \setminus C \neq \varnothing$ and $C \setminus N_1 \neq \varnothing$. Lemmas 4.1–4.4 imply that $N_1 \cap C$ and $N_1 \setminus C$ are committees and also for every i, $i = 2, \cdots, m$, $(N_1 \cap C) \cup N_i$ and $(N_1 \setminus C) \cup N_i$ are committees. Thus, according to Lemma 5.8, $N = (N_1 \cap C) \cup (N_1 \setminus C) \cup N_2 \cup \cdots \cup N_m$ is a perfect decomposition of Γ, in contradiction to our assumption that $N = N_1 \cup \cdots \cup N_m$ is unrefinable.

THEOREM 5.11. *If $\Gamma = (N; v)$ is a dummy-free game satisfying* (2.1)–(2.4), *then there can be no more than one unrefinable perfect decomposition of Γ.*

[3] $\Gamma = (M; u)$ is called additive if for every $T \subset M$, $u(T) = \sum_{i \in T} u(\{i\})$.

Proof. Suppose, per absurdum, that there are two distinct unrefinable perfect decompositions of Γ,

$$N = C_1 \cup \cdots \cup C_r, \tag{5.4}$$

and

$$N = D_1 \cup \cdots \cup D_q. \tag{5.5}$$

Denote

$$E_{ij} = C_i \cap D_j, \qquad i = 1, \cdots, r, \quad j = 1, \cdots, q. \tag{5.6}$$

According to Lemma 4.1, every nonempty coalition of the form E_{ij} is a committee of Γ. Let E_{ij} and E_{kl} be two committees. If $E_{ij} = E_{kj} = \varnothing$, then

$$E_{ij} \cup E_{kl} = (C_i \cup C_k) \cap (D_j \cup D_l), \tag{5.7}$$

and since $C_i \cup C_k$ and $D_j \cup D_l$ are committees, it follows that $E_{ij} \cup E_{kl}$ is a committee (Lemma 4.3). If, for example, $E_{il} \neq \varnothing$ (and E_{kj} is either empty or nonempty), then

$$E_{ij} \cup E_{kl} = (C \setminus D) \cup (D \setminus C), \tag{5.8}$$

where

$$C = C_i \cap (D_j \cup D_l) \tag{5.9}$$

and

$$D = (C_i \cup C_k) \cap D_l. \tag{5.10}$$

Since C and D are committees, it follows that $E_{ij} \cup E_{kl}$ is a committee too. Thus, according to Lemma 5.8,

$$N = \bigcup \{E_{ij} : 1 \leqq i \leqq r, 1 \leqq j \leqq q, E_{ij} \neq \varnothing\} \tag{5.11}$$

is a perfect decomposition which refines (5.4) and (5.5), in contradiction to our assumption that they are unrefinable.

Remark 5.12. A component in an unrefinable perfect decomposition can have a perfect decomposition of its own. For example, in the game[4] $\Gamma_3 = (B_1 \oplus B_1) \otimes B_1$, $N = \{1, 2\} \cup \{3\}$ is an unrefinable perfect decomposition of Γ_3, whereas $\{1, 2\} = \{1\} \cup \{2\}$ is a perfect decomposition of $B_1 \oplus B_1$.

6. The unique decomposition theorem.

Example 6.1. Let B_S denote the unanimity game over the nonempty finite set S (see Example 5.3). It can be easily verified that

$$B_{123} = B_{12} \otimes B_3 = B_{13} \otimes B_2 = B_1 \otimes B_{23} \tag{6.1}$$

(see Remark 5.12). Thus the game B_{123} has at least three different decompositions. Notice that B_{123} is dummy-free. Moreover, since every 2-person game is prime, it follows that these decompositions of B_{123} are all prime (see Definition 5.1).

[4] B_1 is a 1-player unanimity game. $B_1 \oplus B_1 = \oplus^2[B_1, B_1]$ (see Example 5.4), and $B_1 \otimes B_1 = \otimes^2[B_1, B_1]$ (see Example 5.3).

The game B_{123} in Example 6.1 also has a perfect decomposition, because this game is absolutely decomposable (see Definition 5.9). We shall prove that whenever a game has at least two prime decompositions, it has a perfect decomposition.

LEMMA 6.2. *If* $\Gamma = (N; v)$ *is a dummy-free game satisfying* (2.1)–(2.4), *then exactly one of the following statements is true:*

(i) Γ *has a perfect decomposition.*

(ii) *There are at least three maximal*[5] *committees of* Γ *and they are disjoint.*

Proof. (a) Suppose that there are two distinct maximal committees C, D of Γ such that $C \cap D \neq \varnothing$. According to Lemma 4.3, $C \cup D$ is a committee too. Since C and D are maximal, necessarily, $C \cup D = N$. Moreover, Lemmas 4.1–4.4 imply that also $C \setminus D$, $C \cap D$, $D \setminus C$, and $(C \setminus D) \cup (D \setminus C)$ are committees. Thus $N = (C \setminus D) \cup (C \cap D) \cup (D \setminus C)$ is a perfect decomposition of Γ (see Lemma 5.8).

(b) If there are exactly two maximal committees and they are disjoint, then Γ has a perfect decomposition into two components.

(c) If there are at least three maximal disjoint committees, then Γ does not have a perfect decomposition since each committee is contained in a maximal committee and the union of two maximal committees is not a committee under these conditions.

THEOREM 6.3. *If* $\Gamma = (N; v)$ *is a game satisfying* (2.1)–(2.4), *then exactly one of the following statements is true:*

(i) Γ *has a unique unrefinable perfect decomposition.*

(ii) Γ *has a unique prime decomposition with at least three components.*

Proof. (a) If there are no proper committees, then Γ is prime. In this case, the game has a unique prime decomposition, namely, the trivial decomposition with Γ itself as the quotient game. If there are at least three players, then there can be no perfect decomposition of Γ and (ii) is true. If Γ is a 2-person game, then (i) is true.

(b) Suppose that Γ is dummy-free and there is at least one proper committee. Consider the maximal committees of Γ. If C_1 and C_2 are two maximal committees such that $C_1 \cap C_2 \neq \varnothing$, then Γ has a perfect decomposition (Lemma 6.2). According to Theorem 5.11, this implies that Γ has a unique unrefinable perfect decomposition. If all the maximal committees, $C_1, \cdots, C_r$, are disjoint, then Γ can be decomposed as $\Gamma = \Gamma_r[\Gamma_{C_1}, \cdots, \Gamma_{C_r}]$, where Γ_r is an r-player game and Γ_{C_i}, $i = 1, \cdots, r$, is the committee game over C_i. Γ_r is obtained by successive contractions on $C_1, C_2, \cdots, C_r$. Notice that each Γ_{C_i} may happen to be a 1-person game. There are no proper committees in Γ_r since, by Lemma 3.5, the existence of a proper committee of Γ_r would have implied the existence of a proper committee of Γ which would have properly contained a maximal committee of Γ. Thus the above decomposition is prime. Contractions on other committees, or not on all the maximal committees, yield decomposable quotient games. Thus the above decomposition is the unique prime one.

(c) Suppose that there are dummies in Γ. In this case, Γ has the following perfect decomposition:

$$N = \{i_1\} \cup \{i_2\} \cup \cdots \cup \{i_k\} \cup N', \tag{6.2}$$

[5] A committee $C \subsetneqq N$ is called maximal if it is not contained in any other proper committee. Notice that a 1-player committee can be maximal even though it is not a proper committee.

where $i_1, \cdots, i_k$ are the dummies and N' is the set of the nondummies (see Lemma 3.2). Let Γ' be the committee game over N'. If Γ' does not have a perfect decomposition, then (6.2) is the unique unrefinable decomposition of Γ. If Γ' has a perfect decomposition, then it has a unique unrefinable perfect decomposition

$$N' = C_1 \cup \cdots \cup C_m, \qquad m \geqq 2. \tag{6.3}$$

Then

$$N = \{i_1\} \cup \cdots \cup \{i_k\} \cup C_1 \cdots \cup C_m \tag{6.4}$$

is the unique unrefinable perfect decomposition of Γ.

Remark 6.4. According to Theorem 6.3, the component games (in either prime or perfect decomposition) can be decomposed as well. After a finite number of decompositions, each component game will be either a prime or 1-person game. The pattern of this successive decomposition yields a unique hierarchy of committees which is ordered by inclusion. In each grade of this hierarchy, all the committees are disjoint.

Appendix A: Duality.

DEFINITION A.1. The *dual* of a game $\Gamma = (N; v)$ is the game $\Gamma^* = (N; v^*)$, where for every $S \subset N$,

$$v^*(S) = v(N) - v(N \setminus S). \tag{A.1}$$

LEMMA A.2. *Let* Γ_i, $i = 0, 1, \cdots, m$, *be games satisfying* (2.1)–(2.3) *such that* $\Gamma_0 = (M; u)$, $M = \{1, \cdots, m\}$, $\Gamma_i = (N_i; w_i)$, $i = 1, \cdots, m$ *and* $N_i \cap N_j = \varnothing$ *for* $1 \leqq i < j \leqq m$. *Under these conditions,*

$$(\Gamma_0[\Gamma_1, \cdots, \Gamma_m])^* = \Gamma_0^*[\Gamma_1^*, \cdots, \Gamma_m^*]. \tag{A.2}$$

Proof. For every $S \subset N$,

$$\begin{aligned} v^*(S) &= 1 - v(N \setminus S) \\ &= 1 - \sum_{T \subset M} \left\{ \prod_{i \in T} w_i(N_i \setminus S) \prod_{i \notin T} \bar{w}_i(N_i \setminus S) \right\} u(T) \\ &= 1 - \sum_{T \subset M} \left\{ \prod_{i \in T} [1 - w_i^*(S_i \cap N_i)] \prod_{i \notin T} w_i^*(S \cap N_i) \right\} u(T) \\ &= 1 - \sum_{T \subset M} \left\{ \prod_{i \in T} w_i^*(S \cap N_i) \prod_{i \notin T} \overline{w_i^*}(S \cap N_i) \right\} u(M \setminus T). \end{aligned} \tag{A.3}$$

By [9; Lemma 1],

$$\sum_{T \subset M} \left\{ \prod_{i \in T} w_i^*(S \cap N_i) \prod \overline{w_i^*}(S \cap N_i) \right\} = 1. \tag{A.4}$$

Thus (A.3)–(A.4) imply

$$v^*(S) = \sum_{T \subset M} \left\{ \prod_{i \in T} w_i^*(S \cap N_i) \prod_{i \notin T} \overline{w_i^*}(S \cap N_i) \right\} u^*(T) \tag{A.5}$$

COROLLARY A.3. (i) *A coalition $C \subset N$ is a committee of $\Gamma = (N; v)$ if and only if it is a committee in $\Gamma^* = (N; v^*)$. The committee game with respect to Γ^* is the dual of the committee game with respect to Γ.*

(ii) *A game is absolutely decomposable if and only if its dual is absolutely decomposable.*

(iii) (Owen). *The tensor composition of constant-sum*[6] *components with a constant-sum quotient is a constant-sum game. The proof follows from the fact that a game is constant-sum if and only if it is self-dual.*

Appendix B: An interpretation. A *performance indicator* is a binary random variable X_i. A *control unit* is a nonempty finite set of performance indicators $X = \{X_1, \cdots, X_n\}$ (not necessarily independent). A *reliability function* is a function $v(X)$ such that $0 \leqq v(X) \leqq 1$. We call the pair $(X; v)$ a *system* and interpret $v(X)$ to be the probability that the system is functioning when X is the result of control tests. Obviously, a system is isomorphic to a game. A subset $C \subset X$ is called a subsystem if there is a reliability function $c(C)$ such that

$$v(X) = c(C)v(X \vee C) + (1 - c(C))v(X \sim C), \tag{B.1}$$

where $(X \vee C)_i = 1$ either if $X_i = 1$ or if $i \in C$ and $(X \vee C)_i = 0$ otherwise, and $(X \sim C)_i = 1$ if and only if $i \notin C$ and $X_i = 1$. Thus, a subsystem is a set of performance indicators which can be replaced by a single performance indicator. It is easily verified that a subsystem is a committee in the isomorphism between games and systems. A decomposition of a game corresponds to a partition of a system into disjoint subsystems. Our main theorem states that a system decomposes in a unique way into subsystems, every one of which can be replaced by a single performance indicator.

REFERENCES

[1] L. J. BILLERA, *On the composition and decomposition of clutters*, J. Combinatorial Theory, 11 (1971), pp. 234–245.

[2] ———, *Clutter decomposition and monotonic Boolean functions*, Annals of the New York Academy of Sciences, 175 (1970), Article 1, pp. 41–48.

[3] L. J. BILLERA AND R. E. BIXBY, *Decomposition theory for path clutters and flows*, Tech. Rep. 139, Department of Operations Research, Cornell University, Ithaca, N.Y., 1971.

[4] Z. W. BIRNBAUM AND I. D. ESARY, *Modules of coherent binary systems*, J. Soc. Indust. Appl. Math., 13 (1965), pp. 444–462.

[5] R. E. BIXBY, *Composition and decomposition of matroids and related topics*, Ph.D. thesis, Cornell University, Ithaca, N.Y., 1972.

[6] W. CUNNINGHAM, *A combinatorial decomposition theory*, Ph.D. thesis, University of Waterloo, Waterloo, Ontario, 1973.

[7] N. MEGIDDO, *Nucleoluses of compound simple games*, this Journal, 26 (1974), pp. 607–621.

[8] ———, *Kernels of compound games with simple components*, Pacific J. Math., 50 (1974), pp. 531–555.

[9] G. OWEN, *The tensor composition of non-negative games*, Advances in Game Theory, M. Dresher, L. S. Shapley, and H. W. Tucker, eds., Annals of Mathematics Studies, No. 52, Princeton University Press, Princeton, N.J., 1964, pp. 307–326.

[10] ———, *Multilinear extension of games, Part II*, Management Sci., 18 (1972), pp. P-64–P-79.

[6] $\Gamma = (N; v)$ is a constant-sum game if for every $S \subset N$, $v(S) + v(N \setminus S) = v(N)$.

[11] L. S. SHAPLEY, *Compound simple games I: Solutions of sums and products*, RAND Corp., RM-3192, 1962.

[12] ———, *Compound simple games II: Some general composition theorems*, RAND Corp., RM-3643, 1963.

[13] ———, *Solutions of compound simple games*, Advances in Game Theory, M. Dresher, L. S. Shapley, and A. W. Tucker, eds., Annals of Mathematics Studies, No. 52, Princeton University Press, Princeton, N.J., 1964, pp. 267–305.

[14] ———, *Compound simple games III: On committees*, RAND Corp., RM-5438-PR, 1967; also in *New Methods of Thought and Procedure*, F. Zwycky and A. Wilson, eds., Springer, New York, 1968.